On spanning sets and generators of near-vector spaces

Karin-Therese HOWELL\textsuperscript{1,*}, Sogo Pierre SANON\textsuperscript{2,○}
\textsuperscript{1}Department of Mathematical Sciences, Faculty of Science, Stellenbosch University, Stellenbosch, South Africa
\textsuperscript{2}Department of Mathematical Sciences, Faculty of Science, Stellenbosch University, Stellenbosch, South Africa

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Abstract: In this paper we study the quasi-kernel of certain constructions of near-vector spaces and the span of a vector. We characterize those vectors whose span is one-dimensional and those that generate the whole space.

Key words: Field, nearfield, vector space, near-vector space

1. Introduction

The near-vector spaces we study in this paper were first introduced by André in 1974 [1]. His near-vector spaces have less linearity than normal vector spaces. They have been studied in several papers, including [2–6]. More recently, since André did a lot of work in geometry, their geometric structure has come under investigation. In order to construct some incidence structures a good understanding of the span of a vector is necessary. It very quickly became clear that near-vector spaces exhibit some strange behavior, where the span of a vector need not be one-dimensional and it is possible for a single vector to generate the entire space.

In this paper we begin by giving the preliminary material of near-vector spaces. In Section 3 we take a closer look at the class of near-vector spaces of the form \((F^n, F)\), where \(F\) is a nearfield and \(n\) is a natural number, constructed using van der Walt’s important construction theorem in [9] for finite dimensional near-vector spaces. We give conditions for when the quasi-kernel will be the whole space. In the last section we prove that when for a near-vector space \((V, A)\), \(v \in V\), span \(v\) will equal \(vA\). We introduce the dimension of a vector and prove that in the case of a field, it is always less than or equal to the number of maximal regular subspaces in the decomposition of \(V\). We define a generator for \(V\) and give a condition for when \(v\) will be a generator for \(V\). Finally, we characterize the near-vector spaces that have generators.

2. Preliminary material

Definition 2.1 A (right) nearfield is a set \(F\) together with two binary operations \(+\) and \(\cdot\) such that

1. \((F, +)\) is a group;
2. \((F \setminus \{0\}, \cdot)\) is a group;
3. \((a + b) \cdot c = a \cdot c + b \cdot c\) for all \(a, b, c \in F\).

\textsuperscript{*}Correspondence: kthowell@sun.ac.za
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Left nearfields are defined analogously and satisfy the left distributive law. We will use right nearfields throughout this paper. We also have the following definition.

**Definition 2.2** Let $F$ be a nearfield. We define the kernel of $F$ to be the set of all distributive elements of $F$, i.e.

$$F_d := \{ a \in F | a \cdot (b + c) = a \cdot b + a \cdot c \text{ for every } b, c \in F \}.$$ 

If $F$ is a nearfield, $F_d$ is a subfield of it [8]; moreover, $F$ is a vector space over $F_d$. We refer the reader to [7] and [8] for more on nearfields.

**Definition 2.3** ([1]) A near-vector space is a pair $(V, A)$ that satisfies the following conditions:

1. $(V, +)$ is a group and $A$ is a set of endomorphisms of $V$;
2. $A$ contains the endomorphisms $0$, $id$, and $-id$;
3. $A^* = A \setminus \{0\}$ is a subgroup of the group $Aut(V)$;
4. If $x\alpha = x\beta$ with $x \in V$ and $\alpha, \beta \in A$, then $\alpha = \beta$ or $x = 0$, i.e. $A$ acts fixed point free on $V$;
5. The quasi-kernel $Q(V)$ of $V$ generates $V$ as a group. Here, $Q(V) = \{ x \in V | \forall \alpha, \beta \in A, \exists \gamma \in A \text{ such that } x\alpha + x\beta = x\gamma \}$.

We will write $Q(V)^*$ for $Q(V) \setminus \{0\}$ throughout this paper. The *dimension* of the near-vector space, $\dim(V)$, is uniquely determined by the cardinality of an independent generating set for $Q(V)$, called a *basis* of $V$ (see [1]).

**Definition 2.4** ([6]) We say that two near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are isomorphic (written $(V_1, A_1) \cong (V_2, A_2)$) if there are group isomorphisms $\theta : (V_1, +) \to (V_2, +)$ and $\eta : (A_1^*, \cdot) \to (A_2^*, \cdot)$ such that $\theta(x\alpha) = \theta(x)\eta(\alpha)$ for all $x \in V_1$ and $\alpha \in A_1^*$.

We will write a near-vector space isomorphism as a pair $(\theta, \eta)$.

**Example 2.5** ([5]) Consider the field $(GF(3^2), +, \cdot)$ with

$$GF(3^2) := \{ 0, 1, 2, \gamma, 1 + \gamma, 2 + \gamma, 2\gamma, 1 + 2\gamma, 2 + 2\gamma \},$$

where $\gamma$ is a zero of $x^2 + 1 \in \mathbb{Z}_3[x]$. In [8], p. 257, it was observed that $(GF(3^2), +, \circ)$, with

$$x \circ y := \begin{cases} x \cdot y & \text{if } y \text{ is a square in } (GF(3^2), +, \cdot) \\ x^3 \cdot y & \text{otherwise} \end{cases}$$

and

$$+: (a + b\gamma) + (c + d\gamma) = (a + c) \mod 3 + ((b + d) \mod 3) \gamma$$

is a (right) nearfield, but not a field.
The distributive elements of \((GF(3^2), +, \circ)\), denoted by \((GF(3^2), +, \circ)_d\), are the elements 0, 1, 2. From now on when there is no room for confusion, we will write \(x \circ y\) as \(xy\). Now let \(F = (GF(3^2), +, \circ)\), with \(\alpha \in F\) acting as an endomorphism of \(V = F^n\) by defining \((x_1, x_2, x_3)_\alpha = (x_1 \alpha, x_2 \alpha, x_3 \alpha)\). Thus, \(Q(V) = V_1 \cup V_2 \cup V_3\), with \(V_1 = (1, d_1, d_2) F\), \(V_2 = (d_1, 1, d_2) F\) and \(V_3 = (d_1, d_2, 1) F\), with \(d_1, d_2 \in F_d\). We will refer back to this example later in the paper.

In [9] it was proved that finite-dimensional near-vector spaces can be characterized in the following way:

**Theorem 2.6 ([9])** Let \((G, +)\) be a group and let \(A = D \cup \{0\}\), where \(D\) is a fixed point free group of automorphism of \(G\). Then \((G, A)\) is a finite-dimensional near-vector space if and only if there exist a finite number of nearfields \(F_1, \ldots, F_m\), semigroup isomorphisms \(\psi_i : (A, \circ) \to (F_i, \cdot)\), and an additive group isomorphism \(\Phi : G \to F_1 \oplus \ldots \oplus F_m\) such that if \(\Phi(g) = (x_1, \ldots, x_m)\), then \(\Phi(\alpha g) = (x_1 \psi_1(\alpha), \ldots, x_m \psi_m(\alpha))\) for all \(g \in G\), \(\alpha \in A\).

Using this theorem we can specify a finite-dimensional near-vector space by taking \(n\) copies of a nearfield \(F\) for which there are semigroup isomorphisms \(\psi_i : (F, \cdot) \to (F_i, \cdot)\), \(i \in \{1, \ldots, n\}\). We then take \(V := F^n\), \(n\) a positive integer, as the additive group of the near-vector space and define the scalar multiplication by:

\[
(x_1, \ldots, x_n)_\alpha := (x_1 \psi_1(\alpha), \ldots, x_n \psi_n(\alpha)),
\]

for all \(\alpha \in F\) and \(i \in \{1, \ldots, n\}\). This is the type of construction we will use throughout this paper and we will use \((F^n, F)\) to denote an instance of a near-vector space of this form.

The concept of regularity is a central notion in the study of near-vector spaces.

**Definition 2.7 ([1])** A near-vector space is regular if any two vectors of \(Q(V)^*\) are compatible, i.e. if for any two vectors \(u\) and \(v\) of \(Q(V)^*\) there exists a \(\lambda \in A \setminus \{0\}\) such that \(u + v \lambda \in Q(V)\).

**Theorem 2.8 ([1])** Let \(F\) be a (right) nearfield and let \(I\) be a nonempty index set. Then the set

\[
F^{(I)} := \{(n_i)_{i \in I} | n_i \in F, n_i \neq 0 \text{ for at most a finite number of } i \in I\}
\]

with the scalar multiplication defined by

\[
(n_i)\lambda := (n_i \lambda)
\]

gives that \((F^{(I)}, F)\) is a near-vector space.
We describe the quasi-kernel of $F^{(I)}$:

**Theorem 2.9 ([1])** We have

$$Q(F^{(I)}) = \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I \}.$$  

We can also show that the quasi-kernel is not the entire space.

**Theorem 2.10** Letting $F$ be a proper (right) nearfield and let $I$ be a nonempty index set, then the near-vector space $(F^{(I)}, F)$ has $Q(F^{(I)}) \neq F^{(I)}$.

**Proof** Consider the element $v = (a_1, 1, \ldots, 0) \in V$, where $a_1 \notin F_d$. We show that $v$ is in $V \setminus Q(V)$. Suppose that $v \in Q(V)$, and then $(a_1, 1, \ldots, 0) = (d_1\lambda, d_2\lambda, \ldots, 0)$. Thus, we get that $a_1 = d_1\lambda$, $1 = d_2\lambda$, and since $F$ is a nearfield, we can solve this and get that $\lambda = d_2^{-1}$. Substituting this in the first equation we get that $a_1 = d_1d_2^{-1}$, and since $F_d$ is a field, this gives that $a_1 \in F_d$, a contradiction.

The following theorem gives a characterization of regularity in terms of the near-vector space $(F^{(I)}, F)$.

**Theorem 2.11 ([1])** A near-vector space $(V, F)$, with $F$ a nearfield and $V \neq 0$, is a regular near-vector space if and only if $V$ is isomorphic to $F^{(I)}$ for some index set $I$.

The following theorem is central in the theory of near-vector spaces.

**Theorem 2.12 ([1]) (The Decomposition Theorem)** Every near-vector space $V$ is the direct sum of regular near-vector spaces $V_j$ ($j \in J$) such that each $u \in Q(V)^*$ lies in precisely one direct summand $V_j$. The subspaces $V_j$ are maximal regular near-vector spaces.

### 3. Spanning sets and generators

In [5] a study of the subspaces of near-vector spaces was initiated. In this section we add to these results. We begin with some basic definitions.

**Definition 3.1 ([5])** If $(V, A)$ is a near-vector space and $\emptyset \neq V' \subseteq V$ is such that $V'$ is the subgroup of $(V, +)$ generated additively by $X A = \{x a | x \in X, a \in A \}$, where $X$ is an independent subset of $Q(V)$, then we say that $(V', A)$ is a subspace of $(V, A)$, or simply $V'$ is a subspace of $V$ if $A$ is clear from the context.

From the definition, since $X$ is a basis for $V'$, the dimension of $V'$ is $|X|$. It is clear that $V$ is a subspace of itself since it is generated by $X A$ where $X$ denotes a basis of $Q(V)$ and we define the trivial subspace, $\{0\}$, to be the space generated by the empty subset of $Q(V)$.

**Definition 3.2** Letting $(V, A)$ be a near-vector space, then the span of a set $S$ of vectors is defined to be the intersection $W$ of all subspaces of $V$ that contain $S$, denoted span $S$.

It is straightforward to verify that $W$ is a subspace, called the subspace spanned by $S$, or conversely, $S$ is called a spanning set of $W$ and we say that $S$ spans $W$. Moreover, if we define span $\emptyset = \{0\}$, then it is not difficult to check that span $S$ is the set of all possible linear combinations of $S$.
For a vector space \((V, F)\) the span of a single vector \(v\) is always of the form \(vF\), but in general this is not true for near-vector spaces. The following two results were recently proved:

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\textbf{Lemma 3.3} Let \((V, A)\) be a near-vector space. Then for all \(v \in V\), \(\text{span}\{v\} = vA\) if and only if \(Q(V) = V\).

One might wonder if it is possible for a nonzero \(w \in V \setminus Q(V)\) to have \(\text{span}\{w\} = vA\) for some \(v \in Q(V)\).

\textbf{Lemma 3.4} Let \((V, A)\) be a near-vector space. Then for all nonzero \(w \in V \setminus Q(V)\), \(\text{span}\{w\} \neq vA\) for some \(v \in Q(V)\).

\}

We are interested in what the span of a vector outside of \(Q(V)\) looks like.

Let \((V, A)\) be a near-vector space, not necessarily finite-dimensional. By definition, the quasi-kernel \(Q(V)\) generates \(V\), so for any \(v \in V\), there is \(u_1, \ldots, u_m \in Q(V) \setminus \{0\}\) and \(\alpha_1, \ldots, \alpha_m \in A \setminus \{0\}\), such that \(v = u_1\alpha_1 + \cdots + u_m\alpha_m\). This expression is not unique. We can also have \(u_1', \ldots, u_l' \in Q(V) \setminus \{0\}\) and \(\alpha_1', \ldots, \alpha_l' \in A \setminus \{0\}\) such that \(v = u_1'\alpha_1' + \cdots + u_l'\alpha_l'\) with \(m \neq l\).

For \(v \in V \setminus \{0\}\), we consider

\[ n = \min \left\{ m \in \mathbb{N} \mid v = \sum_{i=1}^{m} u_i\alpha_i, \text{ with } u_i \in Q(V) \setminus \{0\}, \alpha_i \in A \setminus \{0\}, i = 1, \ldots, m \right\}. \]

\textbf{Definition 3.5} For \(v \in V \setminus \{0\}\) we define the dimension of \(v\) to be

\[ n = \min \left\{ m \in \mathbb{N} \mid v = \sum_{i=1}^{m} u_i\alpha_i, \text{ with } u_i \in Q(V) \setminus \{0\}, \alpha_i \in A \setminus \{0\}, i = 1, \ldots, m \right\}, \]

and we denote it by \(\text{dim}(v) = n\) and \(\text{dim}(v) = 0\) if \(v\) is the zero vector.

\textbf{Theorem 3.6} We have that \(\text{dim}(\text{span}\{v\}) = \text{dim}(v)\).

\textbf{Proof} Let \(n = \text{dim}(v)\) and \(\{u_1, \ldots, u_n\} \subset Q(V)\), such that \(v = \sum_{i=1}^{n} u_i\alpha_i\) for some \(\alpha_i \in A \setminus \{0\}\). Then \(\text{span}\{v\} \subset \text{span}\{u_1, \ldots, u_n\}\). Since \(\text{span}\{v\}\) is the smallest subset of \(V\) that contains \(v\). Since \(n\) is minimal, \(\{u_1, \ldots, u_n\}\) is a linearly independent subset of \(Q(V)\). Hence, \(\dim(W) = n\) and \(\dim(\text{span}\{v\}) \leq n\).

Let us assume that \(\dim(\text{span}\{v\}) < n\). Since \(v \in \text{span}\{v\}\), there are \(u_1, \ldots, u_m \in Q(V) \setminus \{0\}\) and \(\beta_1, \ldots, \beta_m \in A \setminus \{0\}\) such that \(v = \sum_{i=1}^{m} u_i\beta_i\), with \(m < n\). This a contradiction since \(n\) is the smallest integer that satisfies this condition. Hence, \(\dim(\text{span}\{v\}) = \text{dim}(v)\). \(\square\)

We know that any subspace of \(W\) of \(V\) is generated by \(XA\), with \(X\) a linearly independent subset of \(Q(V)\). For \(\text{span}\{v\}\), \(v\) a vector in \(V \setminus \{0\}\), the subset \(X\) is given by any linearly independent set \(\{u_1, \ldots, u_n\} \subset Q(V)\), such that \(n = \text{dim}(v)\) and \(v = \sum_{i=1}^{n} u_i\alpha_i\) for some \(\alpha_i \in A \setminus \{0\}\).

By Lemma 3.3, we have that:
Proposition 3.7 For any \( v \in V \), \( \dim(v) = 1 \) if and only of \( v \in Q(V) \setminus \{0\} \).

Also, if \( V \) is finite-dimensional, of dimension \( n \), then \( \dim(v) \leq n \), and if \( \dim(v) = n \), then \( \text{span}\{v\} = V \).

Thus, we define:

Definition 3.8 Let \((V, A)\) be a near-vector space. If \( v \in V \) such that \( \text{span}\{v\} = V \), then \( v \) is called a generator of \( V \).

Isomorphisms preserve generators:

Theorem 3.9 Let \((V_1, A_1)\) and \((V_2, A_2)\) be isomorphic near-vector spaces and \( v \in V_1 \). Then \( \dim(v) = \dim(\theta(v)) \), where \((\theta, \eta)\) is the isomorphism.

Proof Let \( \dim(v) = k \) and \( \dim(\theta(v)) = k' \). Then there exist \( u_1, \ldots, u_k \in Q(V_1) \setminus \{0\} \) and \( \alpha_1, \ldots, \alpha_k \in A_1 \setminus \{0\} \) such that \( v = \sum_{i=1}^{k} u_i \alpha_i \). We have

\[
\theta(v) = \theta \left( \sum_{i=1}^{k} u_i \alpha_i \right) = \sum_{i=1}^{k} \theta(u_i) \alpha_i = \sum_{i=1}^{k} \theta(u_i) \eta(\alpha_i).
\]

It follows that \( \dim(\theta(v)) \leq k \).

Assume that \( k' = \dim(\theta(v)) < k \). There are \( v_1, \ldots, v_{k'} \in Q(V_2) \setminus \{0\} \) and \( \beta_1, \ldots, \beta_{k'} \in A_2 \setminus \{0\} \) such that \( \theta(v) = \sum_{i=1}^{k} v_i \beta_i. \) Since \((\theta, \eta)\) is an isomorphism, we have

\[
\theta(v) = \sum_{i=1}^{k'} \theta(v_i') \eta(\beta_i') = \sum_{i=1}^{k'} \theta(v_i' \beta_i') = \theta \left( \sum_{i=1}^{k'} v_i' \beta_i' \right).
\]

It follows that \( v = \sum_{i=1}^{k'} v_i' \beta_i' \) and \( \dim(v) \leq k' < k \), which is a contradiction. \( \Box \)

Corollary 3.10 Let \((V_1, A_1)\) and \((V_2, A_2)\) be isomorphic near-vector spaces. \( v \) is a generator of \( V_1 \) if and only if \( \theta(v) \) is a generator of \( V_2 \), where \((\theta, \eta)\) is the isomorphism.

For \( F \) a field, using the following recently proved result, we can show more.

Theorem 3.11 Let \( F = GF(p^r) \) and \( V = F^n \) be a near-vector space with scalar multiplication defined for all \( \alpha \in F \) by

\[
(x_1, \ldots, x_n) \alpha := (x_1 \psi_1(\alpha), \ldots, x_n \psi_n(\alpha)),
\]

where the \( \psi_i's \) are automorphisms of \((F, \cdot)\). If \( Q(V) \neq V \) and \( V = V_1 \oplus \cdots \oplus V_k \) is the canonical decomposition of \( V \), then \( Q(V) = Q_1 \cup \cdots \cup Q_k \) where \( Q_i = V_i \) for each \( i \in \{1, \ldots, k\} \).
Theorem 3.12 Let $F$ be a field and $V = F^n$ be a near-vector space over $F$ with scalar multiplication defined for all $(x_1, \ldots, x_n) \in F$ and $\alpha \in F$ by

$$(x_1, \ldots, x_n)\alpha := (x_1\psi_1(\alpha), \ldots, x_n\psi_n(\alpha)),$$

where the $\psi_i$s are automorphisms of $(F, \cdot)$ for $i \in \{1, \ldots, n\}$ and they can be equal. If $V_1 \oplus \cdots \oplus V_k$ is the canonical decomposition of $V$, then for all $v \in V$, $\dim(v) \leq k$.

Proof Let $v \in V$ and suppose that $\dim(v) > k$, say $\dim(v) = k'$, where $k' > k$. Then $v = \sum_{i=1}^{k'} u_i \lambda_i$, where $u_i \in Q(V) \setminus \{0\}$, $\lambda_i \in F$ for $i \in 1, \ldots, k'$. However, for all $i \in 1, \ldots, k'$, $u_i \in Q_j$ for some $j$ with $1 \leq j \leq k$, since by Theorem 3.11, $Q(V) = Q_1 \cup \cdots \cup Q_k$ and $k' > k$. Suppose, without loss of generality, that $u_s$ and $u_{s'}$ are in $Q_j$, and then $u_s \lambda_s + u_{s'} \lambda_{s'} \in Q_j$, since $Q_j = V_j$ ($F$ is a field). Now we have that $v$ can be written with fewer than $k'$ elements, i.e. $v = u_1 \lambda_1 + \cdots + u_k \lambda_k$, a contradiction. \hfill \square

Thus, in the case where $F$ is a field, unless the dimension of $V$ is less than or equal to 1, or equal to $k$, where $k$ is the number of maximal regular subspaces in the canonical decomposition of the near-vector space, we cannot have any generators. If the dimension of $V$ is exactly $k$ then the maximal regular spaces have dimension 1 and any element of the form $(1, \ldots, 1)$ will be generator of $V$.

3.1. Generators for regular near-vector spaces

When $F$ is a proper nearfield, we have the following result:

Theorem 3.13 Let $F$ be a proper nearfield and $V' = F^n$ be a near-vector space over $F$ with scalar multiplication defined for all $(x_1, \ldots, x_n) \in V'$, $\alpha \in F$ by

$$(x_1, \ldots, x_n)\alpha := (x_1\alpha, \ldots, x_n\alpha).$$

$v = (a_1, \ldots, a_n)$ is a generator of $V'$ if and only for $d_1, \ldots, d_n \in F_d$,

$$\sum_{i=1}^{n} d_ia_i = 0 \iff d_1 = d_2 = \ldots = d_n = 0.$$

Proof Let us assume that there are $d_1, \ldots, d_n \in F_d$ such that $\sum_{i=1}^{n} d_ia_i = 0$ and $d_{i_0} \neq 0$. We show that $\dim(v) < n$. Without loss of generality let us assume that $i_0 = 1$. Then $a_1 = \sum_{i=2}^{n} d_i^{-1}d_ia_i$, so we get

$$(a_1, \ldots, a_n) = \left(\sum_{i=2}^{n} d_i^{-1}d_ia_i, a_2, \ldots, a_n\right)$$

$$= \sum_{i=2}^{n} u_i, \text{ with } u_i = (d_i^{-1}d_ia_i, \ldots, 0, a_i, 0, \ldots, 0).$$

Since $Q(V') = \{(d_1, \ldots, d_n)\alpha | d_1, \ldots, d_n \in F_d, \alpha \in F\}$, $u_i \in Q(V')$ for all $i = 2, \ldots, n$. It follows that $\dim(v) < n$. Therefore, $\dim(v) = n$ implies that for $d_1, \ldots, d_n \in F_d$,

$$\sum_{i=1}^{n} d_ia_i = 0 \iff d_1 = d_2 = \ldots = d_n = 0.$$
Now let us assume that for \( d_1, \ldots, d_n \in F_d \),

\[
\sum_{i=1}^{n} d_i a_i = 0 \iff d_1 = d_2 = \ldots = d_n = 0,
\]

and that \( \text{dim}(v) < n \). Thus, \( v \) can be written as a linear combination of less than \( k \) vectors of the quasi-kernel with \( k < n \), so there is

\[
(a_i)_{1 \leq i \leq k} \subseteq F \quad \text{and} \quad (d_{i,j})_{1 \leq i \leq n, 1 \leq j \leq k} \subseteq F_d,
\]

such that

\[
(a_1, \ldots, a_n) = \sum_{i=1}^{k} (d_{1,i}, \ldots, d_{n,i}) a_i.
\]

Hence, we get the following system of \( n \) equations with \( k \) unknowns:

\[
\begin{align*}
    d_{1,1} x_1 + d_{1,2} x_2 + \cdots + d_{1,k} x_k &= a_1 \\
    d_{2,1} x_1 + d_{2,2} x_2 + \cdots + d_{2,k} x_k &= a_2 \\
    \vdots & \quad \ddots \\
    d_{n,1} x_1 + d_{n,2} x_2 + \cdots + d_{n,k} x_k &= a_n
\end{align*}
\]

with \((\alpha_1, \ldots, \alpha_k)\) as the solution. Since the equation has a solution, the matrix

\[
A = \begin{pmatrix}
    d_{1,1} & d_{1,2} & d_{1,3} & \ldots & d_{1,k} \\
    d_{2,1} & d_{2,2} & d_{2,3} & \ldots & d_{2,k} \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    d_{n,1} & d_{n,2} & \ldots & d_{n,k-1} & d_{n,k}
\end{pmatrix}
\]

has rank \( k \) in \( F_d \). Therefore, there exist \( \delta_1, \ldots, \delta_n \in F_d \) not all zero such that \( \sum_{i=1}^{n} \delta_i a_i = 0 \). This is a contradiction. \( \square \)

Let \( F \) be a proper nearfield and \( V'' = F^n \) be a regular near-vector space over \( F \).

**Theorem 3.14** \( v = (a_1, \ldots, a_n) \) is a generator of \( V'' \) if and only if for \( d_1, \ldots, d_n \in F_d \),

\[
\sum_{i=1}^{n} d_i a_i = 0 \iff d_1 = \cdots = d_n = 0.
\]

**Proof** It follows from the fact that \((V'', F)\) is isomorphic to \((V', F)\) by Theorem 2.11. \( \square \)

**Theorem 3.15** Let \( V = F^n \) be a near-vector space with \( |F| = |F_d|^m \) and

\[
(x_1, \ldots, x_n) \alpha := (x_1 \alpha, \ldots, x_n \alpha),
\]

for all \( (x_1, \ldots, x_n) \in V \) and \( \alpha \in F \). \( v \) is a generator of \( V \) if and only if \( m \geq n \).
**Theorem** Suppose that there is \( v = (a_1, \ldots, a_n) \in V \) such that \( \dim(v) = n \). By Theorem 3.13 we have that for any \( d_i \in F_d, \ i = 1, \ldots, n, \sum_{i=1}^{n} d_i a_i = 0 \) implies \( d_i = 0 \) for all \( i \). It follows that \( \{a_1, \ldots, a_n\} \) is a linearly independent set of vectors in the vector space \( F \) over \( F_d \). Hence, \( m \geq n \).

To show the converse we assume that \( m < n \). Then for any \( v = (a_1, \ldots, a_n) \in V \) there are \( d_1, \ldots, d_n \) not all zero with \( \sum_{i=1}^{n} d_i a_i = 0 \). Hence, we cannot have \( v \in V \) such that \( \dim(v) = n \).

\[ \square \]

**Example 3.16** Let us consider the Dickson nearfield \( F = DF(3, 2) \) and \( V = F^2 \) a near-vector space with \( (x, y)\alpha := (xa, y\alpha) \). Then the element \( v = (1, \gamma) \) has dimension 2. In fact, \( v \) is not in any of the subspaces. Suppose that \( v \in V_1 \), with \( V_1 \) a one-dimensional subspace of \( V \). Let \( w \) be a basis of \( V^\prime \). It follows that \( v = w\lambda \), with \( \lambda \in F \), since the quasi-kernel is closed under scalar multiplication \( v \in Q(V) \), but \( v \notin Q(V) \). Hence, the smallest subspace of \( V \) that contains \( v \) is \( V \) itself. Hence, \( v \) is a generator of \( V \) and \( \dim(v) = 2 \). Using Theorem 3.15 we can also see that \( \dim(v) = 2 \). For any \( d_1, d_2 \in F_d \), \( d_1 + d_2 \gamma = 0 \) implies that \( d_1 = d_2 = 0 \), since \( \{1, \gamma\} \) is a basis of the vector space \( F \) over \( F_d \).

For three copies of \( F \), \( V = F^3 \), it is not possible to have an element that generates \( V \).

### 3.2. Generators for general near-vector spaces

In this subsection we consider the case where \( F \) is a proper nearfield and \( V = F^n \) is a near-vector space over \( F \) with the canonical decomposition \( V = \bigoplus_{i=1}^{k} V_i \).

**Lemma 3.17** If \( v_i \in V_i \setminus \{0\} \) and \( v_j \in V_j \setminus \{0\} \) with \( i \neq j \), then

\[ \dim(v_i + v_j) = \dim(v_i) + \dim(v_j). \]

**Proof** Let \( \dim(v_i) = l_i, \dim(v_j) = l_j \). It is not difficult to check that \( \dim(v_i + v_j) \leq l_i + l_j \). Suppose that \( l = \dim(v_i + v_j) < l_i + l_j \). There are \( u_1, \ldots, u_l \in Q(V_i) \setminus \{0\} \cup Q(V_j) \setminus \{0\} \) and \( a_1, \ldots, a_l \in F \setminus \{0\} \) such that

\[ v_i + v_j = l m=1 u_m \alpha_m. \]

It follows that we write \( v_i \) as \( v_i = \sum_{m=1}^{l'} u_m \alpha_m, \) with \( l' < l \) or \( v_j = \sum_{m=1}^{l''} u_m \alpha_m \) with \( l'' < l_j \), since \( V_i \cap V_j = \{0\} \). This is a contradiction since \( \dim(v_i) = l_i, \dim(v_j) = l_j \) and we should have \( l_i \geq l' \) and \( l_j \geq l'' \).

\[ \square \]

**Corollary 3.18** If \( v_i \in V_i \setminus \{0\} \) and \( v_j \in V_j \setminus \{0\} \) with \( i \neq j \), then

\[ \text{span}\{v_i + v_j\} = \text{span}\{v_i\} \oplus \text{span}\{v_j\}. \]

**Proof** We have \( \text{span}\{v_i\} \cap \text{span}\{v_j\} = \{0\} \), since \( \text{span}\{v_i\} \subseteq V_i, \text{span}\{v_j\} \subseteq V_j \) and \( V_i \cap V_j = \{0\} \).

We have \( \text{span}\{v_i + v_j\} \subseteq \text{span}\{v_i\} \oplus \text{span}\{v_j\} \). Since \( \dim(v_i + v_j) = \dim(v_i) + \dim(v_j) \), \( \text{span}\{v_i + v_j\} = \text{span}\{v_i\} \oplus \text{span}\{v_j\} \).

\[ \square \]
Corollary 3.19 Let $v_1, \dots, v_m \in V$ such that they are all in distinct maximal regular subspaces. We have

$$\dim(v_1 + \cdots + v_m) = \dim(v_1) + \cdots + \dim(v_m),$$

$$\text{span}\{v_1 + \cdots + v_m\} = \text{span}\{v_1\} \oplus \cdots \oplus \text{span}\{v_m\}.$$ 

Theorem 3.20 A vector $v \in V$ is a generator of $V$ if and only if there are $v_i \in V_i$ generators of $V_i$ for all $i = 1, \ldots, k$, such that $v = v_1 + \cdots + v_k$.

Proof We have $\text{span}\{v\} = \text{span}\{v_1 + \cdots + v_k\} = \text{span}\{v_1\} \oplus \cdots \oplus \text{span}\{v_k\}$. If $v$ is a generator of $v$ we have $\text{span}\{v\} = V$ and so $\text{span}\{v_1\} \oplus \cdots \oplus \text{span}\{v_k\} = V$. Hence, $\text{span}\{v_i\} = V_i$ for all $i = 1, \ldots, k$. Thus, $v_i$ is a generator of $V_i$ for all $i$. Likewise, if $v_i$ is a generator of $V_i$ for all $i$, then $v$ is a generator of $V$.

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