Almost paracontact structures obtained from $G_{2(2)}^*$ structures

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Received: 02.06.2017 • Accepted/Published Online: 03.10.2018 • Final Version: 27.11.2018

Abstract: In this paper, we construct almost paracontact metric structures by using the fundamental 3-forms of manifolds with $G_{2(2)}^*$ structures. The existence of certain almost paracontact metric structures is investigated due to the properties of the 2-fold vector cross-product. Furthermore, we give some relations between the classes of $G_{2(2)}^*$ structures and almost paracontact metric structures.

Key words: $G_{2(2)}^*$ structure, almost paracontact metric structure

1. Introduction

Almost paracontact structures on manifolds of odd dimension, analogues of the almost contact structures on manifolds, were first introduced by Kaneyuki and Williams in [5]. After the work of Zamkovoy in [10], almost paracontact metric structures have been a widely studied research area. In [11], almost paracontact metric structures were classified into $2^{12}$ classes taking into consideration the Levi-Civita covariant derivative of the fundamental 2-form of the structure.

Almost contact metric structures induced by $G_2$ structures were constructed by Matzeu and Munteanu in [7]; see also [1]; and the possible classes that these structures may belong to were considered in [8].

The objective of this manuscript is the investigation of almost paracontact metric structures on manifolds with structure group $G_{2(2)}^*$. First, we construct almost paracontact metric structures induced by $G_{2(2)}^*$ structures. Then we investigate the relation between the classes of almost paracontact metric structures and $G_{2(2)}^*$ structures. In addition, we give an elementary example to support the arguments of the manuscript.

2. Preliminaries

Consider $\mathbb{R}^7$ with the standard basis $\{e_1, ..., e_7\}$. The fundamental 3-form on $\mathbb{R}^7$ is defined as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $\{e^1, ..., e^7\}$ denotes the basis dual to $\{e_1, ..., e_7\}$ and $e^{ijk} = e^i \wedge e^j \wedge e^k$. The Lie group $G_2$ is defined by

$$G_2 := \{ f \in GL(7, \mathbb{R}) \mid f^*\varphi_0 = \varphi_0 \};$$

see [3].

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2010 AMS Mathematics Subject Classification: 53C25, 53D10

This study was supported by the Anadolu University Scientific Research Projects Commission under the Grant No: 1501F017.

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A 7-dimensional oriented manifold $M$ has a $G_2$ structure if and only if its structure group reduces to $G_2$. Then there is a 3-form $\varphi$ on $M$ with the property that $(T_p M, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$, for all $p \in M$, said to be the fundamental 3-form or the $G_2$ structure on $M$. Manifolds $(M, g)$ with $G_2$ structure were classified into 16 classes in [4].

The noncompact dual of $G_2$ is the group

$$G^*_2(2) = \{g \in GL(7, \mathbb{R}) \mid g^* \varphi = \bar{\varphi}\},$$

where

$$\bar{\varphi} = -e^{127} - e^{135} + e^{146} + e^{236} + e^{245} - e^{347} + e^{567}$$

and $\{e^1, ..., e^7\}$ denotes the dual to the standard basis of $\mathbb{R}^{4,3} = (\mathbb{R}^7, g_{4,3})$ with the metric $g_{4,3} = (-1, -1, -1, -1, 1, 1, 1)$. A semi-Riemannian manifold $M$ with the metric of signature $(-, -, -, +, +, +)$ is called a manifold with $G^*_2(2)$ structure. Similar to the $G_2$ case, there is the fundamental 3-form (or the $G^*_2(2)$ structure) $\bar{\varphi}$ on $M$ inducing a metric $g_{4,3}$, a volume form, and a 2-fold vector cross-product $\bar{P}$ on $M$, which can be calculated via

$$\bar{\varphi}(X, Y, Z) = g_{4,3}(\bar{P}(X, Y), Z); \quad (2.1)$$

see [3]. Similar to the $G_2$ case, a $G^*_2(2)$ structure $\bar{\varphi}$ satisfying $\nabla g_{4,3} \bar{\varphi} = 0$ is called a parallel $G^*_2(2)$ structure and a $G^*_2(2)$ structure with $\nabla^X g_{4,3} \bar{\varphi}(X, Y, Z) = 0$ is called nearly parallel [6].

For convenience, throughout the paper, a $G^*_2(2)$ structure and the induced vector cross-product will be denoted by $\varphi$ and $P$, respectively.

A triple $(\phi, \xi, \eta)$ on a $2n+1$-dimensional differentiable manifold $M^{2n+1}$ satisfying

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (2.2)$$

where $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, and $\eta$ is a 1-form $\eta$ on $M$, is called an almost paracontact structure on $M$ and $M$ is called an almost paracontact manifold. As a consequence of (2.2), one can see that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$ on the almost paracontact structure $(\phi, \xi, \eta)$.

If an almost paracontact manifold $M$ has a semi-Riemannian metric $g$ of signature $(n, n+1)$ satisfying

$$g(\phi(X), \phi(Y)) = -g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

then $M$ is an almost paracontact metric manifold having the almost paracontact metric structure $(\phi, \xi, \eta, g)$ and $g$ is said to be a compatible metric.

The 2-form

$$\Phi(X, Y) := g(\phi(X), Y)$$

is said to be the fundamental 2-form of the almost paracontact metric structure. It is known that on an almost paracontact metric manifold there is an orthonormal basis (called a $\phi$-basis) $\{e_1, \phi e_1, \ldots, e_n, \phi e_n, \xi\}$ with

$$g(e_i, e_j) = -g(\phi e_i, \phi e_j) = \delta_{ij}, \quad g(e_i, \phi e_j) = 0, \quad i, j = 1, \ldots, n;$$

see [10]. For the almost contact case, see [2].
Let $F$ be the $(0,3)$ tensor field defined by

$$F(X,Y,Z) = (\nabla_X \phi)(Y,Z) = g((\nabla_X \phi)Y,Z), \quad (2.4)$$

for $X,Y,Z \in TM$. It can be seen that $F$ has the following properties:

$$F(X,Y,Z) = -F(X,Z,Y), \quad (2.5)$$

$$F(X,\phi(Y),\phi(Z)) = F(X,Y,Z) + \eta(Y)F(X,Z,\xi) - \eta(Z)F(X,Y,\xi).$$

In [11], a classification of almost paracontact metric manifolds was obtained by considering the space $\mathcal{F}$ of tensors $F$ that satisfy (2.5). Initially, this space was decomposed into four subspaces

$$W_1 = \left\{ F \in \mathcal{F} \mid \begin{align*}
F(X,Y,Z) &= g(A^F_X Y, Z), \\
F(\xi,Y,Z) &= g(A^F_{\xi} Y, Z) = 0, \\
F(X,\xi,Z) &= g(A^F_X \xi, Y) = 0
\end{align*} \right\}, \quad (2.6)$$

$$W_2 = \left\{ F \in \mathcal{F} \mid \begin{align*}
F(X,Y,Z) &= \eta(Y)g(A^F_X Y, Z) + \eta(Z)g(A^F_{\xi} Y, X), \\
A^F_{\xi} &= 0
\end{align*} \right\}, \quad (2.7)$$

$$W_3 = \mathcal{G}_{11} = \{ F \in \mathcal{F} \mid F(X,Y,Z) = \eta(X)F(\xi,\phi(Y),\phi(Z)) \}, \quad (2.8)$$

$$W_4 = \mathcal{G}_{12} = \{ F \in \mathcal{F} \mid F(X,Y,Z) = \eta(X) (\eta(Y)\omega_F(Z) - \eta(Z)\omega_F(Y)) \}, \quad (2.9)$$

where $A^F_X Y = (\nabla_Y \phi)(X), A^F_{\xi} X = \nabla_X \xi$ and $\omega_F(X) = F(\xi,X)$. Then $W_1$ and $W_2$ were written as sums of $U(n) \times 1$ irreducible components $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, and $\mathcal{G}_5, \cdots, \mathcal{G}_{10}$ respectively, where $U(n)$ is the paraunitary group, with the following defining relations [11]:

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Consider a 7-dimensional smooth manifold $M$.

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where $\theta_F = 0$.

Also, the defining relations of paracontact and almost $K$-paracontact classes are $d\eta = \Phi$ and $\nabla\xi\Phi = 0$, respectively.

Let $(M, \phi, \xi, \eta, g)$ be an almost paracontact metric manifold. $M$ is called normal if

$$\phi((\nabla_X \phi)(Y)) - (\nabla_{\phi X} \phi)(Y) + (\nabla_X \eta)(Y)\xi = 0; \quad (2.10)$$

see [9].

3. Almost paracontact metric structures and $G^*_2(2)$ structures

Consider a 7-dimensional smooth manifold $M$ with a $G^*_2(2)$-structure $\varphi$ inducing the pseudo-Riemannian metric $g_{4,3}$ and the vector cross-product $P$. Let $\xi$ be a nonzero vector field on $M$ such that $g_{4,3}(\xi, \xi) = -1$. Then the quadruple $(\phi, \xi, \eta, g)$, where the endomorphism is

$$\phi(X) = P(\xi, X) \quad (3.1)$$
and \( g = -g_{4,3}, \ \eta(X) = g(\xi, X) \), is an almost paracontact metric structure on \( M \). Indeed, we have
\[
\phi^2 X = \phi(\phi X) = \phi(P(\xi, X)) = P(\xi, P(\xi, X)) \\
= -g_{4,3}(\xi, \xi)X + g_{4,3}(\xi, X)\xi = g(\xi, \xi)X - g(\xi, X)\xi \\
= X - \eta(X)\xi
\]
and
\[
g(\phi X, \phi Y) = -g_{4,3}(P(\xi, X), P(\xi, Y)) \\
= -g_{4,3}(\xi, \xi)g_{4,3}(X, Y) + g_{4,3}(\xi, X)g_{4,3}(\xi, Y) \\
= -g(X, Y) + \eta(X)\eta(Y).
\]

Throughout the paper, unless otherwise stated, \((\phi, \xi, \eta, g)\) corresponds to the almost paracontact metric structure (a.p.m.s.) obtained by a \( G^2_2(2) \) structure \( \phi \) on \( M \). Note that \( \nabla^g = \nabla^{g_{4,3}} \) and we use the notation \( \nabla \) for the Levi-Civita covariant derivative \( \nabla^g \).

The following proposition gives a relation between the covariant derivatives of the fundamental 2-form of the almost paracontact structure and of the \( G^2_2(2) \) structure \( \phi \).

**Proposition 3.1** For an a.p.m.s. \((\phi, \xi, \eta, g)\) on \( M \), the equation
\[
(\nabla_X \Phi)(Y, Z) = -(\nabla_X \varphi)(\xi, Y, Z) - \varphi(\nabla_X \xi, Y, Z)
\] (3.2)
holds.

**Proof**
\[
(\nabla_X \varphi)(\xi, Y, Z) = g_{4,3}(\nabla_X P(\xi, Y), Z) - g_{4,3}(P(\nabla_X \xi, Y), Z) - g_{4,3}(P(\xi, \nabla_X Y), Z) \\
= -g(\nabla_X (\phi Y), Z) - \varphi(\nabla_X \xi, Y, Z) + g(\phi(\nabla_X Y), Z) \\
= -g((\nabla_X \phi)(Y), Z) - \varphi(\nabla_X \xi, Y, Z) \\
= -(\nabla_X \Phi)(Y, Z) - \varphi(\nabla_X \xi, Y, Z).
\]

The following proposition gives a condition for almost paracontact metric structures induced by \( G^2_2(2) \) structures to be paracontact.

**Proposition 3.2** An a.p.m.s. \((\phi, \xi, \eta, g)\) induced by a \( G^2_2(2) \) structure is paracontact (i.e. \( d\eta = \Phi \)) if and only if \( \xi \) satisfies
\[
g_{4,3}(P(\xi, X), Y) = \frac{1}{2}(g_{4,3}(\nabla_X \xi, Y) - g_{4,3}(\nabla_Y \xi, X)).
\] (3.3)

**Proof** The exterior derivative of \( \eta \) is:
\[
2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X.
\] (3.4)
After some calculations, the following is obtained:

\[ d\eta(X,Y) = \frac{1}{2}(-g_{4,3}(\nabla_X \xi, Y) + g_{4,3}(\nabla_Y \xi, X)). \]

Besides, for the corresponding almost paracontact metric structure, we have \( \Phi(X,Y) = g(\phi(X),Y) = -g_{4,3}(P(X,Y)). \)
Thus, \( d\eta = \Phi \) if the relation (3.3) holds.

**Theorem 1** An a.p.m.s. \((\phi, \xi, \eta, g)\) induced by a parallel \(G_{2(2)}^*\) structure \(\varphi\) on \((M, g_{4,3})\) (i.e. \(\nabla^{g_{4,3}} \varphi = 0\)) is in the class \(G_0(\nabla \Phi = 0)\) (paracosymplectic) if and only if the vector field \(\xi\) is parallel.

**Proof** Let \(\varphi\) be a parallel structure; that is, \(\nabla \varphi = 0\). Then from the equation (3.2), we have

\[ (\nabla_X \Phi)(Y,Z) = -\varphi(Y,Z,\nabla_X \xi) = -g_{4,3}(P(Y,Z), \nabla_X \xi), \]

which implies

\[ \nabla \Phi = 0 \iff \nabla \xi = 0. \]

**Theorem 2** For an a.p.m.s. \((\phi, \xi, \eta, g)\), if \(\xi\) is not parallel, then the structure is not in \(W_1\).

**Proof** Consider the equation

\[ g(A^F_\xi X, \phi Z) = g(\nabla_X \xi, \phi Z). \]

Letting the vector field \(\xi\) not be parallel, then there exists \(X_0\) such that \(\nabla_{X_0} \xi \neq 0\) and obviously the third condition of the defining relation (2.6) of \(W_1\) fails.

Note that, under the assumption of Theorem 2, the structure is not an element of any subclass of \(W_1 = G_1 \oplus G_2 \oplus G_3 \oplus G_4\).

**Theorem 3** If the \(G_{2(2)}^*\) structure \(\varphi\) is nearly parallel and \(\xi\) is parallel, then \((\phi, \xi, \eta, g)\) is in \(W_1\).

**Proof** Let \(\varphi\) be nearly parallel; that is,

\[ (\nabla_X \varphi)(X,Y,Z) = 0, \]

and let \(\xi\) be parallel, i.e. \(\nabla \xi = 0\). Then, from equation (3.2),

\[ F(\xi,Y,Z) = -(\nabla_\xi \varphi)(\xi,Y,Z) - \varphi(\nabla_\xi \xi, Y, Z) = 0 \]

and

\[ F(X,\xi,Z) = -(\nabla_X \varphi)(\xi,\xi,Z) - \varphi(\nabla_X \xi, \xi, Z) = 0. \]

Thus, the definition of \(W_1\) is satisfied.

**Theorem 4** An a.p.m.s. \((\phi, \xi, \eta, g)\) from a nearly parallel structure \(\varphi\) satisfies \(\nabla_\xi \Phi = 0\) (almost K-paracontact) if and only if \(\nabla_\xi \xi = 0\).
Proof Let $\varphi$ be nearly parallel. Then this is an immediate consequence of formula (3.2) and of the definition of the nearly parallel $G_{2(2)}^2$ structure. Indeed,

\[(\nabla_{\xi}\Phi)(X, Y) = -(\nabla_{\xi})(\varphi(X, Y) - \varphi(\nabla_{\xi}X, Y)) = -\varphi(\nabla_{\xi}X, Y).\]

Then

\[\nabla_{\xi}\Phi = 0 \iff \nabla_{\xi} = 0.\]

Note that an a.p.m.s. $(\phi, \xi, \eta, g)$ such that $\nabla_{\xi} \neq 0$ cannot be in the class $W_2$ by the definition of $W_2$. In addition, if $\xi$ is not Killing, the structure is not in the class $G_5 \oplus G_8$.

Theorem 5 If $\xi$ is not parallel, then the structure $(\phi, \xi, \eta, g)$ is not an element of $W_3(= G_{11})$.

Proof Take $Y = \xi$ in the defining relation (2.8) of the class $W_3$. Then, as a consequence of the formula (3.2), the left-hand side of (2.8) is

\[(\nabla_X\Phi)(\xi, Z) = X[\Phi(\xi, Z)] - \Phi(\nabla_X\xi, Z) - \Phi(\xi, \nabla_X Z) = g(\phi Z, \nabla_X\xi),\]

while the right-hand side vanishes since $\phi(\xi) = 0$. Thus, if $\nabla_{\xi} \neq 0$ (i.e. $\xi$ is not parallel), the structure cannot be in the class $W_3$.

Theorem 6 If there exists a vector field $X \in \{\xi\}^\perp$ with the property $\nabla_X\xi \neq 0$, then the structure $(\phi, \xi, \eta, g)$ is not in $W_4(= G_{12})$.

Proof Let $X \in \{\xi\}^\perp$ with $\nabla_X\xi \neq 0$. Take $Y = \xi$ in the defining relation (2.9) of the class $W_4$. Then $\eta(X) = 0$ since $X \in \{\xi\}^\perp$, so the right-hand side of the relation (2.9) is zero. On the other hand, from formula (3.2),

\[(\nabla_X\Phi)(Y, Z) = (\nabla_X\Phi)(\xi, Z) = X[\Phi(\xi, Z)] - \Phi(\nabla_X\xi, Z) - \Phi(\xi, \nabla_X Z) = g(\phi Z, \nabla_X\xi).\]

Therefore, $(\nabla_X\Phi)(Y, Z)$ does not have to be zero since $\nabla_X\xi \neq 0$. Hence, the defining relation is not satisfied under the given conditions.

Example 7 Consider the seven-dimensional Lie algebra $\mathfrak{L}$ with nonzero brackets

\[[e_1, e_2] = e_5, \quad [e_1, e_3] = e_6.\]

Then $\mathfrak{L}$ admits the $G_{2(2)}^2$ structure

\[\varphi = e_5^{e_7} - e_5^{e_1^2} - e_5^{e_3^4} - e_5^{e_1^3} + e_5^{e_1^4} + e_5^{e_1^7} + e_5^{e_1^4}.\]

The metric $g_{4,3}$ induced by $\varphi$ is

\[g_{4,3}(x, y) = x_5y_5 + x_6y_6 + x_7y_7 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4.\]
for any vector fields \( x = \sum x_i e_i, \ y = \sum y_i e_i; \) see [3]. Note that \( g_{4,3}(e_i, e_i) = -1 \) for \( i = 1, 2, 3, 4 \) and \( g_{4,3}(e_i, e_i) = 1 \) otherwise. The cross-product of frame elements are obtained via (2.1):

\[
P(e_1, e_2) = -e_5, \quad P(e_1, e_3) = -e_6, \quad P(e_1, e_4) = e_7, \quad P(e_1, e_5) = -e_2, \\
P(e_1, e_6) = -e_3, \quad P(e_1, e_7) = e_4, \quad P(e_2, e_3) = e_7, \quad P(e_2, e_4) = e_6, \\
P(e_2, e_5) = e_1, \quad P(e_2, e_6) = e_4, \quad P(e_2, e_7) = e_3, \quad P(e_3, e_4) = -e_5, \\
P(e_3, e_5) = -e_4, \quad P(e_3, e_6) = e_1, \quad P(e_3, e_7) = -e_2, \quad P(e_4, e_5) = e_3, \\
P(e_4, e_6) = -e_2, \quad P(e_4, e_7) = -e_1, \quad P(e_5, e_6) = e_7, \quad P(e_5, e_7) = -e_6, \quad P(e_6, e_7) = e_5.
\]

The nonzero Levi-Civita covariant derivatives evaluated by Koszul’s formula are

\[
\nabla_{e_1} e_2 = \frac{1}{2} e_5, \quad \nabla_{e_1} e_3 = \frac{1}{2} e_6, \quad \nabla_{e_1} e_4 = \frac{1}{2} e_2, \quad \nabla_{e_1} e_5 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_6 = \frac{1}{2} e_7, \quad \nabla_{e_2} e_1 = \frac{1}{2} e_5, \quad \nabla_{e_2} e_5 = -\frac{1}{2} e_1 \\
\nabla_{e_3} e_1 = -\frac{1}{2} e_6, \quad \nabla_{e_3} e_6 = -\frac{1}{2} e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2} e_2, \quad \nabla_{e_5} e_1 = -\frac{1}{2} e_1, \quad \nabla_{e_6} e_1 = \frac{1}{2} e_3, \quad \nabla_{e_6} e_3 = -\frac{1}{2} e_1.
\]

Now we investigate the existence of certain classes on \( \mathcal{L} \).

Assume that a nonzero vector field \( X = a_1 e_1 + \cdots + a_7 e_7 \) is parallel. Then,

\[
\nabla_{e_1} X = a_1 \nabla_{e_1} e_1 + a_2 \nabla_{e_1} e_2 + a_3 \nabla_{e_1} e_3 + a_4 \nabla_{e_1} e_4 + a_5 \nabla_{e_1} e_5 + a_6 \nabla_{e_1} e_6 + a_7 \nabla_{e_1} e_7 \\
= \frac{a_2}{2} e_5 + \frac{a_3}{2} e_6 + \frac{a_5}{2} e_2 + \frac{a_6}{2} e_3 \\
= 0 \iff a_2 = a_3 = a_5 = a_6 = 0.
\]

On the other hand,

\[
\nabla_{e_2} X = -\frac{a_1}{2} e_5 = 0 \iff a_1 = 0
\]

and there is no other restriction on the coefficients \( a_i \). Thus, \( X = \sum a_i e_i \) is parallel iff \( X = a_4 e_4 + a_7 e_7 \), that is, iff \( X \in \text{span}\{e_4, e_7\} \).

Note that the \( G^*_{2(2)} \) structure (3.5) is neither parallel (since \( (\nabla_{e_1} \varphi)(e_2, e_3, e_4) = 1 \neq 0 \)) nor nearly parallel (since \( (\nabla_{e_1} \varphi)(e_2, e_3, e_4) + (\nabla_{e_2} \varphi)(e_1, e_3, e_4) = \frac{1}{2} \neq 0 \)).

Now we give an example of an a.p.m.s. such that the characteristic vector field is parallel. Let \( (\phi, \xi, \eta, g) \) be the a.p.m.s. induced by the \( G^*_{2(2)} \) structure (3.5), where \( \xi = e_4 \) and \( g = -g_{4,3} \). Then from the equation (3.1), we get \( \phi(e_1) = P(e_4, e_1) = -e_7, \phi(e_2) = -e_6, \phi(e_3) = e_5, \phi(e_4) = 0, \phi(e_5) = e_3, \phi(e_6) = -e_2, \phi(e_7) = -e_1 \). Since \( (\nabla_{e_1} \phi)(e_2) = -e_3 \neq 0 \), this structure is not paracosymplectic. Theorem 1 states that an a.p.m.s. induced by a parallel \( G^*_{2(2)} \) structure is paracosymplectic if and only if the characteristic vector field is parallel. This example shows that if the \( G^*_{2(2)} \) structure is not parallel, we can obtain a.p.m. structures that are not paracosymplectic but have parallel characteristic vector fields.

It is easy to check that this structure is in \( W_1 \), although the \( G^*_{2(2)} \) structure is not nearly parallel, comparing with Theorem 3.

Now let \( (\phi, \xi, \eta, g) \) be the a.p.m.s. induced by the \( G^*_{2(2)} \) structure (3.5), where \( \xi = e_2 \) (\( \xi \) is not parallel in this case) and \( g = -g_{4,3} \). By Theorem 2, this structure is not in \( W_1 \). In addition, it is not in \( W_3 \) by Theorem 5. Also, since \( \nabla_{e_1} e_2 = \frac{1}{2} e_3 \neq 0 \), this structure is not in \( W_4 \) by Theorem 6. From the equation (3.1), we have \( \phi(e_1) = e_5 \),
\( \phi(e_2) = 0, \phi(e_3) = e_7, \phi(e_4) = e_6, \phi(e_5) = e_1, \phi(e_6) = e_4, \phi(e_7) = e_3. \) Since \((\nabla e_i \phi)(e_1) = \frac{1}{3} e_2 \neq 0, \) this structure is not paracosymplectic. One can check that \( \nabla \xi \phi = \nabla e_2 \phi = 0; \) that is, this structure is almost-K-paracontact.

Now we investigate the existence of paracontact structures on \( \Sigma \) induced by the \( G^*_2(2) \) structure \((3.5). \) Let \((\phi, \xi, \eta, g)\) be such a structure with fundamental 2-form \( \Phi; \) that is, \( \eta \xi = \Phi. \) Since \( de^5 = e^{12}, \) \( de^6 = e^{13}, \) for \( \eta = \sum b_i e^i, \) \( i = 1, \ldots, 7, \) we have \( d\eta = b_5 e^{12} + b_6 e^{13} = \Phi. \) This implies \( \phi(e_5) = 0. \) From the equation
\[
g(\phi(e_5), \phi(e_5)) = -g(e_5, e_5) + \eta^2(e_5),
\]
we obtain \( \eta^2(e_5) = -1, \) which is a contradiction. Therefore, there is no paracontact structure on \( \Sigma \) induced by the given \( G^*_2(2) \) structure.

Finally, we study the existence of \( \alpha \)-para-Sasakian structures on \( \Sigma \) induced by the \( G^*_2(2) \) structure \((3.5). \) Let \((\phi, \xi, \eta, g)\) be an \( \alpha \)-para-Sasakian structure induced by \((3.5). \) Note that \( g = -g_{4,3}. \) The characteristic vector field \( \xi \) is Killing. From the equation
\[
g(\nabla e_i \xi, e_j) + g(\nabla e_j \xi, e_i) = 0, \tag{3.7}
\]
we obtain that \( \xi \) is Killing if and only if \( a_1 = a_2 = a_3 = 0. \) Thus, \( \xi = a_4 e_4 + \ldots + a_7 e_7. \) From the definition of an \( \alpha \)-para-Sasakian structure, we have \( \phi(X) = \frac{1}{\alpha} \nabla X \xi \) for all vector fields \( X. \) Then \( \phi(e_2) = -\frac{a_2}{2\alpha} e_1 \) and \( \phi(e_3) = -\frac{a_3}{2\alpha} e_1. \) The equation
\[
g(\phi(e_2), \phi(e_3)) = -g(e_2, e_3) + \eta(e_2) \eta(e_3)
\]
implies \( a_5 a_6 = 0. \) Thus, \( \phi(e_1) = 0 \) or \( \phi(e_2) = 0. \) Assume without loss of generality that \( \phi(e_1) = 0. \) Since
\[
0 = g(\phi(e_1), \phi(e_1)) \neq -g(e_1, e_1) + \eta^2(e_1) = -1,
\]
there is no \( \alpha \)-para-Sasakian structure induced by \((3.5). \)

References