Some properties of $e$-symmetric rings

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Abstract: In this paper, we first give some characterizations of $e$-symmetric rings. We prove that $R$ is an $e$-symmetric ring if and only if $a_1a_2a_3 = 0$ implies that $a_\sigma(1)a_\sigma(2)a_\sigma(3)c = 0$, where $\sigma$ is any transformation of $\{1, 2, 3\}$. With the help of the Bott–Duffin inverse, we show that for $e \in ME_1(R)$, $R$ is an $e$-symmetric ring if and only if for any $a \in R$ and $g \in E(R)$, if $a$ has a Bott–Duffin $(e, g)$-inverse, then $g = eg$. Using the solution of the equation $axe = c$, we show that for $e \in ME_1(R)$, $R$ is an $e$-symmetric ring if and only if for any $a, c \in R$, if the equation $axe = c$ has a solution, then $c = ec$. Next, we study the properties of $e$-symmetric $*$-rings. Finally we discuss when the upper triangular matrix ring $T_2(R)$ (resp. $T_3(R, I)$) becomes an $e$-symmetric ring, where $e \in E(T_2(R))$ (resp. $e \in E(T_3(R, I))$).

Key words: $e$-Symmetric ring, $*$-ring, left semicentral, left min-abel ring, Bott–Duffin inverse, upper triangular matrix ring

1. Introduction
Throughout this paper, all rings are associative with unity. For a ring $R$, $T_2(R)$ denotes the $2 \times 2$ upper triangular matrix ring over $R$, and $E(R), U(R), Z(R)$, and $N(R)$ denote the set of all idempotents, the set of all invertible elements, the center of $R$, and the set of all nilpotent elements of $R$, respectively. An element $e \in E(R)$ is called left minimal idempotent of $R$ if $Re$ is a minimal left ideal of $R$. Write $ME_1(R)$ to denote the set of all left minimal idempotents of $R$. An idempotent $e$ of a ring $R$ is called left (right) semicentral if $ae = eae$ ($ea = eae$) for each $a \in R$. A ring $R$ is called (strongly) left min-abel [10] if either $ME_1(R) = \emptyset$ or every element $e$ in $ME_1(R)$ is (right) left semicentral.

A ring $R$ is symmetric [5] if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. The study of symmetric rings also can be found in [6]. Symmetric rings were generalized by Ouyang and Chen to weak symmetric rings in [8]; that is, a ring $R$ is said to be weak symmetric if for all $a, b, c \in R$, if $abc \in N(R)$, then $acb \in N(R)$. Following [3], a ring $R$ is called central symmetric if for any $a, b, c \in R$, $abc = 0$ implies $bac \in Z(R)$. Central symmetric rings are another form of generalization of symmetric rings. In [11], Wei introduced generalized weakly symmetric rings, which further generalized the concept of symmetric rings. In [7], a ring $R$ is called (strongly) $e$-symmetric if for any $a, b, c \in R$, $abc = 0$ implies $aceb = 0$ if $ae \in Z(R)$. In the paper, it was shown that a ring $R$ is $e$-symmetric if and only if $e$ is left semicentral and $eRe$ is symmetric [7, Theorem 2.2].

In [7, Theorem 3.1], it was shown that a ring $R$ is strongly $e$-symmetric if and only if $e$ is central and $eRe$
is symmetric. Also, using \( e \)-symmetric rings, we gave some new characterizations of left min-abel rings in [10] and [11].

This paper is organized as follows. In Section 2, we first discuss many properties of \( e \)-symmetric rings and strongly \( e \)-symmetric rings. Then, with the help of \( e \)-symmetric rings, we give some characterizations of left min-abel rings. In Section 3, we study the \( e \)-symmetricity of \(*\)-rings. We show that for a \(*\)-ring \( R \), if \( R \) is \( e \)-symmetric and \( 1 + (e^* - e)^*(e^* - e) \in U(R) \), then \( R \) is strongly \( e \)-symmetric and \( e \) is a projection. In Section 4, we discuss when the upper triangular matrix ring \( T_2(R) \) (resp. \( T_3(R, I) \)) becomes an \( e \)-symmetric ring, where \( e \in E(T_2(R)) \) (resp. \( e \in E(T_3(R, I)) \)).

## 2. Some characterizations of \( e \)-symmetric rings

**Proposition 2.1** The following conditions are equivalent for a ring \( R \):

1. \( R \) is an \( e \)-symmetric ring;
2. \( abc = 0 \) implies \( bace = 0 \) for all \( a, b, c \in R \).

**Proof**  
(1) \( \Rightarrow \) (2) Since \( R \) is an \( e \)-symmetric ring, by [7, Theorem 2.2], \( e \) is left semicentral. Let \( a, b, c \in R \) and satisfy \( abc = 0 \). Then we have \( 1a(bc) = 0 \), \( 1bcae = 0 \); that is, \( bcae = 0 \). Again, the \( e \)-symmetricity of \( R \) gives that \( b(ace) = 0 \). Noting that \( e \) is left semicentral, then we get \( bace = 0 \).

(2) \( \Rightarrow \) (1) Let \( x \in R \). We have \( xc(1 - e)e = 0 \); by hypothesis, one obtains \( (1 - e)xeee = 0 \), and it follows that \( (1 - e)Re = 0 \). Thus, \( e \) is left semicentral. By (2) we know that \( eRe \) is a symmetric ring. By [7, Theorem 2.2], \( R \) is an \( e \)-symmetric ring.

By Proposition 2.1, we get the following corollaries.

**Corollary 2.2** Let \( R \) be an \( e \)-symmetric ring. If \( abc = 0 \), then we have

1. \( bace = 0 \);
2. \( cbae = 0 \);
3. \( cbae = 0 \).

**Corollary 2.3** \( R \) is an \( e \)-symmetric ring if and only if for any \( a_1, a_2, a_3 \in R \), \( a_1a_2a_3 = 0 \) implies that \( a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}e = 0 \), where \( \sigma \) is any transformation of \( \{1, 2, 3\} \).

Let \( e, g \in E(R) \). If \( Re \cong Rg \) as left \( R \)-modules, then we say \( e \) and \( g \) are left isomorphic. Similarly, if \( eR \cong gR \) as right \( R \)-modules, then we say \( e \) and \( g \) are right isomorphic.

**Theorem 2.4** Let \( R \) be an \( e \)-symmetric ring.

1. If \( g \) and \( e \) are left isomorphic, then \( R \) is a \( g \)-symmetric ring.
2. If \( g \) and \( e \) are right isomorphic, then \( R \) is a \( g \)-symmetric ring.
3. If \( g \) and \( e \) are left isomorphic, then \( eR = gR \).

**Proof** Since \( R \) is an \( e \)-symmetric ring, by [7, Theorem 2.2], \( e \) is left semicentral.

1. Let \( \sigma : Re \to Rg \) be the left \( R \)-module isomorphism and \( g = \sigma(xe) \) where \( x \in R \); then \( eg = e\sigma(xe) = \sigma(exe) = \sigma(xe) = g \). Let \( a, b, c \in R \) and satisfy \( abc = 0 \). Then \( acbe = 0 \) (since \( R \) is an \( e \)-symmetric ring), so we have \( acbg = acbeg = 0 \). Thus, \( R \) is a \( g \)-symmetric ring.

2. Let \( \tau : eR \to gR \) be the right \( R \)-module isomorphism. Then there exist \( x, y \in R \) such that \( \tau(e) = gx \) and \( \tau(ey) = g \), so we have \( g = \tau(e)y = gx \). Let \( f = ygx \). Then

\[
f^2 = ygxygx = yg^2x = ygx = f,
\]
\[ ef = eygx = \tau^{-1}(g)gx = \tau^{-1}(gx) = e, \]
\[ fe = ygxe = yr(e)e = y(e)ygx = f, \]
and so \( Re = Rf \), and \( e \) and \( f \) are left isomorphic. By (1), \( R \) is an \( f \)-symmetric ring. Then by [7, Theorem 2.2], \( f \) is left semicentral. Therefore,
\[ g = g^2 = gxygxy = gx fy = fgxfy = fg. \]

Let \( a, b, c \in R \) and satisfy \( abc = 0 \); then \( acbf = 0 \) (since \( R \) is an \( f \)-symmetric ring). We have \( acbg = acbf g = 0 \). Thus, \( R \) is a \( g \)-symmetric ring.

(3) Since \( g \) and \( e \) are left isomorphic, by (1), \( R \) is a \( g \)-symmetric ring. Hence, \( g \) is left semicentral by [7, Theorem 2.2]. Observing the proof of (1), we have \( e = ge \) and \( g = eg \), and this gives \( eR = gR \). \( \square \)

**Corollary 2.5** Let \( R \) be a strongly \( e \)-symmetric ring.

(1) If \( g \) and \( e \) are left isomorphic, then \( e = g \).

(2) If \( g \) and \( e \) are right isomorphic, then \( e = g \).

**Proof** Since \( R \) is a strongly \( e \)-symmetric ring, by [7, Theorem 3.1], \( e \) is a central element and \( R \) is \( e \)-symmetric.

(1) If \( g \) and \( e \) are left isomorphic, then \( eR = gR \) by Theorem 2.4(3). Hence, \( g = eg \) and \( ge = e \). Noting that \( e \) is central, then \( g = ge = e \).

(2) If \( g \) and \( e \) are right isomorphic, then the proof of Theorem 2.4(2) implies that \( eR = gR \), and by (1), we know that \( e = g \). \( \square \)

Let \( R \) be a ring and \( a \in R \) and \( e, f \in E(R) \). If there exists an element \( y \in R \) satisfying
\[ y = ey = yf, \quad yae = e, \quad fay = f, \]
then \( y \) is called a Bott–Duffin \((e, f)\)-inverse of \( a \) (see [2]). If \( y \) exists, then it is unique. Denote it by \( a^{(e,f)}_{BD} \).

**Proposition 2.6** Let \( a \in R \) and \( e, f \in E(R) \). If \( R \) is \( e \)-symmetric and \( a \) has a Bott–Duffin \((e, f)\)-inverse \( y \), then:

(1) \( R \) is \( f \)-symmetric and \( eR = fR \);

(2) \( y^{(e,f)}_{BD} = eaf \).

**Proof** (1) Since \( a \) has a Bott–Duffin \((e, f)\)-inverse \( y \), \( y = ey = yf, \ yae = e, \) and \( fay = f \). Noting that \( R \) is \( e \)-symmetric, then \( e \) is left semicentral by [7, Theorem 2.2], so \( f = fay = fa(ey) = e(aye) = ef \), and this implies that \( R \) is \( f \)-symmetric. Hence, \( f \) is left semicentral, and it follows that \( e = yae = (yf)ae = (fyf)ae = f(yf)ae = f(e)ae = fe \). Therefore, \( eR = fR \).

(2) Noting that \( e \) and \( f \) are left semicentral, then \( eafye = eafye = eaye = eaye = (fe)aye = faye = fe = e \) and \( fyeaf = feafy = yeaf = yfa = yfa = yae = ef = f \). Then \( y^{(e,f)}_{BD} = eaf. \square \)

**Proposition 2.7** Let \( R \) be an \( e \)-symmetric ring and \( f \in E(R) \). If \( R \) satisfies one of the following conditions, then \( R \) is \( f \)-symmetric:
(1) \(eR + (1 - f)R = R\);
(2) \(ea + 1 - f \in U(R)\) for some \(a \in R\);
(3) \(Re + R(1 - f) = R\);
(4) \(ae + 1 - f \in U(R)\) for some \(a \in R\).

**Proof** (1) Since \(R\) is \(e\)-symmetric, \(e\) is left semicentral by [7, Theorem 2.2]. Noting that \(eR + (1 - f)R = R\), then \(fR = feR = efR \subseteq eR\), and it follows that \(f = ef\). The proof of Theorem 2.4(1) implies that \(R\) is \(f\)-symmetric.

(2) Set \(ea + 1 - f = u \in U(R)\). Then \(fu = fea\) and one obtains \(f = feau^{-1}\). Noting that \(e\) is left semicentral, then \(f = ef\), and this gives that \(R\) is \(f\)-symmetric.

(3) If \(Re + R(1 - f) = R\), then \(Rf = Ref\). Set \(f = eaf\) for some \(c \in R\). Then \(f = ecf = ef\) because \(e\) is left semicentral. Therefore, \(R\) is \(f\)-symmetric.

(4) Set \(ae + 1 - f = v \in U(R)\). Then \(fv = fae\) and one obtains \(f = faev^{-1}\). Noting that \(e\) is left semicentral, then \(f = ef\), so \(R\) is \(f\)-symmetric. \(\square\)

**Proposition 2.8** A ring \(R\) is a strongly left min-abel ring if and only if for \(e \in ME_1(R)\) and \(a, b \in R\), \(e = eab\) implies that \(e = eba\).

**Proof** \((\Rightarrow)\) Assume that \(R\) is strongly left min-abel and \(e = eab\). Then \(e\) is central, and it follows that \(e = ee = eabeab = e(aeba)b\). This implies that \(eba \neq 0\), and one has \(Re = Reba\). Set \(e = eaba\) for some \(c \in R\).

Noting that \(e = eab\), then \(be = beab = ebab\). This gives \(eba = bea = eba\), so \(e = eca = (eaba)ba = eba\).

\((\Leftarrow)\) Let \(e \in ME_1(R)\) and \(x \in R\). Set \(g = e + ex(1 - e)\). Then \(eg = g, ge = e,\) and \(g^2 = g \in ME_1(R)\).

Since \(e = ege\), by hypothesis, \(e = eeg = eg = g\). Thus, \(ex(1 - e) = g - e = 0\) for each \(x \in R\), and this gives that \(e\) is right semicentral. By [7, Lemma 3.3], \(e\) is central. Hence, \(R\) is strongly left min-abel. \(\square\)

**Theorem 2.9** Let \(e \in ME_1(R)\). Then \(R\) is an \(e\)-symmetric ring if and only if for any \(a \in R\) and \(g \in E(R)\), if \(a\) has a Bott–Duffin \((e,g)\)-inverse, then \(g = eg\).

**Proof** \((\Rightarrow)\) Let \(R\) be an \(e\)-symmetric ring. Then \(e\) is left semicentral. Assume that \(a \in R\) and \(g \in E(R)\) and \(a\) has a Bott–Duffin \((e,g)\)-inverse. Letting \(a_{BD}^{(e,g)} = y\), then \(y = ey = yg\) and \(g = gay = gaey = egaey = eg\).

\((\Leftarrow)\) First we prove that \(e\) is left semicentral. For any \(x \in R\), set \(g = e + (1 - e)xe\); then \(eg = e, ge = g, g^2 = g\). Obviously, \(e\) is a Bott–Duffin \((e,g)\)-invertible element and \(e_{BD}^{(e,g)} = e\). By hypothesis \(g = eg = e\), and then \((1 - e)xe\) for any \(x \in R\). Thus, \(e\) is left semicentral.

Next, we prove that \(eRe\) is a symmetric ring. Any \(a, b, c \in eRe\) satisfy \(abc = 0\). Assuming that \(acb \neq 0\), then \(a \neq 0\) and \(b \neq 0\), and so we have \(Ra = Re = Rb\). Let \(e = ra = sb\) for some \(r, s \in R\); then \(acb = aecb = ashcb = asecb = fathercb = 0\), which is a contradiction. Thus, \(eRe\) is a symmetric ring, and hence \(R\) is an \(e\)-symmetric ring by [7, Theorem 2.2]. \(\square\)

**Proposition 2.10** Let \(e \in ME_1(R)\). Then \(R\) is an \(e\)-symmetric ring if and only if for any \(a \in R\) and \(g \in E(R)\), if \(a\) has a Bott–Duffin \((e,g)\)-inverse, then \(e = ge\).
Proof (⇒) From Theorem 2.9, we know $g = eg$. Since $e \in ME_l(R)$, $g \in ME_l(R)$, and $R$ is a $g$-symmetric ring, $g$ is left semicentral. Thus, $e = yae = ygae = gygae = ge$, where $y = \sigma^{(e,g)}_{BD}$. (⇐) The proof is similar to Theorem 2.9.

Let $R$ be a ring and $a \in R$. If there exists $b \in R$ such that $a = aba$, then $a$ is called a regular element of $R$ and $b$ is called an inner inverse. Clearly, if $b$ exists, it is not unique. We denote by $a\{1\}$ the set of all inner inverses of a regular element $a$. Let $b \in a\{1\}$. Then $ab, ba \in E(R)$.

Proposition 2.11 Let $a$ be a regular element of $R$ and $b \in a\{1\}$. If $R$ is $ab$-symmetric, then $R$ is $ba$-symmetric.

Proof Since $b \in a\{1\}$, we have $a = aba$. Let $e = ab$ and $g = ba$; then $e, g \in E(R)$ and $ea = a = ag$. Denote $\sigma : Re \to Ra$ by $\sigma(re) = rea$ for any $r \in R$. It is easy to prove that $\sigma$ is a left $R$-module isomorphism. Since $Ra = Rg$, we have $Re \cong Rg$ as left $R$-modules. By hypothesis, $R$ is an $e$-symmetric ring, and thus $R$ is a $g$-symmetric ring by Theorem 2.4. That is, $R$ is a $ba$-symmetric ring.

Lemma 2.12 Let $a, b \in R$ and $e \in ME_l(R)$ satisfy $abe = e$. If $e$ is left semicentral, then $e = bae$.

Proof Since $abe = e$ and $e$ is left semicentral, we have $e = aceb$. Then $Re = Rae$. Letting $e = cae$ for some $c \in R$, then $e = c(abe) = caeb = eb = be$ and $b = beae = ceae = ca = e$.

Lemma 2.13 Let $e \in ME_l(R)$. If $e$ is left semicentral, then $eRe$ is a symmetric ring.

Proof Let $a, b, c \in eRe$ and satisfy $abc = 0$. If $acb \neq 0$, then $Racb = Re$, so $e = dacbe$ for some $d \in R$. By Lemma 2.12, $e = bdaec = cbdae$. Thus, $e = dacbe = dacbe = dabaedacecbe = d(ab)daedacecbe = 0$, which is a contradiction. Hence, $acb = 0$ and so $eRe$ is a symmetric ring.

Proposition 2.14 Let $e \in ME_l(R)$. Then $R$ is an $e$-symmetric ring if and only if for any $a \in R$ either $aRe = 0$ or the equation $axe = e$ has a solution.

Proof (⇒) Since $R$ is an $e$-symmetric ring, $e$ is left semicentral. Let $a \in R$. If $aRe \neq 0$, then $abe \neq 0$ for some $b \in R$. Thus, $Rabe = Re$. Set $e = dabe$ for some $d \in R$. By Lemma 2.12, $e = abde$. Hence, $x = bd$ is a solution of the equation $axe = e$.

(⇐) Let $e \in ME_l(R)$. If $(1 - e)Re \neq 0$, then by hypothesis we know $(1 - e)xe = e$ has a solution. However, $(1 - e)xe = e$ does not have a solution and that is a contradiction. Thus, $(1 - e)Re = 0$, $e$ is left semicentral. By Lemma 2.13, $eRe$ is a symmetric ring. Hence, $R$ is an $e$-symmetric ring by [7, Theorem 2.2].

Theorem 2.15 Let $e \in ME_l(R)$. Then $R$ is an $e$-symmetric ring if and only if for any $a, c \in R$, if the equation $axe = c$ has a solution, then $c = ec$.

Proof (⇒) Since $R$ is an $e$-symmetric ring, $e$ is left semicentral. If the equation $axe = c$ has a solution $x = b$, then $c = abe = eabe = ec$. 

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(⇐) For any \( a \in R \), denote \( h = (1 - e)ae \). If \( h \neq 0 \), then \( Rh = Re \). Let \( e = ch \) for some \( c \in R \). Then \( h(ch)e = hch = he = h \). Thus, the equation \( hxe = h \) has a solution, and then \( h = eh = e(1 - e)ae = 0 \), which is a contradiction. Then \( (1 - e)ae = 0 \) for any \( a \in R \). Hence, \( e \) is left semicentral. By Lemma 2.13, \( eRe \) is a symmetric ring. Hence, \( R \) is an \( e \)-symmetric ring.

\[ \square \]

3. Symmetricity of \( * \)-rings

An involution \( a \mapsto a^{*} \) in a ring \( R \) is an antiisomorphism of degree 2; that is,

\[(a^{*})^{*} = a, \quad (a + b)^{*} = a^{*} + b^{*}, \quad (ab)^{*} = b^{*}a^{*}.\]

A ring \( R \) with an involution \( * \) is called a \( * \)-ring (see [1]).

Let \( R \) be a \( * \)-ring and \( e \in E(R) \). If \( e^{*} = e \), then \( e \) is called projection.

Let \( R \) be a ring and \( e \in E(R) \). \( R \) is called left \( e \)-reflexive if \( aRe = 0 \) implies \( eRa = 0 \) for any \( a \in R \).

**Proposition 3.1** (1) \( R \) is strongly \( e \)-symmetric if and only if \( R \) is \( e \)-symmetric and left \( e \)-reflexive.

(2) If \( e \) is a projection element of a \( * \)-ring \( R \), then \( R \) is strongly \( e \)-symmetric if and only if \( R \) is \( e \)-symmetric.

**Proof**  (1) \((\Rightarrow)\) Assume that \( aRe = 0 \). Since \( R \) is a strongly \( e \)-symmetric ring, by [7, Theorem 3.1], \( e \) is a central element. Then we get \( eRa = eRa = 0 \). Thus, \( R \) is left \( e \)-reflexive.

\((\Leftarrow)\) Suppose that \( R \) is \( e \)-symmetric and left \( e \)-reflexive; by [7, Theorem 2.2], \( e \) is left semicentral. Then we have \( (1 - e)Re = 0 \). Since \( R \) is left \( e \)-reflexive, we have \( eR(1 - e) = 0 \), so \( e \) is a central element. Thus, \( R \) is a strongly \( e \)-symmetric ring by [7, Theorem 3.1].

(2) Noting that a projection element \( e \) in a \( * \)-ring \( R \) is left semicentral if and only if it is central, (2) holds.

\[ \square \]

**Proposition 3.2** Let \( R \) be a \( * \)-ring and \( e \in E(R) \). If \( R \) is an \( e \)-symmetric ring, then:

1. \( e^{*}e \) is an idempotent element;
2. the following conditions are equivalent:
   a. \( R \) is \( e^{*}e \)-symmetric,
   b. for each \( x \in R \), \( e^{*}xe = xe^{*}e \),
   c. \( e^{*}e \) is central,
   d. \( ee^{*}e = e^{*}e \).

**Proof**  (1) Since \( R \) is an \( e \)-symmetric ring, \( e \) is left semicentral, and it follows that \( (e^{*}e)^{2} = e^{*}ee^{*}e = e^{*}e = e^{*}e \).

(2) \((a) \Rightarrow (c)\) By (1), we know that \( e^{*}e \) is a projection. Since \( R \) is \( e^{*}e \)-symmetric, by Proposition 3.1(2), \( R \) is strongly \( e^{*}e \)-symmetric, and by [7, Theorem 3.1], \( e^{*}e \) is central.

\((c) \Rightarrow (b)\) For each \( x \in R \), by (c), we have \( e^{*}ex = xe^{*}e \), and this gives \( xe^{*}e = e^{*}exe \). Noting that \( e \) is left semicentral, \( xe^{*}e = e^{*}xe \).

\((b) \Rightarrow (d)\) Choose \( x = e \); we are done.

\((d) \Rightarrow (a)\) Let \( a, b, c \in R \) and satisfy \( abc = 0 \). Since \( R \) is \( e \)-symmetric, \( acbe = 0 \), and this leads to \( acbe^{*}e = acbee^{*}e = 0 \). Hence, \( R \) is \( e^{*}e \)-symmetric.

\[ \square \]
Proposition 3.3 Let $R$ be a *-ring. If $R$ is $e$-symmetric, then the following conditions are equivalent:

(1) $ee^* \in E(R)$;
(2) $xxee^* = e^*xe$ for each $x \in R$;
(3) $ee^* = e^*e$;
(4) $ee^*$ is central.

Proof (1) $\implies$ (2) Since $R$ is an $e$-symmetric ring and $ee^* \in E(R)$, $R$ is $ee^*$-symmetric, and it follows that $ee^*$ is left semicentral. Hence, $xee^* = ee^*xee^*$ for each $x \in R$. Noting that $e$ is left semicentral and $e^*$ is right semicentral, then $xee^* = e^*xe$.

(2) $\implies$ (3) Choose $x = e$. Then, by (2), we have $ee^* = e^*e$.

(3) $\implies$ (4) Since $R$ is $e$-symmetric and $ee^* = e^*e$, $R$ is $e^*e$-symmetric, and by Proposition 3.2(2), $e^*e$ is central. Hence, $ee^*$ is central.

(4) $\implies$ (1) Trivial. \hfill \qedsymbol

Theorem 3.4 Let $R$ be a *-ring and an $e$-symmetric ring. If $1 + (e^* - e)(e^* - e) \in U(R)$, then $R$ is a strongly $e$-symmetric ring and $e$ is a projection.

Proof Set $u = 1 + (e^* - e)(e^* - e)$ and $v = u^{-1}$. Then $u^* = u$, $eu = ee^*e = ue$, and it follows that $ev = ve$ and $v^* = v$, so $e^*v = ve^*$. Choose $f = ee^*v = vee^*$. Then $f^2 = (vee^*)(vee^*) = v(vee^*)e^*v = vee^*v = vee^*v = ee^*v = f$ and $f^* = f$, and this gives that $f$ is a projection. Since $R$ is $e$-symmetric and $f = ef$, $R$ is $f$-symmetric. By Proposition 3.1(2), $R$ is strongly $f$-symmetric, so $f$ is central and it follows that $f = ef = fe = vee^*v = vee = e$. Hence, $e$ is projection and $R$ is a strongly $e$-symmetric ring. \hfill \qedsymbol

Let $R$ be a *-ring and $e \in E(R)$. $p \in R$ is said to be a range projection [4] if $p$ is a projection satisfying $pe = e$ and $ep = p$. The range projection of $e$ is denoted by $e^\perp$.

Proposition 3.5 Let $R$ be a *-ring. If $R$ is $e$-symmetric, then the following conditions are equivalent:

(1) $1 + (e^* - e)(e^* - e) \in U(R)$;
(2) $e + e^* - 1 \in U(R)$;
(3) $e^\perp$ exists.

Proof (1) $\implies$ (2) By Theorem 3.4, $e$ is a projection. Hence, $e + e^* - 1 = 2e - 1 \in U(R)$.

(2) $\implies$ (3) Follows from [4, Theorem 2.1].

(3) $\implies$ (1) Let $p = e^\perp$. Then $ep = p$. Noting that $R$ is $e$-symmetric, then $R$ is $p$-symmetric, and by Proposition 3.1, $p$ is central. It follows that $e = pe = ep = p$. Hence, $1 + (e^* - e)(e^* - e) = 1 \in U(R)$. \hfill \qedsymbol

An element $a^\dagger$ in a *-ring $R$ is called the Moore–Pensor inverse (or MP-inverse) of $a$ [9] if

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad aa^\dagger = (aa^\dagger)^*, \quad a^\dagger a = (a^\dagger a)^*.$$ 

In this case, we call $a$ MP-invertible in $R$. The set of all MP-invertible elements of $R$ is denoted by $R^\dagger$.

Corollary 3.6 Let $R$ be a *-ring and an $e$-symmetric ring. Then $e \in R^\dagger$ if and only if $e$ is a projection.
Proof  \((\Rightarrow)\) Assume that \(e \in R^\dagger\). By [4, Theorem 3.1], \(e + e^* - 1 \in U(R)\). By Proposition 3.5, \(1 + (e^* - e)(e^* - e) \in U(R)\), and by Theorem 3.4, \(e\) is a projection.

\((\Leftarrow)\) Suppose that \(e\) is a projection. Then \(e + e^* - 1 = 2e - 1 \in U(R)\). By [4, Theorem 3.1], \(e \in R^\dagger\). \(\square\)

**Theorem 3.7** Let \(R\) be a \(*\)-ring and \(a \in R^\dagger\). If \(R\) is \(aa^\dagger\)-symmetric, then \(a\) is EP.

**Proof** Note that \(R\) is \(aa^\dagger\)-symmetric and \(aa^\dagger\) is projection. Hence, by Proposition 3.1(2), \(R\) is strongly \(aa^\dagger\)-symmetric, and it follows that \(aa^\dagger\) is central from [7, Theorem 3.1]. This gives that \(a = (aa^\dagger)a = a(aa^\dagger) = a^2a^\dagger\).

Noting that \(Ra = R(a^\dagger a)\) and \(Raa^\dagger \cong Ra\) as left \(R\)-module, then \(Raa^\dagger \cong Ra^\dagger a\) as left \(R\)-module. Since \(R\) is \(aa^\dagger\)-symmetric, it follows that \(R\) is \(a^\dagger a\)-symmetric from Theorem 2.4. Noting that \(a^\dagger a\) is projection, then \(a^\dagger a\) is central, which implies that \(a = a(a^\dagger a) = a^\dagger a^2\). Hence, \(a \in a^2R \cap Ra^2\), and one obtains that \(a \in R^\dagger\) and so \(a^2\) exists. Now we have \(a^2a = a^2a^2a^\dagger = aa^\dagger\); hence, \(a\) is EP. \(\square\)

### 4. Upper triangular matrix ring

**Proposition 4.1** Let \(R\) be a ring and \(e \in E(R), r \in R\). Then we have the following results:

1. \(T_2(R)\) is a \(\left( \begin{array}{cc} 1 & r \\ 0 & 0 \end{array} \right)\)-symmetric ring if and only if \(R\) is a symmetric ring.
2. \(T_2(R)\) is a \(\left( \begin{array}{cc} e & 0 \\ 0 & 0 \end{array} \right)\)-symmetric ring if and only if \(R\) is an \(e\)-symmetric ring.
3. \(T_2(R)\) is a \(\left( \begin{array}{cc} e & e \\ 0 & 0 \end{array} \right)\)-symmetric ring if and only if \(R\) is an \(e\)-symmetric ring.

**Proof** \((1)\) \((\Rightarrow)\) Let \(a, b, c \in R\) and satisfy \(abc = 0\). Then we have

\[
\left( \begin{array}{ccc} a & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} b & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} c & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right);
\]

Since \(T_2(R)\) is a \(\left( \begin{array}{cc} 1 & r \\ 0 & 0 \end{array} \right)\)-symmetric ring, we have

\[
\left( \begin{array}{ccc} a & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} b & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & r \\ 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right);
\]

this is \(\left( \begin{array}{ccc} abc & acbr \\ 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right)\). Then we get \(acb = 0\), and so \(R\) is a symmetric ring.

\((\Leftarrow)\) Let \(A = \left( \begin{array}{ccc} a_1 & b_1 \\ 0 & c_1 \end{array} \right), B = \left( \begin{array}{ccc} a_2 & b_2 \\ 0 & c_2 \end{array} \right), C = \left( \begin{array}{ccc} a_3 & b_3 \\ 0 & c_3 \end{array} \right) \in T_2(R)\), and \(ABC = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right)\); this is \(ABC = \left( \begin{array}{ccc} a_1a_2a_3 & * \\ 0 & c_1c_2c_3 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right)\), and then we get \(a_1a_2a_3 = c_1c_2c_3 = 0\). Since \(R\) is a symmetric ring, we have \(a_1a_3a_2 = c_1c_3c_2 = 0\), so \(ACB\left( \begin{array}{ccc} 1 & r \\ 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} a_1a_3a_2 & a_1a_3a_2r \\ 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right)\). Hence, \(T_2(R)\) is a \(\left( \begin{array}{cc} 1 & r \\ 0 & 0 \end{array} \right)\)-symmetric ring.
Similarly, we can prove (2) and (3).

Let $R$ be a ring and $I$ an ideal of $R$,

$$T_3(R, I) = \{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} | a_1, a_3, a_4, a_5, a_6 \in R \text{ and } a_2 \in I \}. $$

Then, by the usual matrix addition and multiplication, $T_3(R, I)$ is a ring.

**Proposition 4.2** Let $R$ be a ring, $I$ an ideal of $R$, and $e \in E(R)$. Then $T_3(R, I)$ is a $e$-symmetric ring if and only if $R$ is an $e$-symmetric ring and $Ie = 0$.

**Proof** Let $a \in I$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$, $B = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$, and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$. Then $ABC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $T_3(R, I)$ is a $e$-symmetric ring, we have $ACB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $ae = 0$, and so $Ie = 0$.

Let $x, y, z \in R$ and satisfy $xyz = 0$. Choose $A = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$, $B = \begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$, and $C = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, I)$. Then $ABC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $T_3(R, I)$ is a $e$-symmetric ring, we have $ACB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $xyz = 0$, and $R$ is an $e$-symmetric ring.

Conversely, let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in T_3(R, I)$, $B = \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in T_3(R, I)$, $C = \begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 \\ 0 & 0 & c_6 \end{pmatrix} \in T_3(R, I)$, and $ABC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} a_1b_1c_1 & * & * \\ 0 & a_4b_4c_4 & * \\ 0 & 0 & a_5b_5c_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and then

$$a_1b_1c_1 = a_4b_4c_4 = a_5b_5c_6 = 0.$$
Since $R$ is an $e$-symmetric ring, we get that $a_1c_1b_1e = a_4c_4b_4e = a_6c_6b_6e = 0$. Since $a_2, b_2, c_2 \in I$, $a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4 \in I$, by hypothesis $(a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4)e = 0$. Hence,

$$ACB \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1c_1b_1e & (a_1c_1b_2 + a_1c_2b_4 + a_2c_4b_4)e & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Thus, $T_3(R, I)$ is a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring. \hfill \Box

The following corollary follows from Proposition 4.2.

**Corollary 4.3** $T_3(R, I)$ is a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$-symmetric ring if and only if $R$ is a symmetric ring and $I = 0$.

**Example 4.4** Let $R$ be a symmetric ring and $I = 0$. Take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$, and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, 0)$. Then $ABC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, but $ACB \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This shows that $T_3(R, 0)$ is not a symmetric ring. Similarly, we can prove that for an $e$-symmetric ring $R$, $T_3(R, 0)$ need not be a $\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}$-symmetric ring.

Let $R$ be a ring,

$$WV_3(R) = \{ \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} | a, b, c \in R \},$$

$$WT_3(R) = \{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} | a, b, c, d \in R \}.$$

Then by the usual matrix addition and multiplication, $WV_3(R)$ and $WT_3(R)$ are rings. Obviously, $WV_3(R)$ and $WT_3(R)$ are subrings of $T_3(R, I)$. Similarly, we can prove that Proposition 4.2 and Corollary 4.3 also hold for $WV_3(R)$ and $WT_3(R)$.

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