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Iteration method of approximate solution of the Cauchy problem for a singularly perturbed weakly nonlinear differential equation of an arbitrary order

Alexey R. ALIMOV[1,2,*], **Evgeny E. BUKZHALEV**[3]
[1]Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
[2]Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia
[3]Faculty of Physics, Moscow State University, Moscow, Russia

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Abstract: We construct an iteration sequence converging (in the uniform norm in the space of continuous functions) to the solution of the Cauchy problem for a singularly perturbed weakly nonlinear differential equation of an arbitrary order (the weak nonlinearity means the presence of a small parameter in the nonlinear term). The sequence thus constructed is also asymptotic in the sense that the departure of its nth element from the solution of the problem is proportional to the $(n+1)$th power of the perturbation parameter.

Key words: Singular perturbations, Banach contraction principle, method of asymptotic iterations, Routh–Hurwitz stability criterion

1. Introduction

Let $f^i : \mathbb{A} \to \mathbb{B}$, where $i \in \overline{1,m}$. Throughout, $(f^1, \dots, f^m)$ will denote the vector function $f : \mathbb{A} \to \mathbb{B}^m$, $x \mapsto (f^1(x), \dots f^m(x))$.

In the present paper we propose a method of constructing a sequence $\{\psi_n(\cdot\,;\varepsilon)\}_{n=0}^\infty$ of functions $\psi_n(\cdot\,;\varepsilon) := (\psi_n^1(\cdot\,;\varepsilon), \dots, \psi_n^m(\cdot\,;\varepsilon))$, which is convergent, for any $\varepsilon \in (0, \varepsilon_0)$, in the norm of the space $C_m[0,X]$ of m-dimensional continuous vector functions on $[0,X]$, to a function $\psi(\cdot\,;\varepsilon) := (y(\cdot\,;\varepsilon),\ y'(\cdot\,;\varepsilon),\ \dots, y^{(m-1)}(\cdot\,;\varepsilon))$, where $y(\cdot\,;\varepsilon)$ is the classical solution to the problem (1)–(2) (here and in what follows, by the derivative we mean the derivative with respect to the first argument), and ε_0 is expressible in terms of the input data of the problem (see (38)). The construction and proof of the convergence of the sequence $\{\psi_n(\cdot\,;\varepsilon)\}_{n=0}^\infty$ are based on the Banach contraction principle in complete metric spaces (see, for example, [13]). Since in our setting the contraction factor $k(\varepsilon)$ of the mapping is of order ε ($k(\varepsilon) \le \varepsilon/\varepsilon_0$), the departure of $\psi_n(\cdot\,;\varepsilon)$ from $\psi(\cdot\,;\varepsilon)$ (here by the departure we mean the departure in the norm of $C_m[0,X]$) is $O(\varepsilon^{n+1})$ (for $0 < \varepsilon < \varepsilon_0$), and hence the result obtained is also asymptotical.

Each successive element of the sequence $\{\psi_n(\cdot\,;\varepsilon)\}_{n=0}^\infty$ is the action of some operator on the previous element. Elements of such sequences are called iterations, and such sequences are called iteration sequences. In our setting, the convergence rate of the iteration $\psi_n(\cdot\,;\varepsilon)$ to $\psi(\cdot\,;\varepsilon)$ is asymptotically large (is inversely proportional to ε). Hence, the above algorithm for construction of the sequence $\{\psi_n(\cdot\,;\varepsilon)\}_{n=0}^\infty$ is not only

*Correspondence: alexey.alimov@gmail.com

iterative, but can also be subsumed into the class of asymptotical methods of investigation of singularly perturbed equations. Such a method is sometimes called the asymptotic iteration method or the method of asymptotic iterations (see, for example, [1]). The sequence $\{\psi_n^i(\,\cdot\,;\varepsilon)\}_{n=0}^{\infty}$ will be called the *asymptotic sequence of the* $(i-1)$ *th derivative of the solution* $y(\,\cdot\,;\varepsilon)$ to the above problem.

Asymptotic integration of problem (1)–(2) can also be carried out with the help of various asymptotic methods (see [12]), for example, using the method of boundary functions (see [15]). However, the method of asymptotic iteration is capable of constructing approximations that converge (for sufficiently small ε) not only in the asymptotic but also in the usual sense (in the norm of $C_m[0,X]$). Such a duality is the principal advantage of this method over other asymptotic methods (in particular, over the method of boundary functions), which are capable of constructing sequences or series that are asymptotic but still diverging (in particular, for arbitrarily small ε).

The idea of application of the iteration approach to perturbed equations is not new per se. For example, in the papers [2, 3] an iteration process is used to construct asymptotic approximations to the solution of the Cauchy problem for a system of fast and slow equations. Under this approach, the simplification achieved by the application of the iteration method consists of the reduction of the dimension of the system under consideration. However, it should be noted first that these two papers contain a principle mistake (which shall be examined and rectified in a separate paper), and second, in contrast to these two papers, in the present study the simplification comes from the linearization and autonomation of the original equations. We also note that the principal advantage of iteration procedures is that the smoothness requirements on the input data are very modest. In the case of problem (1)–(2), to construct all $\psi_n(\,\cdot\,;\varepsilon)$ it suffices that conditions (3) on the functions a_i, b, and g be satisfied; however, when using, for example, the method of boundary functions, to construct all terms of the asymptotic expansion it is required that a_i, b, and g be all infinitely differentiable.

The present paper extends a number of results obtained earlier for more simple classes of singularly perturbed differential equations. A similar approach was used to study the Cauchy problems for weakly nonlinear first-order equations with one or two small parameters (see [5, 7]), for linear and weakly nonlinear second-order equations (see [4, 6]), and also for linear homogeneous and inhomogeneous equations of arbitrary order (see [8, 10]). A passage from linear to even weakly nonlinear equations brings to light new questions and issues, whose solution requires additional estimates and a considerable number of transformations.

2. Statement of the problem and auxiliary estimates

Consider the Cauchy problem for the singularly perturbed weakly nonlinear differential equation of order m:

$$\begin{aligned}\varepsilon^m\, y^{(m)}(x;\varepsilon) =&\varepsilon^{m-1}\, a_{m-1}(x)\, y^{(m-1)}(x;\varepsilon) + \cdots + a_0(x)\, y(x;\varepsilon) + b(x)\\ &+ \varepsilon\, g(\varepsilon^{m-1}\, y^{(m-1)}(x;\varepsilon),\dots,y(x;\varepsilon),x), \quad x\in(0,X]; \end{aligned}\tag{1}$$

$$y(0;\varepsilon) = y^0,\ \dots,\ y^{(m-1)}(0;\varepsilon) = y^{m-1}/\varepsilon^{m-1},\tag{2}$$

where $\varepsilon>0$ is a perturbation parameter, $X>0$, $y^0,\dots,y^{m-1}\in\mathbb{R}$,

$$a_0,\dots,a_{m-1},b\in C^1[0,X], \quad g\in C^{1,\dots,1,0}(\mathbb{R}^m\times[0,X]).\tag{3}$$

Besides, we assume that for all $x \in [0, X]$ the coefficients $a_i(x)$ satisfy the Routh–Hurwitz condition (see, for example, [11]):

$$-a_{00}(x) > 0, \quad \begin{vmatrix} a_{00}(x) & a_{01}(x) \\ a_{10}(x) & a_{11}(x) \end{vmatrix} > 0, \quad \dots, \quad (-1)^m \begin{vmatrix} a_{00}(x) & \dots & a_{0(m-1)}(x) \\ \vdots & \ddots & \vdots \\ a_{(m-1)0}(x) & \dots & a_{(m-1)(m-1)}(x) \end{vmatrix} > 0, \tag{4}$$

where

$$a_{ij}(x) := \begin{cases} a_{2i-j}(x), & 0 \le 2\,i - j < m, \\ -1, & 2\,i - j = m, \\ 0, & 2\,i - j < 0 \text{ or } 2\,i - j > m. \end{cases}$$

We recall (see also [11]) that for condition (4) to hold it is necessary (and for $m \in \{1, 2\}$ it is also sufficient) that all $a_i(x)$ be negative.

Let p be the mapping that assigns with each $x \in [0, X]$ the polynomial

$$p(x) := \lambda^m - a_{m-1}(x)\,\lambda^{m-1} - \ldots - a_1(x)\,\lambda - a_0(x). \tag{5}$$

Since the degree of the polynomial $p(x)$ is m on the entire interval $[0, X]$, there exist functions λ^1, ..., λ^m: $[0, X] \to \mathbb{C}$ such that, for any $x \in [0, X]$,

$$p(x) = (\lambda - \lambda^1(x)) \ldots (\lambda - \lambda^m(x))$$

($\lambda^1(x)$, ..., $\lambda^m(x)$ are roots of the polynomial $p(x)$). The function $(\lambda^1, \dots, \lambda^m)$ will be called the vector function of the roots of the mapping p. In general, the set Λ of vector functions of the roots of the mapping p is infinite, because for any $x \in [0, X]$ the roots of the polynomial $p(x)$ can be labeled differently. It can be proved (see, for example, [14]) that at least one of the functions from Λ is continuous on $[0, X]$ (here the fact that the argument x is scalar is essential). Next, by λ^1, ..., λ^m we shall imply the components of the same (arbitrarily chosen) continuous vector function of the roots of the mapping p.

According to the Routh–Hurwitz criterion (see [11]), a necessary and sufficient condition that the real parts of the roots of the polynomial $p(x)$ be negative is that its coefficients $a_i(x)$ satisfy inequalities (4). Thus, for all $(i, x) \in \{1, \dots, m\} \times [0, X]$, we have

$$\operatorname{Re} \lambda^i(x) < 0.$$

It can be easily shown that each of the functions $\operatorname{Re} \lambda^i$ is bounded from above on the interval $[0, X]$ by some negative constant. Indeed, by the Weierstrass extreme value theorem on the maximum of a continuous function, there exists $x_0 \in [0, X]$ such that

$$\varkappa := -\max_{x \in [0,X]} \max\{\operatorname{Re} \lambda^1(x), \dots, \operatorname{Re} \lambda^m(x)\} = -\max\{\operatorname{Re} \lambda^1(x_0), \dots, \operatorname{Re} \lambda^m(x_0)\} > 0, \tag{6}$$

and hence $\operatorname{Re} \lambda^i(x) < -\varkappa$ for all $(i, x) \in \{1, \dots, m\} \times [0, X]$.

Consider the auxiliary problem

$$a_0(x)\,\bar{y}(x) + b(x) = 0, \quad x \in [0, X]; \tag{7}$$

$$\Pi^{(m)}(\xi) = a_{m-1}(0)\,\Pi^{(m-1)}(\xi) + \ldots + a_0(0)\,\Pi(\xi), \quad \xi \in (0, X/\varepsilon]; \tag{8}$$

$$\Pi(0) = y^0 - \bar{y}(0), \quad \Pi'(0) = y^1, \quad \dots, \quad \Pi^{(m-1)}(0) = y^{m-1}. \tag{9}$$

Equation (7) is a first-order algebraic equation for $\bar{y}(x)$, and (8) is a homogeneous linear autonomous differential equation for the function Π. The solution to problem (7)–(9) reads as

$$\bar{y}(x) = -b(x)/a_0(x), \quad \Pi(\xi) = \alpha_{11}\, e^{\lambda^1(0)\,\xi} + \ldots + \alpha_{1m_1}\, \xi^{m_1-1}\, e^{\lambda^{m_1}(0)\,\xi} + \ldots$$
$$+ \alpha_{q1}\, e^{\lambda^{m_1+\ldots+m_{q-1}+1}(0)\,\xi} + \ldots + \alpha_{qm_q}\, \xi^{m_q-1}\, e^{\lambda^{m_1+\ldots+m_{q-1}+m_q}(0)\,\xi}, \tag{10}$$

where $\lambda^1(0) = \cdots = \lambda^{m_1}(0)$, ..., $\lambda^{m_1+\ldots+m_{q-1}+1}(0) = \cdots = \lambda^{m_1+\ldots+m_q}(0)$ are roots of the polynomial $p(0)$ (see (5)), and α_{11}, ..., α_{qm_q} are the constants uniquely expressible in terms of $y^0 - \bar{y}(0)$, y^1, ..., y^{m-1} and $\lambda^1(0)$, ..., $\lambda^m(0)$ $(m_1 + \cdots + m_q = m)$.

Using (10) and (6) we see that, for sufficiently large $\tilde{C}$ and $\bar{C}$,

$$|\Pi^{(i)}(\xi)| \le \tilde{C}\,(1 + \xi^{m-1})\, e^{-\varkappa\,\xi} \le \bar{C}, \quad (i, \xi) \in \{0, \ldots, m-1\} \times [0, +\infty). \tag{11}$$

For the sake of brevity, consider the function

$$f(y^m, \ldots, y^2, y^1, x) := g(y^m, \ldots, y^2, \bar{y}(x) + y^1, x), \quad (y^1, \ldots, y^m, x) \in \mathbb{R}^m \times [0, X].$$

It is clear that f has the same smoothness as g (see (3)). The subscript y_{m+1-i} will be used to denote the partial derivative of f in the ith argument $(1 \le i \le m)$; that is, $\partial_1 f =: f_{y^m}$, $\partial_2 f =: f_{y^{m-1}}$, ..., $\partial_m f =: f_{y^1}$.

We change variables in problem (1)–(2):

$$\begin{gathered} x = \varepsilon\,\xi, \quad y(x;\varepsilon) = \tilde{y}(\xi, x) + \varepsilon\, z^1(\xi;\varepsilon), \\ y^{(i-1)}(x;\varepsilon) = \varepsilon^{1-i}\, \Pi^{(i-1)}(\xi) + \varepsilon^{2-i} z^i(\xi;\varepsilon), \quad i \in \overline{2, m}, \end{gathered} \tag{12}$$

where $\tilde{y}(\xi, x) := \bar{y}(x) + \Pi(\xi)$. Note that, for $i \ge 2$,

$$\Pi^{(i-1)}(\xi) = \tilde{y}_{\xi^{i-1}}(\xi, x) \tag{13}$$

(here and in what follows, the subscript ξ denotes the partial derivative in the first argument).

For the new functions $z^i(\xi;\varepsilon)$, we have the following initial problem:

$$(z^1)'(\xi;\varepsilon) = z^2(\xi;\varepsilon) - \bar{y}'(\varepsilon\,\xi), \quad \xi \in (0, X/\varepsilon]; \tag{14}$$

$$(z^i)'(\xi;\varepsilon) = z^{i+1}(\xi;\varepsilon), \quad (i, \xi) \in \{2, \ldots, m-1\} \times (0, X/\varepsilon]; \tag{15}$$

$$(z^m)'(\xi;\varepsilon) = a_{m-1}(\varepsilon\,\xi)\, z^m(\xi;\varepsilon) + \cdots + a_0(\varepsilon\,\xi)\, z^1(\xi;\varepsilon) +$$
$$+ \tilde{f}(z^m(\xi;\varepsilon), \ldots, z^1(\xi;\varepsilon), \xi;\varepsilon), \quad \xi \in (0, X/\varepsilon]; \tag{16}$$

$$z^1(0;\varepsilon) = \ldots = z^m(0;\varepsilon) = 0 \tag{17}$$

(here (14) applies only for $m \ge 2$, and (15) only for $m \ge 3$), where

$$\tilde{f}(z^m, \ldots, z^1, \xi; \varepsilon) := \begin{cases} \varepsilon^{-1}\left\{[a_{m-1}(\varepsilon\,\xi) - a_{m-1}(0)]\, \Pi^{(m-1)}(\xi) + \cdots + [a_0(\varepsilon\,\xi) - a_0(0)]\, \Pi(\xi)\right\} \\ \quad + f(\Pi^{(m-1)}(\xi) + \varepsilon\, z^m, \ldots, \Pi(\xi) + \varepsilon\, z^1, \varepsilon\,\xi), & m \ge 2; \\ \varepsilon^{-1}\, [a_0(\varepsilon\,\xi) - a_0(0)]\, \Pi(\xi) + f(\Pi(\xi) + \varepsilon\, z^1, \varepsilon\,\xi) - \bar{y}'(\varepsilon\,\xi), & m = 1. \end{cases} \tag{18}$$

We change equation (16) by introducing $x \in [0, X]$ as a new parameter:

$$(z^1)'(\xi;\varepsilon,x) = z^2(\xi;\varepsilon,x) - \bar{y}'(\varepsilon\,\xi), \quad \xi \in (0, X/\varepsilon];$$

$$(z^i)'(\xi;\varepsilon,x) = z^{i+1}(\xi;\varepsilon,x), \quad (i,\xi) \in \{2,\dots,m-1\} \times (0, X/\varepsilon];$$

$$(z^m)'(\xi;\varepsilon,x) = a_{m-1}(x)\,z^m(\xi;\varepsilon,x) + \dots + a_0(x)\,z^1(\xi;\varepsilon,x) \tag{19}$$

$$+ [a_{m-1}(\varepsilon\,\xi) - a_{m-1}(x)]\,z^m(\xi;\varepsilon,x) + \dots + [a_0(\varepsilon\,\xi) - a_0(x)]\,z^1(\xi;\varepsilon,x)$$

$$+ \tilde{f}(z^m(\xi;\varepsilon,x),\dots,z^1(\xi;\varepsilon,x),\xi;\varepsilon), \quad \xi \in (0, X/\varepsilon];$$

$$z^1(0;\varepsilon,x) = \ldots = z^m(0;\varepsilon,x) = 0 \tag{20}$$

(it is clear that (14)–(17) and (19)–(20) are equivalent for each x under consideration).

Problem (19)–(20) is equivalent to the system of integral equations (with the parameters ε and x)

$$z^i(\xi;\varepsilon,x) = -\int_0^\xi \Phi^1_{\xi^{i-1}}(\xi-\zeta;\,x)\,\bar{y}'(\varepsilon\,\zeta)\,d\zeta + \int_0^\xi \Phi^m_{\xi^{i-1}}(\xi-\zeta;\,x)\,\Big\{[a_{m-1}(\varepsilon\,\zeta) - a_{m-1}(x)]$$

$$\times\, z^m(\zeta;\varepsilon,x) + \dots + [a_0(\varepsilon\,\zeta) - a_0(x)]\,z^1(\zeta;\varepsilon,x) + \tilde{f}(z^m(\zeta;\varepsilon,x),\dots,z^1(\zeta;\varepsilon,x),\zeta;\varepsilon)\Big\}\,d\zeta, \tag{21}$$

$$(i,\xi) \in \overline{1,m} \times [0, X/\varepsilon],$$

where $\Phi^j_{\xi^{i-1}}(\,\cdot\,;x)$ are the components of the matricant $\Phi(\,\cdot\,;x)$ (see (24)) of the corresponding homogeneous system

$$(Z^1)'(\xi;x) = Z^2(\xi;x), \quad \dots, \quad (Z^{m-1})'(\xi;x) = Z^m(\xi;x),$$

$$(Z^m)'(\xi;x) = a_{m-1}(x)\,Z^m(\xi;x) + \dots + a_0(x)\,Z^1(\xi;x), \quad \xi \in \mathbb{R} \tag{22}$$

(with the parameter x).

Remark 1 Since for each $x \in [0, X]$ system (22) is a system of differential equations with constant coefficients, we have, for the Cauchy matrix $K(\,\cdot\,,\,\cdot\,;x)$ of system (22),

$$K(\xi,\zeta;x) = \Phi(\xi;x)\,\Phi(\zeta;x)^{-1} = \Phi(\xi-\zeta;\,x), \quad \xi,\zeta \in \mathbb{R}.$$

Remark 2 The equivalence of problem (19)–(20) and system (21) is a trivial corollary to the definition of $\Phi(\,\cdot\,;x)$ (see (23)).

Recall that by the definition of a matricant

$$\Phi_\xi(\,\cdot\,;x) = J(x)\,\Phi(\,\cdot\,;x), \quad \Phi(0;x) = E, \tag{23}$$

where

$$J(x) := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0(x) & a_1(x) & a_2(x) & \dots & a_{m-1}(x) \end{bmatrix}, \quad E := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Using (23) we readily obtain for $\Phi(\,\cdot\,;x)$

$$\Phi(\,\cdot\,;x):\mathbb{R}\ni\xi\mapsto\begin{bmatrix}\Phi^1(\xi;\,x) & \dots & \Phi^m(\xi;\,x)\\ \Phi^1_\xi(\xi;\,x) & \dots & \Phi^m_\xi(\xi;\,x)\\ \vdots & \ddots & \vdots\\ \Phi^1_{\xi^{m-1}}(\xi;\,x) & \dots & \Phi^m_{\xi^{m-1}}(\xi;\,x)\end{bmatrix}, \tag{24}$$

where the functions $\Phi^j(\,\cdot\,;x)$ are the solutions to the following initial problems:

$$\Phi^j_{\xi^m}(\xi;x)=a_{m-1}(x)\,\Phi^j_{\xi^{m-1}}(\xi;x)+\dots+a_0(x)\,\Phi^j(\xi;x),\quad \xi\in\mathbb{R}; \tag{25}$$

$$\Phi^j(0;x)=\ldots=\Phi^j_{\xi^{j-2}}(0;x)=\Phi^j_{\xi^j}(0;x)=\ldots=\Phi^j_{\xi^{m-1}}(0;x)=0,\quad \Phi^j_{\xi^{j-1}}(0;x)=1. \tag{26}$$

Remark 3 Using (25)–(26), taking into account the smoothness condition (3), and applying theorems on the continuity and differentiability with respect to the parameter of the solution of an initial problem (see, for example, [13]), we see that $\Phi^j\in C^{\infty,1}(\mathbb{R}\times[0,X])$ for all $j\in\overline{1,m}$.

Since for each $x\in[0,X]$ the solution $(z^1(\,\cdot\,;\varepsilon,x),\dots,z^m(\,\cdot\,;\varepsilon,x))$ of system (21) coincides with the solution $(z^1(\,\cdot\,;\varepsilon),\ \dots,z^m(\,\cdot\,;\varepsilon))$ of problem (14)–(17) (and hence is certainly independent of the parameter x), it follows that $z^i(\,\cdot\,;\varepsilon)$ satisfy any system that is obtained from system (21) in which x is replaced by a function of ξ and ε with values in $[0,X]$. In particular, it satisfies the system

$$\begin{gathered}z^i(\xi;\varepsilon)=\,-\int\limits_0^\xi\Phi^1_{\xi^{i-1}}(\xi-\zeta;\,\varepsilon\,\xi)\,\bar{y}'(\varepsilon\,\zeta)\,d\zeta+\int\limits_0^\xi\Phi^m_{\xi^{i-1}}(\xi-\zeta;\,\varepsilon\,\xi)\\ \times\Big\{[a_{m-1}(\varepsilon\,\zeta)-a_{m-1}(\varepsilon\,\xi)]\,z^m(\zeta;\varepsilon)+\dots+[a_0(\varepsilon\,\zeta)-a_0(\varepsilon\,\xi)]\,z^1(\zeta;\varepsilon)\\ +\tilde{f}(z^m(\zeta;\varepsilon),\dots,z^1(\zeta;\varepsilon),\zeta;\varepsilon)\Big\}\,d\zeta=:\hat{A}_i(\varepsilon)(z^1(\,\cdot\,;\varepsilon),\dots,z^m(\,\cdot\,;\varepsilon))(\xi),\\ (i,\xi)\in\overline{1,m}\times[0,X/\varepsilon]\end{gathered} \tag{27}$$

(the first integral appears only for $m\geq 2$), which is obtained from (21) by putting $x=\varepsilon\,\xi$. Thus, system (27) is a corollary to problem (14)–(17). However, the above does not imply per se the converse implication, and so to prove the required equivalence one needs to show that any solution to system (27) satisfies problem (14)–(17).

Remark 4 The left-hand sides of the equations in system (21) are independent of the parameter x, and so surely are the right-hand sides of these equations, in spite of the fact that they explicitly involve this parameter; here it is essential that the functions $z^i(\,\cdot\,;\varepsilon,x)$ from the integrals satisfy equations (21) (if $z^i(\,\cdot\,;\varepsilon,x)$ are replaced by functions not satisfying equations (21), then the integrals from the right-hand sides of these equations will depend in general on the parameter x).

System (27) can be written in the abbreviated vector form

$$\begin{gathered}z(\xi;\varepsilon)=\big(\hat{A}_1(\varepsilon)(z(\,\cdot\,;\varepsilon))(\xi),\dots,\hat{A}_m(\varepsilon)(z(\,\cdot\,;\varepsilon))(\xi)\big)=:\hat{A}(\varepsilon)(z(\,\cdot\,;\varepsilon))(\xi),\\ \xi\in[0,X/\varepsilon],\end{gathered} \tag{28}$$

where $z:=(z^1,\dots,z^m)$.

Remark 5 For any fixed $\varepsilon \in (0, +\infty)$, by the domain of the operator $\hat{A}(\varepsilon)$ we mean $C_m[0, X/\varepsilon]$, which is the space of m-dimensional vector functions continuous on the interval $[0, X/\varepsilon]$. It is clear that $\hat{A}(\varepsilon) : C_m[0, X/\varepsilon] \to C_m[0, X/\varepsilon]$.

Proposition 1 *Problem (14)–(17) is equivalent to system (28).*

Proof Since by the above system (28) is a corollary to problem (14)–(17), it remains to show that problem (14)–(17) is also a corollary to system (28). Let $z(\,\cdot\,;\varepsilon) = (z^1(\,\cdot\,;\varepsilon), \dots, z^m(\,\cdot\,;\varepsilon))$ be the solution to system (28). We claim that $z(\,\cdot\,;\varepsilon)$ is a solution to problem (14)–(17). Let $\Delta^i(\,\cdot\,;\varepsilon)$ be the residuals with which the vector function $z(\,\cdot\,;\varepsilon)$ satisfies equations (14)–(16):

$$(z^1)'(\xi;\varepsilon) = z^2(\xi;\varepsilon) - \bar{y}'(\varepsilon\,\xi) + \Delta^1(\xi;\varepsilon), \quad \xi \in (0, X/\varepsilon];$$

$$(z^i)'(\xi;\varepsilon) = z^{i+1}(\xi;\varepsilon) + \Delta^i(\xi;\varepsilon), \quad (i,\xi) \in \{2, \dots, m-1\} \times (0, X/\varepsilon];$$

$$(z^m)'(\xi;\varepsilon) = a_{m-1}(\varepsilon\,\xi)\, z^m(\xi;\varepsilon) + \dots + a_0(\varepsilon\,\xi)\, z^1(\xi;\varepsilon)$$

$$+ \tilde{f}(z^m(\xi;\varepsilon), \dots, z^1(\xi;\varepsilon), \xi; \varepsilon) + \Delta^m(\xi;\varepsilon), \quad \xi \in (0, X/\varepsilon].$$

Note that (see (27))

$$z^1(0;\varepsilon) = \ldots = z^m(0;\varepsilon) = 0.$$

We need to show that all $\Delta^i(\,\cdot\,;\varepsilon)$ vanish on the interval $[0, X/\varepsilon]$. By the definition of the matriciant $\Phi(\,\cdot\,;x)$ of system (22) (see (23)) and the operator $\hat{A}(\varepsilon)$ (see (28) and (27)), we have

$$z(\xi;\varepsilon) = \hat{A}(\varepsilon)(z(\,\cdot\,;\varepsilon))(\xi) + \int_0^{\xi} \Phi(\xi - \zeta;\, \varepsilon\,\xi)\, \Delta(\zeta;\varepsilon)\, d\zeta, \quad \xi \in [0, X/\varepsilon], \tag{29}$$

where $\Delta(\,\cdot\,;\varepsilon) := (\Delta^1(\,\cdot\,;\varepsilon), \dots, \Delta^m(\,\cdot\,;\varepsilon))^{\mathrm{T}}$ (the proof of the fact that $z(\,\cdot\,;\varepsilon)$ satisfies system (29) is completely similar to the proof that the solution to problem (14)–(17) obeys system (28)).

From (29) and (28) we have the following relation for $\Delta(\,\cdot\,;\varepsilon)$:

$$\int_0^{\xi} K(\xi, \zeta; \varepsilon)\, \Delta(\zeta;\varepsilon)\, d\zeta = \Theta, \quad \xi \in [0, X/\varepsilon],$$

where $K(\xi, \zeta; \varepsilon) := \Phi(\xi - \zeta;\, \varepsilon\,\xi)$, $\Theta := (0, \dots, 0)^{\mathrm{T}}$. Note that (see (23))

$$\det K(\xi, \xi; \varepsilon) = \det \Phi(0;\, \varepsilon\,\xi) = 1 \neq 0, \quad \xi \in [0, X/\varepsilon].$$

Thus, $\Delta(\,\cdot\,;\varepsilon)$ is a solution to the system of first-order homogeneous integral Volterra equations of the first kind with nondegenerate kernel. However, since any such system has only the trivial solution, we have $\Delta^i(\xi;\varepsilon) = 0$ for all $(i,\xi) \in \overline{1,m} \times [0, X/\varepsilon]$. □

Below we shall require one auxiliary estimate of the solution $w(\,\cdot\,; M_m, N_m)$ to the Cauchy problem for the linear differential equation with constant coefficients M_m and initial values N_m considered as parameters for w:

$$w^{(m)}(\xi; M_m, N_m) = a_{m-1}\, w^{(m-1)}(\xi; M_m, N_m) + \ldots + a_0\, w(\xi; M_m, N_m), \quad \xi \in (0, +\infty); \tag{30}$$

$$w(0; M_m, N_m) = w^0,\ \ldots,\ w^{(m-1)}(0; M_m, N_m) = w^{m-1}, \tag{31}$$

where $M_m = (a_0, \dots, a_{m-1}) \in \mathbb{C}^m$, $N_m = (w^0, \dots, w^{m-1}) \in \mathbb{C}^m$.

We set

$$\bar{\Lambda}_m(M_m) := \max\{\operatorname{Re}\Lambda^1(M_m), \dots, \operatorname{Re}\Lambda^m(M_m)\},$$

where $\Lambda^1(M_m)$, ..., $\Lambda^m(M_m)$ are the roots of the characteristic polynomial of equation (30),

$$\Pi_m(C) := \{(x_1, \dots, x_m) \in \mathbb{C}^m : |x_1| \le C, \dots, |x_m| \le C\}.$$

Lemma 1 *Let $C_a \ge 0$, $C_w \ge 0$. Then there exists $\tilde{C}_m \ge 0$ such that*

$$\left|w^{(i)}(\xi; M_m, N_m)\right| \le \tilde{C}_m\,(1 + \xi^{m-1})\, e^{\bar{\Lambda}_m(M_m)\,\xi}$$

for all $(i, \xi, M_m, N_m) \in \{0, \dots, m-1\} \times [0, +\infty) \times \Pi_m(C_a) \times \Pi_m(C_w)$, where $w(\,\cdot\,; M_m, N_m)$ is the solution to problem (30)–(31).

The lemma can be proved by induction in m (see [8, 9]). From the proof one can also derive a recurrence formula for the coefficients $\tilde{C}_m$, which shows that they can be looked upon as known values.

Corollary 1 *There exist $\varkappa > 0$ and $C_\Phi > 0$ such that*

$$|\Phi^1_{\xi^i}(\xi; x)|,\ |\Phi^m_{\xi^i}(\xi; x)| \le C_\Phi\,(1 + \xi^{m-1})\, e^{-\varkappa\,\xi} \tag{32}$$

for all $(i, \xi, x) \in \{0, \dots, m-1\} \times [0, +\infty) \times [0, X]$, where $\Phi^j(\,\cdot\,; x)$ is the solution to problem (25)–(26).

Proof To prove estimate (32) it suffices to put

$$\varkappa := -\max_{x\in[0,X]} \max\{\operatorname{Re}\lambda^1(x), \dots, \operatorname{Re}\lambda^m(x)\}$$

(see (6)), employ Weierstrass's first theorem on the boundedness of continuous functions for a_i, and use Lemma 1. □

3. Proof of the existence of the solution

For any $C \ge 0$ let $O(C, \varepsilon) := \big\{(z^1, \dots, z^m) \in C_m[0, X/\varepsilon] \mid \forall \xi \in [0, X/\varepsilon]\ \ (z^1(\xi), \dots, z^m(\xi)) \in [-C, +C]^m\big\}$ be the closed C-neighborhood of the vector function $\vartheta : \xi \mapsto (0, \dots, 0)$ in the space $C_m[0, X/\varepsilon]$ and let $\hat{A}(C, \varepsilon)$ be the restriction of the operator $\hat{A}(\varepsilon)$ to $O(C, \varepsilon)$ (for the definition of $\hat{A}(\varepsilon)$, see (27) and (28)).

Proposition 2 *There exist $\varepsilon_0 > 0$ and $C_0 \ge 0$ such that $\hat{A}(C_0, \varepsilon) : O(C_0, \varepsilon) \to O(C_0, \varepsilon)$ for any $\varepsilon \in (0, \varepsilon_0]$.*

Proof We fix arbitrary $\varepsilon > 0$ and $C_0 \ge 0$, apply the operators $\hat{A}_i(\varepsilon)$ (the components of $\hat{A}(\varepsilon)$) to an arbitrary vector function $\varphi = (z^1, \dots, z^m)$ from $O(C_0, \varepsilon)$, and using (27) and (32) estimate the result. We have

$$\begin{aligned}
&\big|\hat{A}_i(\varepsilon)(\varphi)(\xi)\big| \\
&\quad \le C_\Phi\, e^{-\varkappa\,\xi} \Big\{ C_0 \int_0^\xi e^{\varkappa\,\zeta} \left[1 + (\xi - \zeta)^{m-1}\right] \left[|a_{m-1}(\varepsilon\,\zeta) - a_{m-1}(\varepsilon\,\xi)| + \ldots + |a_0(\varepsilon\,\zeta) - a_0(\varepsilon\,\xi)|\right] d\zeta \\
&\qquad + \int_0^\xi e^{\varkappa\,\zeta} \left[1 + (\xi - \zeta)^{m-1}\right] \left[|\tilde{f}(z^m(\zeta), \dots, z^1(\zeta), \zeta; \varepsilon)| + |\bar{y}'(\varepsilon\,\zeta)|\right] d\zeta \Big\}
\end{aligned} \tag{33}$$

(the term $|\bar{y}'(\varepsilon\,\zeta)|$ appears only for $m \ge 2$), where $i \in \overline{1, m}$, $\xi \in (0, X/\varepsilon]$.

For the first integral in (33) we have

$$\int\limits_0^\xi e^{\varkappa\,\zeta}\left[1+(\xi-\zeta)^{m-1}\right]\left[|a_{m-1}(\varepsilon\,\zeta)-a_{m-1}(\varepsilon\,\xi)|+\cdots+|a_0(\varepsilon\,\zeta)-a_0(\varepsilon\,\xi)|\right]d\zeta$$

$$\le\varepsilon\left\{\|a'_{m-1}\|+\cdots+\|a'_0\|\right\}\int\limits_0^\xi e^{\varkappa\,\zeta}\left[(\xi-\zeta)+(\xi-\zeta)^m\right]d\zeta$$

$$=\varepsilon\,\alpha\left\{\tfrac{1}{\varkappa^2}\left[e^{\varkappa\,\xi}-1-\varkappa\,\xi\right]+\tfrac{m!}{\varkappa^{m+1}}\left[e^{\varkappa\,\xi}-1-\varkappa\,\xi-\cdots-\tfrac{1}{m!}\,(\varkappa\,\xi)^m\right]\right\}\le\varepsilon\,\beta\,e^{\varkappa\,\xi},\tag{34}$$

where $\|\cdot\|$ is the norm of the space $C[0,X]$, $\alpha:=\|a'_{m-1}\|+\cdots+\|a'_0\|$, $\beta:=\alpha\,\frac{\varkappa^{m-1}+m!}{\varkappa^{m+1}}$.

For the second integral in (33) we have (see (18) and (11))

$$\int\limits_0^\xi e^{\varkappa\,\zeta}\left[1+(\xi-\zeta)^{m-1}\right]\left[|\tilde f(z^m(\zeta),\ldots,z^1(\zeta),\zeta;\varepsilon)|+|\bar y'(\varepsilon\,\zeta)|\right]d\zeta\le\int\limits_0^\xi e^{\varkappa\,\zeta}\left[1+(\xi-\zeta)^{m-1}\right]$$

$$\times\Big\{\tilde C\left[\|a'_{m-1}\|+\cdots+\|a'_0\|\right](\zeta+\zeta^m)\,e^{-\varkappa\,\zeta}+\left|f(\Pi^{(m-1)}(\zeta),\,\ldots,\,\Pi(\zeta),\,\varepsilon\,\zeta)\right|$$

$$+\,\varepsilon\left|z^m(\zeta)\right|\left|f_{y_m}(\Pi^{(m-1)}(\zeta)+\varepsilon\,\theta\,z^m(\zeta),\,\ldots,\,\Pi(\zeta)+\varepsilon\,\theta\,z^1(\zeta),\,\varepsilon\,\zeta)\right|+\ldots$$

$$+\,\varepsilon\left|z^1(\zeta)\right|\left|f_{y_1}(\Pi^{(m-1)}(\zeta)+\varepsilon\,\theta\,z^m(\zeta),\,\ldots,\,\Pi(\zeta)+\varepsilon\,\theta\,z^1(\zeta),\,\varepsilon\,\zeta)\right|+\left|\bar y'(\varepsilon\,\zeta)\right|\Big\}\,d\zeta$$

$$\le\Big\{\tilde C\,\alpha\max_{\zeta>0}\left[(\zeta+\zeta^m)\,e^{-\varkappa\,\zeta}\right]+\|f\|_0+C_0\,\varepsilon\left[\|f_{y_m}\|_{C_0\varepsilon}+\cdots+\|f_{y_1}\|_{C_0\varepsilon}\right]+\|\bar y'\|\Big\}$$

$$\times\int\limits_0^\xi e^{\varkappa\,\zeta}\left[1+(\xi-\zeta)^{m-1}\right]d\zeta=\Big\{\ldots\Big\}\Big\{\tfrac{1}{\varkappa}\left[e^{\varkappa\,\xi}-1\right]$$

$$+\tfrac{(m-1)!}{\varkappa^m}\left[e^{\varkappa\,\xi}-1-\varkappa\,\xi-\cdots-\tfrac{1}{(m-1)!}\,(\varkappa\,\xi)^{m-1}\right]\Big\}\le\left[C_0\,\varepsilon\,h(C_0\,\varepsilon)+\gamma\right]e^{\varkappa\,\xi},\tag{35}$$

where $\theta=\theta(\zeta;\varepsilon)\in(0,1)$, $\|\cdot\|_\delta$ is the norm of the space $C([-\bar C-\delta,\,+\bar C+\delta]^m\times[0,X])$,

$$h(\delta):=\left[\|f_{y_m}\|_\delta+\cdots+\|f_{y_1}\|_\delta\right]\tfrac{\varkappa^{m-1}+(m-1)!}{\varkappa^m},$$

$$\gamma:=\Big\{\tilde C\,\alpha\max_{\zeta>0}\left[(\zeta+\zeta^m)\,e^{-\varkappa\,\zeta}\right]+\|f\|_0+\|\bar y'\|\Big\}\,\tfrac{\varkappa^{m-1}+(m-1)!}{\varkappa^m}.$$

From (33), (34), and (35) we see that if C_0 and ε satisfy the inequalities

$$0\le l(C_0,\varepsilon):=C_0\,\varepsilon\,C_\Phi\left[\beta+h(C_0\,\varepsilon)\right]+C_\Phi\,\gamma\le C_0,\tag{36}$$

then $\hat A(C_0,\varepsilon)(\varphi)(\xi):=\hat A(\varepsilon)(\varphi)(\xi)=(\hat A_1(\varepsilon)(\varphi)(\xi),\ldots,\hat A_m(\varepsilon)(\varphi)(\xi))\in O(C_0,\varepsilon)$, and hence $\hat A(C_0,\varepsilon):O(C_0,\varepsilon)\to O(C_0,\varepsilon)$.

Assume that

$$C_0>C_\Phi\,\gamma.\tag{37}$$

Since $l(C_0,\varepsilon)$ is a nondecreasing function of ε and $0\le l(C_0,0)<C_0$, it follows that, first, the equation

$$l(C_0,\varepsilon_0)=C_0\tag{38}$$

has at most one root ε_0 and this root ε_0 is a fortiori positive (if there are no roots we assume that $\varepsilon_0 = +\infty$), and second, that inequalities (36) hold for all $\varepsilon \in (0, \varepsilon_0]$. □

We next require the following estimate, which follows directly from the definition of ε_0:

$$\varepsilon_0 \leq C_\Phi^{-1} \left[\beta + h(C_0\,\varepsilon_0)\right]^{-1}. \tag{39}$$

Remark 6 Inequality (39) holds formally also for $\varepsilon_0 = +\infty$, because the equation $l(C_0, \varepsilon) = C_0$ has no roots only in the case $\beta = h(+\infty) = 0$; that is, if $a_i = \text{const}$ on $[0, X]$, $g(y_m, \ldots, y_1, x) = \tilde{g}(x)$ for all $(y_m, \ldots, y_1, x) \in \mathbb{R}^m \times [0, X]$.

For any $\varepsilon > 0$ and any $\varphi_1 = (z_1^1, \ldots, z_1^m)$ and $\varphi_2 = (z_2^1, \ldots, z_2^m)$ from $C_m[0, X/\varepsilon]$, consider the distance

$$\rho_\varepsilon(\varphi_1, \varphi_2) := \|\varphi_2 - \varphi_1\|_{C_m[0,X/\varepsilon]} := \max_{\xi\in X(\varepsilon)} \max_{1\leq i\leq m} |z_2^i(\xi) - z_1^i(\xi)|$$

between φ_1 and φ_2, where $X(\varepsilon) := [0, X/\varepsilon]$. Note that $C_m[0, X/\varepsilon]$ and $O(C_0, \varepsilon)$ are complete spaces with respect to ρ_ε.

Proposition 3 *$\hat{A}(C_0, \varepsilon)$ is a contraction for any $\varepsilon \in (0, \varepsilon_0)$.*

Proof Given two arbitrary functions $\varphi_1 = (z_1^1, \ldots, z_1^m)$ and $\varphi_2 = (z_2^1, \ldots, z_2^m)$ from $O(C_0, \varepsilon)$, we use (27) and (32) to estimate the distance between $\hat{A}(C_0, \varepsilon)(\varphi_1)$ and $\hat{A}(C_0, \varepsilon)(\varphi_2)$. We have

$$\rho_\varepsilon(\hat{A}(C_0,\varepsilon)(\varphi_1), \hat{A}(C_0,\varepsilon)(\varphi_2)) = \max_{\xi\in X(\varepsilon)} \max_{1\leq i\leq m} \left|\hat{A}_i(\varepsilon)(\varphi_2)(\xi) - \hat{A}_i(\varepsilon)(\varphi_1)(\xi)\right|$$

$$= \max_{\xi\in X(\varepsilon)} \max_{1\leq i\leq m} \int_0^\xi \left|\Phi^m_{\xi^{i-1}}(\xi - \zeta; \varepsilon\,\xi)\right| \Big(\Big\{\left|a_{m-1}(\varepsilon\,\zeta) - a_{m-1}(\varepsilon\,\xi)\right| + \varepsilon\left|f_{y_m}(\Pi^{(m-1)}(\zeta) + \varepsilon\, z_{12}^m(\zeta;\varepsilon),\right.$$

$$\left.\ldots, \Pi(\zeta) + \varepsilon\, z_{12}^1(\zeta;\varepsilon), \varepsilon\,\zeta)\right|\Big\} \left|z_2^m(\zeta) - z_1^m(\zeta)\right| + \cdots + \Big\{\left|a_0(\varepsilon\,\zeta) - a_0(\varepsilon\,\xi)\right|$$

$$+ \varepsilon\left|f_{y_1}(\Pi^{(m-1)}(\zeta) + \varepsilon\, z_{12}^m(\zeta;\varepsilon), \ldots, \Pi(\zeta) + \varepsilon\, z_{12}^1(\zeta;\varepsilon), \varepsilon\,\zeta)\right|\Big\} \left|z_2^1(\zeta) - z_1^1(\zeta)\right|\Big)\, d\zeta$$

$$\leq \rho_\varepsilon(\varphi_1, \varphi_2)\, C_\Phi \max_{\xi\in X(\varepsilon)} \int_0^\xi e^{\varkappa(\zeta-\xi)} \left[1 + (\xi - \zeta)^{m-1}\right] \Big\{\left|a_{m-1}(\varepsilon\,\zeta) - a_{m-1}(\varepsilon\,\xi)\right| + \ldots$$

$$+ \left|a_0(\varepsilon\,\zeta) - a_0(\varepsilon\,\xi)\right| + \varepsilon\left|f_{y_m}(\Pi^{(m-1)}(\zeta) + \varepsilon\, z_{12}^m(\zeta;\varepsilon), \ldots, \Pi(\zeta) + \varepsilon\, z_{12}^1(\zeta;\varepsilon), \varepsilon\,\zeta)\right| + \ldots$$

$$+ \varepsilon\left|f_{y_1}(\Pi^{(m-1)}(\zeta) + \varepsilon\, z_{12}^m(\zeta;\varepsilon), \ldots, \Pi(\zeta) + \varepsilon\, z_{12}^1(\zeta;\varepsilon), \varepsilon\,\zeta)\right|\Big\}\, d\zeta$$

$$\leq \rho_\varepsilon(\varphi_1, \varphi_2)\, \varepsilon\, C_\Phi \left[\beta + h(C_0\,\varepsilon)\right] \tag{40}$$

(cf. (34) and (35)), where

$$z_{12}^i(\zeta;\varepsilon) = (1-\theta)\, z_1^i(\zeta) + \theta\, z_2^i(\zeta), \quad \theta = \theta(\zeta;\varepsilon) \in (0,1).$$

From (40) and (39) it follows that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$k(C_0, \varepsilon) \leq \varepsilon\, C_\Phi \left[\beta + h(C_0\,\varepsilon)\right] \leq \varepsilon\, C_\Phi \left[\beta + h(C_0\,\varepsilon_0)\right] \leq \varepsilon/\varepsilon_0 < 1, \tag{41}$$

for the contraction factor $k(C_0, \varepsilon)$ of the operator $\hat{A}(C_0, \varepsilon)$. □

By the Banach contraction principle, as applied to the operator $\hat{A}(C_0,\varepsilon)$, for any $\varepsilon \in (0,\varepsilon_0)$ in $O(C_0,\varepsilon)$ there exists a unique solution $(z^1(\,\cdot\,;\varepsilon),\ \ldots,\ z^m(\,\cdot\,;\varepsilon)) =: \varphi(\,\cdot\,;\varepsilon)$ of equation (28) (which in view of Proposition 1 is equivalent to problem (14)–(17)). The uniqueness of the solution $\varphi(\,\cdot\,;\varepsilon)$ (for all $\varepsilon \in \mathbb{R}$), which is in fact the global uniqueness (that is, the uniqueness in the class $C_m[0,X/\varepsilon]$), is immediate from condition (3) on the functions a_i, b, and g, and so in this result only the existence of the solution and its membership in $O(C_0,\varepsilon)$ are nontrivial.

4. Estimate of the convergence rate of iterations

The contraction property of the operator $\hat{A}(C_0,\varepsilon)$ also allows one to construct an iteration sequence $\{\varphi_n(\,\cdot\,;\varepsilon)\}_{n=0}^{\infty}$ of functions $\varphi_n(\,\cdot\,;\varepsilon) = (z_n^1(\,\cdot\,;\varepsilon),\ldots,z_n^m(\,\cdot\,;\varepsilon))$, which converges in the norm of the space $C_m[0,X/\varepsilon]$ to the exact solution $\varphi(\,\cdot\,;\varepsilon) = (z^1(\,\cdot\,;\varepsilon),\ldots,z^m(\,\cdot\,;\varepsilon))$ of problem (14)–(17) for any $\varepsilon \in (0,\varepsilon_0)$:

$$\|\varphi(\,\cdot\,;\varepsilon) - \varphi_n(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} := \max_{\xi \in X(\varepsilon)} \max_{1\le i\le m} |z^i(\xi;\varepsilon) - z_n^i(\xi;\varepsilon)| \to 0, \quad n \to \infty$$

(for any ε_0' from the interval $(0,\varepsilon_0)$ this convergence is uniform with respect to ε on $(0,\varepsilon_0']$).

We set $\varphi_0(\xi;\varepsilon) := (0,\ldots,0)$ for $\xi \in [0,X/\varepsilon]$. Since $\varphi(\,\cdot\,;\varepsilon) \in O(C_0,\varepsilon)$, we have

$$\|\varphi(\,\cdot\,;\varepsilon) - \varphi_0(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} = \|\varphi(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} \le C_0. \tag{42}$$

Next, for any natural n, we define

$$\varphi_n(\xi;\varepsilon) := \hat{A}(C_0,\varepsilon)(\varphi_{n-1}(\,\cdot\,;\varepsilon))(\xi) = \hat{A}(\varepsilon)(\varphi_{n-1}(\,\cdot\,;\varepsilon))(\xi), \quad \xi \in [0,X/\varepsilon]. \tag{43}$$

Hence, using (41) and (42), we have, for any $n \in \{0\} \cup \mathbb{N} =: \mathbb{N}_0$,

$$\|\varphi(\,\cdot\,;\varepsilon) - \varphi_n(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} \le k(C_0,\varepsilon)^n\, \|\varphi(\,\cdot\,;\varepsilon) - \varphi_0(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} \le C_0\,(\varepsilon/\varepsilon_0)^n. \tag{44}$$

Let us return to problem (1)–(2) for $y(\,\cdot\,;\varepsilon)$. Taking into account (12) and (13), we define the sequences $\{\psi_n^1(\,\cdot\,;\varepsilon)\}_{n=0}^{\infty}$, ..., $\{\psi_n^m(\,\cdot\,;\varepsilon)\}_{n=0}^{\infty}$,

$$\psi_n^i(x;\varepsilon) := \varepsilon^{1-i}\,\tilde{y}_{\xi^{i-1}}(x/\varepsilon,x) + \varepsilon^{2-i}\,z_n^i(x/\varepsilon;\,\varepsilon), \tag{45}$$

which converge, respectively, to the solution $y(\,\cdot\,;\varepsilon)$ and its derivatives $y'(\,\cdot\,;\varepsilon)$, ..., $y^{(m-1)}(\,\cdot\,;\varepsilon)$, where $x \in [0,X]$, $i \in \overline{1,m}$, $n \in \mathbb{N}_0$.

For $n \ge 1$, from (45) and (43) one can express $\psi_n^i(\,\cdot\,;\varepsilon)$ directly in terms of $\psi_{n-1}^i(\,\cdot\,;\varepsilon)$:

$$\begin{aligned}\psi_n^i(x;\varepsilon) &= \varepsilon^{1-i}\,\tilde{y}_{\xi^{i-1}}(x/\varepsilon,x) + \varepsilon^{2-i}\,\hat{A}_i(\varepsilon)(z_{n-1}^1(\,\cdot\,;\varepsilon),\ldots,z_{n-1}^m(\,\cdot\,;\varepsilon))(x/\varepsilon)\\ &= \varepsilon^{1-i}\,\tilde{y}_{\xi^{i-1}}(x/\varepsilon,x) + \varepsilon^{2-i}\,\hat{A}_i(\varepsilon)(\hat{Z}_1(\varepsilon)(\psi_{n-1}^1(\,\cdot\,;\varepsilon)),\ldots,\hat{Z}_m(\varepsilon)(\psi_{n-1}^m(\,\cdot\,;\varepsilon)))(x/\varepsilon)\\ &=: \hat{B}_i(\varepsilon)(\psi_{n-1}^1(\,\cdot\,;\varepsilon),\ldots,\psi_{n-1}^m(\,\cdot\,;\varepsilon))(x),\end{aligned}$$

where $\hat{Z}_i(\varepsilon)(\psi) : [0,X/\varepsilon] \ni \xi \mapsto \varepsilon^{i-2}\,\psi(\varepsilon\,\xi) - \varepsilon^{-1}\,\tilde{y}_{\xi^{i-1}}(\xi,\,\varepsilon\,\xi)$, or briefly,

$$\begin{aligned}\psi_n(x;\varepsilon) &= \big(\hat{B}_1(\varepsilon)(\psi_{n-1}(\,\cdot\,;\varepsilon))(x),\ldots,\hat{B}_m(\varepsilon)(\psi_{n-1}(\,\cdot\,;\varepsilon))(x)\big)\\ &=: \hat{B}(\varepsilon)(\psi_{n-1}(\,\cdot\,;\varepsilon))(x), \quad x \in [0,X],\end{aligned}$$

where $\psi_n := (\psi_n^1, \ldots, \psi_n^m)$. We note that $\hat{B}(C_0,\varepsilon) : \tilde{O}(C_0,\varepsilon) \to \tilde{O}(C_0,\varepsilon)$ for $\varepsilon \in (0,\varepsilon_0]$ (that is, with the same ε as in Proposition 2), where

$$\tilde{O}(C_0,\varepsilon) := \Big\{\psi \in C_m[0,X] \mid \forall x \in [0,X]$$

$$\psi(x) \in \prod_{i=1}^{m} \left[\varepsilon^{1-i}\, \tilde{y}_{\xi^{i-1}}(x/\varepsilon, x) - \varepsilon^{2-i}\, C_0,\ \varepsilon^{1-i}\, \tilde{y}_{\xi^{i-1}}(x/\varepsilon, x) + \varepsilon^{2-i}\, C_0\right]\Big\}$$

is the closed $(\varepsilon\, C_0, C_0, \ldots, \varepsilon^{2-m}\, C_0)$-neighborhood of the vector function $\tilde{\psi}(\,\cdot\,;\varepsilon) : x \mapsto (\tilde{y}(x/\varepsilon, x), \ \ldots, \ \varepsilon^{1-m}\, \tilde{y}_{\xi^{m-1}}(x/\varepsilon, x))$ in the space $C_m[0,X]$, and $\hat{B}(C_0,\varepsilon)$ is the restriction of the operator $\hat{B}(\varepsilon)$ to $\tilde{O}(C_0,\varepsilon)$. Moreover, the operator $\hat{B}(C_0,\varepsilon)$ is a contraction for any $\varepsilon \in (0,\varepsilon_0)$ (that is, for the same ε as in Proposition 3).

Theorem 1 *Let conditions (3) and (4) hold and let ε_0 and C_0 satisfy (38) and (37). Then, first, for any $\varepsilon \in (0,\varepsilon_0)$ there exists a unique solution $y(\,\cdot\,;\varepsilon)$ of problem (1)–(2), and second, for any $\varepsilon \in (0,\varepsilon_0)$, $n \in \mathbb{N}_0$, and $i \in \overline{1,m}$,*

$$\|y^{(i-1)}(\,\cdot\,;\varepsilon) - \psi_n^i(\,\cdot\,;\varepsilon)\| \le C_0\, \varepsilon^{2-i}\, (\varepsilon/\varepsilon_0)^n.$$

Proof Since the existence and uniqueness of the solution $y(\,\cdot\,;\varepsilon)$ to problem (1)–(2) are direct corollaries to the existence and uniqueness of the solution $(z^1(\,\cdot\,;\varepsilon),\ \ldots,\ z^m(\,\cdot\,;\varepsilon))$ to problem (14)–(17) (which were justified in the previous section), it remains to estimate the accuracy of approximation of $y^{(i-1)}(\,\cdot\,;\varepsilon)$ by $\psi_n^i(\,\cdot\,;\varepsilon)$. For any $n \in \mathbb{N}_0$, $i \in \overline{1,m}$, $\varepsilon \in (0,\varepsilon_0)$, and $x \in [0,X]$ we have (see (45), (12), (13), and (44))

$$\begin{aligned}|y^{(i-1)}(x;\varepsilon) - \psi_n^i(x;\varepsilon)| &= |y^{(i-1)}(x;\varepsilon) - \varepsilon^{1-i}\, \tilde{y}_{\xi^{i-1}}(x/\varepsilon, x) - \varepsilon^{2-i}\, z_n^i(x/\varepsilon;\, \varepsilon)| \\ &= \varepsilon^{2-i}\, |z^i(x/\varepsilon;\, \varepsilon) - z_n^i(x/\varepsilon;\, \varepsilon)| \le \varepsilon^{2-i}\, \|\varphi(\,\cdot\,;\varepsilon) - \varphi_n(\,\cdot\,;\varepsilon)\|_{C_m[0,X/\varepsilon]} \\ &\le C_0\, \varepsilon^{2-i}\, (\varepsilon/\varepsilon_0)^n.\end{aligned}$$

□

Remark 7 The norm of the function $y^{(i-1)}(\,\cdot\,;\varepsilon)$ itself is $O(\varepsilon^{1-i})$, and hence the relative accuracy provided by the function $\psi_n^i(\,\cdot\,;\varepsilon)$ is of the same order ε^{n+1} for all $i \in \overline{1,m}$.

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