Certain strongly clean matrices over local rings

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Abstract: We are concerned about various strongly clean properties of a kind of matrix subrings \( L(\alpha)(R) \) over a local ring \( R \). Let \( R \) be a local ring, and let \( s \in C(R) \). We prove that \( A \in L(\alpha)(R) \) is strongly clean if and only if \( A \) or \( I_2 - A \) is invertible, or \( A \) is similar to a diagonal matrix in \( L(\alpha)(R) \). Furthermore, we prove that \( A \in L(\alpha)(R) \) is quasipolar if and only if \( A \in GL_2(R) \) or \( A \in L(\alpha)(R)_{qnil} \), or \( A \) is similar to a diagonal matrix \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) in \( L(\alpha)(R) \), where \( \lambda \in J(R) \), \( \mu \in U(R) \) or \( \lambda \in U(R) \), \( \mu \in J(R) \), and \( l_\mu - r_\lambda \), \( l_\lambda - r_\mu \) are injective. Pseudopolarity of such matrix subrings is also obtained.

Key words: Matrix ring, strongly clean matrix, quasipolar matrix

1. Introduction
Throughout, all rings are associative with an identity. An element \( a \) in a ring \( R \) is strongly clean provided that it is the sum of an idempotent and a unit that commute. The commutant of \( a \in R \) is defined by \( comm(a) = \{ x \in R \mid xa = ax \} \). Set \( R^{qnil} = \{ a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a) \} \). We say \( a \in R \) is quasinilpotent if \( a \in R^{qnil} \). The double commutant of \( a \in R \) is defined by \( comm^2(a) = \{ x \in R \mid xy = yx \text{ for all } y \in comm(a) \} \).

In [6], an element \( a \) in a ring \( R \) is called quasipolar if for any \( a \in R \) there exists \( c^2 = e \in comm^2(a) \) such that \( a + e \in U(R) \) and \( ae \in R^{qnil} \). As is well known, an element \( a \in R \) is quasipolar if and only if it has generalized Drazin inverse, i.e. there exists \( b \in comm^2(a) \) such that \( b = b^2a, a - a^2b \in R^{qnil} \) (see [6]).

Following [8], an element \( a \) in a ring \( R \) has a pseudo Drazin inverse if and only if there exists \( b \in comm^2(a) \) such that \( b = bab, a^k - a^{k+1}b \in J(R) \) for some \( k \geq 1 \). In a ring \( R \), evidently, \{ elements having pseudo Drazin inverses \} \( \subseteq \{ \text{quasipolar elements} \} \subseteq \{ \text{strongly clean elements} \} \). The subjects of strongly clean rings, quasipolar rings, and pseudo Drazin inverses were extensively studied in \([1–5, 7]\) and \([10–12]\).

Evidently, the clean property for a ring does not transfer to its subrings. For example, \( \mathbb{Z}(2) \) is strongly clean, but the subring \( \mathbb{Z} \) of \( \mathbb{Z}(2) \) is not strongly clean where \( \mathbb{Z}(2) \) is the localization of \( \mathbb{Z} \) at the prime ideal \( (2) \). For instance, \( 4 \in \mathbb{Z} \) cannot be written as the sum of an idempotent and a unit that commute.

The motivation of this paper is to investigate the behave of subrings of a strongly clean ring. This enables us to construct related counterexamples as well. For this purpose, we introduce a kind of matrix subrings over

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a local ring $R$. Here, a ring $R$ is local if it has only one maximal right ideal. Let $R$ be a local ring, and let $s \in C(R)$. Let

$$L_{(s)}(R) = \left\{ \begin{pmatrix} a & b \\ sc & d \end{pmatrix} \in M_2(R) \mid a, b, c, d \in R \right\},$$

where the operations are defined as those in $M_2(R)$. We prove that $A \in L_{(s)}(R)$ is strongly clean if and only if $A$ or $I_2 - A$ is invertible, or $A$ is similar to a diagonal matrix in $L_{(s)}(R)$. Moreover, we prove that $A \in L_{(s)}(R)$ is quasipolar if and only if $A \in GL_2(R)$ or $A \in L_{(s)}(R)^{qnil}$, or $A$ is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$, where $\lambda \in J(R)$, $\mu \in U(R)$ or $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Note that for $a \in R$, $l_a$ and $r_a$ denote the abelian group endomorphisms of $R$ given by left and right multiplication by $a$, respectively. Pseudopolarity of such matrix subrings is also obtained.

We use $J(R)$ to denote the Jacobson radical of $R$ and $U(R)$ to denote the group of units of $R$. Furthermore, $C(R)$ is the center of a ring $R$ and $\mathbb{N}$ stands for the set of all natural numbers.

2. Strongly clean matrices

The goal of this section is to investigate strong cleanness of $L_{(s)}(R)$. We begin with the following.

**Proposition 2.1** Let $R$ be a ring, and let $s \in C(R) \cap U(R)$. Then $L_{(s)}(R) \cong M_2(R)$.

**Proof** Let $\varphi : M_2(R) \rightarrow L_{(s)}(R)$, defined by

$$\varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & b \\ sc & d \end{pmatrix} = \begin{pmatrix} a & b \\ s^{-1}c & d \end{pmatrix}.$$ 

As $s \in C(R) \cap U(R)$, one directly checks that $\varphi$ is a ring isomorphism, as asserted.

**Lemma 2.2** Let $R$ be a ring, and let $s \in C(R) \cap J(R)$. Then $U(L_{(s)}(R)) = \left\{ \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c, d \in U(R), x, y \in R \right\}$.

**Proof** As $s \in J(R)$, we easily see that $U(L_{(s)}(R)) \subseteq \left\{ \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c, d \in U(R), x, y \in R \right\}$. To see the converse inclusion, suppose that $c, d \in U(R)$. Then

$$\begin{pmatrix} c & x \\ sy & d \end{pmatrix}^{-1} = \begin{pmatrix} c^{-1} & -c^{-1}xd^{-1} \\ sd^{-1}yc^{-1} & d^{-1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where $a = (1-sxd^{-1}yc^{-1})^{-1}$ and $b = (1-syc^{-1}xd^{-1})^{-1}$, as desired.

**Lemma 2.3** Let $R$ be a ring, and let $s \in C(R) \cap J(R)$. Then $J(L_{(s)}(R)) = \left\{ \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \in M_2(R) \mid c, d \in J(R), x, y \in R \right\}$.
Proof As is well known, \( r \in J(R) \) if and only if \( 1 - rx \in U(R) \) for any \( x \in R \). We easily obtain the result by Lemma 2.2.

Lemma 2.4 Let \( R \) be a ring, and let \( e, f \in R \) be idempotents. Then the following are equivalent:

1. There exist \( u, v \in U(R) \) such that \( uev = f \).
2. \( e \) is similar to \( f \), i.e. \( w^{-1}ew = f \) for some \( w \in U(R) \).

Proof (1) \( \Rightarrow \) (2) By hypothesis, there exist \( u, v \in U(R) \) such that \( uev = f \). Let \( w = u^{-1}(-1 + f + ueu^{-1}) \). Then \( w^{-1} = (-1 + f + ueu^{-1})u \). Therefore, \( f w^{-1} = w^{-1}e \), and so \( w^{-1}ew = f \).

(2) \( \Leftarrow \) (1) This is trivial.

Lemma 2.5 Let \( R \) be a local ring, let \( s \in C(R) \cap J(R) \), and let \( E^2 = E \in L(s)(R) \). Then \( E = 0 \), \( E = I_2 \), or \( E \) is similar to \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

Proof Assume that \( E = \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \neq 0 \) and \( E \neq I_2 \). Then \( c \) or \( d \) is invertible.

Case I. \( c \in U(R) \). Then

\[
\begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} E \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}.
\]

This implies that \( \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix} \in L(s)(R) \) is regular, and then so is \( d - syc^{-1}x \in R \). As \( R \) is local, we easily check that \( d - syc^{-1}x \) is zero or invertible. If \( d - syc^{-1}x = 0 \), then \( P_1EQ_1 \) is an idempotent diagonal matrix where \( P_1 = \begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} \in U(L(s)(R)) \) and \( Q_1 = \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} \in U(L(s)(R)) \). If \( d - syc^{-1}x \) is invertible, then there exist

\[
P_2 = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -syc^{-1} & 1 \end{pmatrix} \in U(L(s)(R))
\]

and

\[
Q_2 = \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} \in U(L(s)(R))
\]

such that \( P_2EQ_2 \) is an idempotent diagonal matrix. In light of Lemma 2.4, \( E \) is similar to a diagonal matrix.

Case II. \( d \in U(R) \). Then

\[
\begin{pmatrix} 1 & -sxd^{-1} \\ 0 & 1 \end{pmatrix} E \begin{pmatrix} 1 & 0 \\ -sxd^{-1}y & d^{-1} \end{pmatrix} = \begin{pmatrix} c - sxd^{-1}y & 0 \\ 0 & 1 \end{pmatrix}.
\]

This implies that \( \begin{pmatrix} c - sxd^{-1}y & 0 \\ 0 & 1 \end{pmatrix} \in L(s)(R) \) is regular, and then so is \( c - sxd^{-1}y \in R \). Hence, \( c - sxd^{-1}y \) is zero or invertible. Thus, similar to Case I, we have \( P, Q \in U(L(s)(R)) \) such that \( PEQ \) is an idempotent diagonal matrix. In light of Lemma 2.4, \( E \) is similar to a diagonal matrix.
Write \( P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \) for some \( P \in U(L(s)(R)) \). We may assume that \( e = 1, f = 0 \) or \( e = 0, f = 1 \). This completes the proof. \( \square \)

**Theorem 2.6** Let \( R \) be a local ring, and let \( s \in C(R) \). Then \( A \in L(s)(R) \) is strongly clean if and only if

1. \( A \) or \( I_2 - A \) is invertible; or

2. \( A \) is similar to a diagonal matrix in \( L(s)(R) \).

**Proof** Since \( R \) is local, we see that \( s \in U(R) \) or \( s \in J(R) \).

Case I. \( s \in U(R) \). By virtue of Proposition 2.1, \( L(s)(R) \cong M_2(R) \). Then the result follows from [9].

Case II. \( s \in J(R) \).

\( \Rightarrow \) If \( A \) or \( I_2 - A \) is invertible, then \( A \in L(s)(R) \) is strongly clean. Suppose that \( A \) is similar to a diagonal matrix in \( L(s)(R) \). Then there exists \( U \in U(L(s)(R)) \) such that \( U^{-1}AU = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \). As \( R \) is local, it is strongly clean. Hence, we can find idempotents \( e, f \in R \) such that \( a - e, b - f \in U(R) \), \( ea = ae \), and \( bf = fb \). Set \( E = U \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} U^{-1} \). Then \( E^2 = E \in L(s)(R) \) and \( EA = AE \). Furthermore, \( A - E \in U(L(s)(R)) \), as desired.

\( \Rightarrow \) Suppose that \( A \) and \( I_2 - A \) are not invertible. Write \( A = E + U \) with \( EA = AE, E^2 = E, U \in U(L(s)(R)) \). Set \( E = \begin{pmatrix} c & x \\ sy & d \end{pmatrix} \). In light of Lemma 2.5, \( E \) is similar to a diagonal matrix.

Write \( P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \) for some \( P \in U(L(s)(R)) \). We may assume that \( e = 1, f = 0 \) or \( e = 0, f = 1 \). If \( e = 1, f = 0 \), then

\[ P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + P^{-1}UP \]

and

\[ P^{-1}AP \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}AP. \]

This forces that \( P^{-1}AP \) is diagonal. If \( e = 0, f = 1 \), we prove that \( A \) is similar to a diagonal matrix in a similar way. This completes the proof. \( \square \)

3. Quasipolar and pseudopolar matrices

The aim of this section is to extend Theorem 2.6 to quasipolar and pseudopolar matrices over a local ring. The following lemma is crucial.

**Lemma 3.1** Let \( R \) be a local ring, and \( s \in C(R) \), and let \( a, b \in R \). Then all matrices that commute with

\( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) or \( \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \) must be diagonal in \( L(s)(R) \) if and only if both \( l_a - r_b \) and \( r_a - l_b \) are injective.
Proof $\implies$ Assume that $ax = xb$. Then
\[
\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} \in \text{comm}\left(\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix}\right).
\]
Hence, $x = 0$. Thus, $l_a - r_b : R \to R$ is injective. Assume that $bx = xa$. Then
\[
\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} \in \text{comm}\left(\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix}\right);
\]
hence, $x = 0$. Thus, $l_b - r_a : R \to R$ is injective.

$\implies$ This is clear by [3, Lemma 3.4], as $L_{(s)}(R)$ is a subring of $M_2(R)$.

\begin{theorem}
Let $R$ be a local ring, and let $s \in C(R)$. Then $A \in L_{(s)}(R)$ is quasipolar if and only if
\begin{enumerate}
\item $A \in GL_2(R)$; or
\item $A \in L_{(s)}(R)^{\text{qnil}}$; or
\item $A$ is similar to a diagonal matrix $\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}$ in $L_{(s)}(R)$, where $\lambda \in J(R)$, $\mu \in U(R)$ or $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective.
\end{enumerate}
\end{theorem}

Proof We may assume that $s \in U(R)$ or $s \in J(R)$.

Case I. $s \in U(R)$. By virtue of Proposition 2.1, $L_{(s)}(R) \cong M_2(R)$. Hence, the result follows from [4].

Case II. $s \in J(R)$.

$\implies$ Since $A \in L_{(s)}(R)$ is quasipolar. Write $A + E = U$ with $E^2 = E \in \text{comm}^2(A)$, $U \in U(L_{(s)}(R))$ and $AE \in L_{(s)}(R)^{\text{qnil}}$. In view of Lemma 2.5, $E = 0$, $E = I_2$, or $E$ is similar to $\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}$ or $\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}$.

Case 1. $E = 0$. Then $A \in GL_2(R)$.

Case 2. $E = I_2$. Then $A \in L_{(s)}(R)^{\text{qnil}}$.

Case 3. $E$ is similar to $\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}$. Write $P^{-1}EP = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}$ where $P \in U(L_{(s)}(R))$. Then
\[
P^{-1}AP + \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} = P^{-1}UP
\]
and
\[
P^{-1}AP \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} P^{-1}AP.
\]

As in the proof of Theorem 2.6, $P^{-1}AP$ is a diagonal matrix $\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}$ where $\lambda \in J(R)$, $\mu \in U(R)$. Let $x \in R$ such that $\lambda x = x\mu$. Then
\[
\begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} \in \text{comm}(P^{-1}AP), \text{ and so } \begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} \in \text{comm}\left(\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}\right).
\]
Therefore, $x = 0$, and so $l_\lambda - r_\mu$ is injective. Likewise, $l_\mu - r_\lambda$ is injective.
Case 4. $E$ is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in $L_{(s)}(R)$ where $\lambda \in U(R)$, $\mu \in J(R)$. The rest is also similar to the Case 3.

$\iff$ Case 1. $A \in GL_2(R)$. Then $A$ is quasipolar.

Case 2. $A \in L_{(s)}(R)^{qnil}$. Then $A$ is quasipolar with the spectral idempotent $I_2$.

Case 3. $A$ is similar to a diagonal matrix $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right)$, where $\lambda \in J(R)$, $\mu \in U(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Then

$$\left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} \lambda + 1 & 0 \\ 0 & \mu \end{array} \right) \in U(L_{(s)}(R)).$$

Let $\left( \begin{array}{cc} x & q \\ sp & y \end{array} \right) \in \mathsf{comm} \left( \left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right) \right)$. In view of Lemma 3.1, $sp = q = 0$. Then $\left( \begin{array}{cc} x & q \\ sp & y \end{array} \right) \in \mathsf{comm} \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right)$. Therefore,

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \in \mathsf{comm}^2 \left( \left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right) \right).$$

Furthermore, $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \in L_{(s)}(R)^{qnil}$ and so $A$ is quasipolar.

Case 4. $A$ is similar to a diagonal matrix $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right)$, where $\lambda \in U(R)$, $\mu \in J(R)$, and $l_\mu - r_\lambda$, $l_\lambda - r_\mu$ are injective. Then similar to Case 3, $A$ is quasipolar with the spectral idempotent $\left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$. This completes the proof. $\Box$

**Corollary 3.3** Let $R$ be a commutative local ring, and let $s \in R$. Then $A \in L_{(s)}(R)$ is quasipolar if and only if

(1) $A \in GL_2(R)$; or

(2) $A^2 \in J(L_{(s)}(R))$; or

(3) $A$ is similar to a diagonal matrix in $L_{(s)}(R)$.

**Proof** \[ \implies \] This is obvious by Theorem 3.2.

$\iff$ If $A \in GL_2(R)$ or $A^2 \in J(L_{(s)}(R))$, then $A$ is quasipolar by Theorem 3.2. Suppose that $A$ is similar to a diagonal matrix $\left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right)$. If $\alpha, \beta \in U(R)$, then $A \in U(L_{(s)}(R))$ and so $A$ is quasipolar. If $\alpha, \beta \in J(R)$, then $A \in J(L_{(s)}(R))$ and so $A$ is quasipolar. Otherwise, $A$ is similar to $\left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right)$, where $\alpha \in U(R)$, $\beta \in J(R)$ or $\alpha \in J(R)$, $\beta \in U(R)$. According to Theorem 3.2, $A \in L_{(s)}(R)$ is quasipolar, as asserted. $\Box$

**Theorem 3.4** Let $R$ be a local ring, and let $s \in C(R)$. Then $A \in L_{(s)}(R)$ is pseudopolar if and only if
(1) \( A \in GL_2(R) \) or \( A^2 \in J(L(s)(R)) \); or

(2) \( A \) is similar to a diagonal matrix \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) in \( L(s)(R) \), where \( \lambda \in J(R) \), \( \mu \in U(R) \) or \( \lambda \in U(R) \), \( \mu \in J(R) \), and \( l_\mu - r_\lambda, l_\lambda - r_\mu \) are injective.

**Proof** Let \( A \) be pseudopolar. It is clear that \( A \in GL_2(R) \) if and only if \( A \) is pseudopolar with an idempotent \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). Also, \( A^2 \in J(L(s)(R)) \) if and only if \( A \) is pseudopolar with an idempotent \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Hence, we may assume \( A \notin GL_2(R) \) or \( A^2 \notin J(L(s)(R)) \). Since \( A \) is pseudopolar, there exists \( P^2 = P \in L(s)(R) \) such that \( P \in \text{comm}^2(A) \), \( A + P \in U(L(s)(R)) \), and for some \( k \geq 1 \), \( A^k P \in J(L(s)(R)) \). Then by Lemma 2.5, there exists \( V \in U(L(s)(R)) \) such that \( V^{-1} PV = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \). We may take \( e = 1, f = 0 \) or \( e = 0, f = 1 \). If \( e = 1 \) and \( f = 0 \), since \( AP = PA \), \( A \) is similar to \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) where \( \lambda \in J(R) \), \( \mu \in U(R) \). Thus, \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) is pseudopolar since \( A \) is pseudopolar. Hence, the strongly spectral idempotent of \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) is \( Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

To see that \( l_\lambda - r_\mu \) is injective, let \( (l_\lambda - r_\mu)(x) = 0 \). Then

\[
\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.
\]

Since \( Q \in \text{comm}^2 \left( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right) \), we have \( x = 0 \) as asserted. If \( (l_\mu - r_\lambda)(y) = 0 \), then for \( B = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \),

\[
B \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} B.
\]

Hence, similarly, \( l_\mu - r_\lambda \) is injective. If \( e = 0 \) and \( f = 1 \), then \( A \) is similar to \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) where \( \lambda \in U(R) \) and \( \mu \in J(R) \). Furthermore, \( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \) is pseudopolar with the strongly spectral idempotent \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Hence, \( l_\mu - r_\lambda \) and \( l_\lambda - r_\mu \) are injective. By Lemma 3.1, the converse is obvious. \( \square \)

**Corollary 3.5** Let \( R \) be a commutative local ring, and let \( s \in R \). Then \( A \in L(s)(R) \) is quasipolar if and only if it is pseudopolar.

**Proof** \( \iff \) This is obvious by Theorem 3.2 and Theorem 3.4, as \( A^2 \in J(L(s)(R)) \) implies that \( A \in L(s)(R)^{nil} \).

\( \implies \) If \( s \in U(R) \), we obtain the result by [3] and Proposition 2.1. We may assume that \( s \in J(R) \).

Write \( A = \begin{pmatrix} x & y \\ sp & y \end{pmatrix} \in L(s)(R)^{nil} \). Then \( xy = \text{det}(A) + spq \in J(R) \). Suppose \( x + y \in U(R) \). Choose \( Y = \begin{pmatrix} -(x+y)^{-1} & 0 \\ 0 & -(x+y)^{-1} \end{pmatrix} \). Then \( Y \in \text{comm}(A) \). Hence, \( I_2 + AY \in U(L(s)(R)) \). This shows that \( 1 - x(x+y)^{-1}, 1 - y(x+y)^{-1} \in U(R) \). Thus, \( x, y \in U(R) \), a contradiction. Therefore, \( x + y \in J(R) \). Regarding \( A \) as a matrix in \( M_2(R) \), by the Cayley–Hamilton theorem, \( A^2 = \text{tr}(A)A - \text{det}(A)I_2 \in M_2(J(R)) \). Moreover, we have \( A^2 \in J(L(s)(R)) \). Therefore, we complete the proof by Theorem 3.4 and Theorem 3.2. \( \square \)
Let $R$ be a commutative local ring, let $s \in R$, and let $A \in L_{(s)}(R)$. Evidently, $A \in L_{(s)}(R)^{qnil}$ if and only if $A \in M_2(R)^{qnil}$ if and only if $A^2 \in M_2(J(R))$.

Example 3.6 Let $R = \mathbb{Z}_4$. Then we have

$$
\begin{pmatrix}
1 & 3 \\
1 & 2 \\
\end{pmatrix}^{-1} \begin{pmatrix}
3 & 3 \\
2 & 0 \\
\end{pmatrix} \begin{pmatrix}
1 & 3 \\
1 & 2 \\
\end{pmatrix} = \begin{pmatrix}
2 & 0 \\
0 & 1 \\
\end{pmatrix}.
$$

Hence, $\begin{pmatrix}
3 & 3 \\
2 & 0 \\
\end{pmatrix}$ is isomorphic to the diagonal matrix $\begin{pmatrix}
2 & 0 \\
0 & 1 \\
\end{pmatrix}$ in $M_2(\mathbb{Z}_4)$. However, $\begin{pmatrix}
3 & 3 \\
2 & 0 \\
\end{pmatrix}$ is not isomorphic to the diagonal matrix $\begin{pmatrix}
2 & 0 \\
0 & 1 \\
\end{pmatrix}$ in $L_2(\mathbb{Z}_4)$. Otherwise, we can find some $p, q \in \mathbb{Z}_4$ such that

$$
\begin{pmatrix}
3 & 3 \\
2 & 0 \\
\end{pmatrix} \begin{pmatrix}
x & q \\
2p & y \\
\end{pmatrix} = \begin{pmatrix}
x & q \\
2p & y \\
\end{pmatrix} \begin{pmatrix}
2 & 0 \\
0 & 1 \\
\end{pmatrix},
$$

where $x, y$ are $-1$ or $1$. Thus, $2x = 0$, and so $2 = 0$, which is absurd. In this case, $l_2 - r_1$ and $l_1 - r_2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ are injective.

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