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Integral representation for solutions of the pseudoparabolic equation in matrix form

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Abstract: In this paper, an integral representation is given for special bounded solutions of pseudoparabolic equations of the form

$$Lw := \frac{\partial}{\partial t}\left(w_{\overline{\phi}} + aw + b\overline{w}\right) + cw + d\overline{w}$$

by means of a generating pair of the corresponding class of the generalized Q-holomorphic functions in $L_{p,2}(\mathbb{C})$, for $p > 2$, where a, b, c, d are functions of z alone.

Key words: Cauchy-type integral representation, generalized Beltrami systems

1. Introduction

An analogue of analytic function theory was developed by Douglis [4] for a more general elliptic system in the plane while the Cauchy–Riemann equations assume the complex form

$$w_x + iw_y + aEw_x + bEw_y = 0.$$

Later Bojarskiĭ [3] extended the function theory of Douglis to a more general system, which was written in following form:

$$Dw(z) := w_{\overline{z}}(z) - Q(z)w_z(z) = 0, \tag{1.1}$$

where w is an $m \times 1$ vector and Q is an $m \times m$ quasidiagonal matrix. He also presumed that all of eigenvalues of Q are less than 1. Subsequently, Hile [6] took into consideration Eq. (1.1), taking Q as an $m \times m$ complex matrix and w as an $m \times s$ complex matrix. If $Q(z)$ satisfies the property

$$Q(z_1)\, Q(z_2) = Q(z_2)\, Q(z_1)$$

for any two points z_1, z_2 in the domain Ω_0 of Q, we say that $Q(z)$ is self-commuting. More generally, if A and B are matrix valued functions defined in Ω_0 and satisfy the condition

$$A(z_1)\, B(z_2) = B(z_2)\, A(z_1)\,, \text{ for all } z_1, z_2 \text{ in } \Omega_0,$$

then we say that A and B commute in Ω_0. If $Q(z)$ is self-commuting and has no eigenvalues of magnitude 1 for each z in Ω_0, then Hile called the system (1.1) a generalized Beltrami system and the solutions of such a system

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were called Q-holomorphic functions, which have properties similar to those of the complex analytic functions. Following Douglis and Bojarskiĭ, Hile introduced the concept of the generating solution for the investigated Q-holomorphic functions. He referred to the matrix valued function $\phi(z) := \phi_0(z) + N(z)$ defined in a domain Ω_0 for a generating solution of (1.1), where N is the nilpotent part of ϕ and ϕ_0 is the main diagonal term of ϕ satisfying the Beltrami equation

$$\frac{\partial \phi_0}{\partial \overline{z}} - \mu \frac{\partial \phi_0}{\partial z} = 0.$$

Moreover, if a function $\Phi(z)$ satisfies the equation $D\Phi(z) = 0$, then it is called a Q-holomorphic function and may be written merely as an analytic function of a generating solution, namely $\Phi(z) \equiv f(\phi(z))$ [6]. Hence, this relation proposes defining differentiation formally with respect to ϕ as

$$\frac{\partial}{\partial \phi} = \left(\phi_z \overline{\phi}_{\overline{z}} - \phi_{\overline{z}} \overline{\phi}_z\right)^{-1} \left[\overline{\phi}_z \frac{\partial}{\partial \overline{z}} - \overline{\phi}_{\overline{z}} \frac{\partial}{\partial z}\right]$$

and differentiation with respect to the conjugate of ϕ as

$$\frac{\partial}{\partial \overline{\phi}} = \left(\phi_z \overline{\phi}_{\overline{z}} - \phi_{\overline{z}} \overline{\phi}_z\right)^{-1} \phi_z D.$$

From the above equation, we may write (1.1) in the following form:

$$\frac{\partial w}{\partial \overline{\phi}} = 0.$$

Later in [8, 9], by means of the techniques of Vekua and Bers, a theory of functions was given for the equation

$$\frac{\partial w}{\partial \overline{\phi}} + aw + b\overline{w} = 0, \tag{1.2}$$

where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix, $Q(z) = \{q_{ij}(z)\}$ is a self-commuting complex matrix with dimensions $m \times m$, and $q_{k,k-1} \neq 0$ for $k = 2, \dots m$. The matrices $a = \{a_{ij}(z)\}$ and $b = \{b_{ij}(z)\}$, which belong to $L_p(\Omega_0)$, commute with Q. Solutions of such an equation were called *generalized Q-holomorphic functions*. The results obtained in the case of equation (1.2) closely resemble those in the classical theory of Vekua [10] and Bers [2].

Integral representations of analytic functions appeared in the early phases of the development of mathematical analysis and function theory mainly as proper devices for the explicit representation of analytic solutions of differential equations. Gilbert and Schneider investigated pseudoparabolic equations in [5] by making use of the generalized analytic function theory of Vekua [10], which have the following form:

$$Lw := \frac{\partial}{\partial t}[w_{\bar{z}} + \tilde{A}w + \tilde{B}\bar{w}] + \tilde{C}w + \tilde{D}\bar{w},$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are functions of z only and belong to $L_{p,2}(\mathbb{C})$, $p > 2$. $L_{p,2}(\mathbb{C})$ is the space of functions w that satisfies the following conditions: $w(z)$ is defined in $\mathbb{C}$ and

$$w(z) \in L_p(\mathbb{C}_0), \; w^{(2)}(z) := |z|^{-2} w(1/z) \in L_p(\mathbb{C}_0).$$

Here, $\mathbb{C}_0 := \{z : |z| \leq 1\}$. The validity of integral representations reminiscent of representations of generalized analytic functions was shown. This helps in solving initial-boundary value problems.

In this work, by utilizing a generating pair of the corresponding class of generalized Q-holomorphic functions, the general solution is given for a pseudoparabolic equation of the form

$$L[w] := E[w_t] + A[w] = 0, \tag{1.3}$$

where E is an elliptic operator

$$E[w] := w_{\overline{\phi}}(z,t) + a(z)w(z,t) + b(z)\overline{w(z,t)}$$

and A is an algebraic operator that has the following form:

$$A[w] := c(z)w(z,t) + d(z)\overline{w(z,t)},$$

where the unknown $w(z,t) = \{w_{ij}(z,t)\}$ is an $m \times s$ complex matrix. The coefficients a, b, c, and d are $m \times m$ complex matrices commuting with Q and do not depend on the time variable t. They vanish identically in the unbounded component of $\mathbb{C}\backslash\hat{D}$ where $D \subset \mathbb{C}$ is a bounded domain.

2. Representations of solutions via fundamental solutions

By means of the Pompeiu operator (see [8], p. 433), the pseudoparabolic equation (1.3) may be formulated as the integral equation

$$\begin{aligned} &w(z,t) + J\left(a(z)\,w(z,t) + b(z)\,\overline{w(z,t)}\right) + J\left(\int_0^t \left(c(z)\,w(z,\tau) + d(z)\,\overline{w(z,\tau)}\right)d\tau\right) \\ &= w(z,0) + J\left(a(z)\,w(z,0) + b(z)\,\overline{w(z,0)}\right) + \Psi(z,t) \end{aligned} \tag{2.1}$$

where

$$(JF)(z) = P^{-1}\int_{\mathbb{C}} d\phi(\zeta)\overline{d\phi(\zeta)}\left(\phi(\zeta) - \phi(z)\right)^{-1}F(\zeta)$$

and $\frac{\partial^2\Psi}{\partial\overline{\phi}\partial t} \equiv 0$, where the constant matrix P is defined by

$$P = \int_{|z|=1}(zI + \overline{z}Q)^{-1}(Idz + Qd\overline{z}),$$

called the P-value for the generalized Beltrami system ([6], p. 107). Here

$$\Psi(z,t) = \sum_{k=0}^{\infty}\phi^k(z)\,a_k(t) \quad (z \in \mathbb{C}, t \in \mathbb{R}), \quad \Psi(z,0) = 0,$$

and $\Psi(z,t)$ is a differentiable function of t for an $m \times s$ matrix function a_k of t. A Q-holomorphic version of Liouville's theorem can be proven in the same way as in the complex case as we know that

$$\frac{d^n w(z)}{d\phi^n} = n!P^{-1}\int_{|\zeta - z|=r} d\phi(\zeta)(\phi(\zeta) - \phi(z))^{-n-1}w(\zeta)$$

for all n and for any $r > 0$ ([6], p. 115).

For each $t \in \mathbb{R}$, $\Psi(z,t)$ is a bounded Q-holomorphic function in $\mathbb{C}$ provided that $w(z,t)$ is a bounded function in $\mathbb{C}$. Thus, $\Psi(z,t)$ is a function of only t. This means that $\Psi(z,t) = \Psi(t)$ by the help of Liouville's theorem. In Eq. (2.1), when z goes to ∞, we easily obtain

$$\Psi(t) := w(\infty, t) - w(\infty, 0), \text{ i.e. } \Psi(0) = 0.$$

Therefore, (1.3) is equivalent to the following equation:

$$w(z,t) + J\left(a(z)w(z,t) + b(z)\overline{w(z,t)}\right) = -J\left(\int_0^t \left(c(z)w(z,\tau) + d(z)\overline{w(z,\tau)}\right) d\tau\right) + \Psi(t) + \varphi(z) \tag{2.2}$$

for bounded solutions. Here

$$\varphi(z) := w(z,0) + J\left(a(z)\,w(z,0) + b(z)\,\overline{w(z,0)}\right). \tag{2.3}$$

If w is a continuous solution of (2.2), which is bounded in $z \in \mathbb{C}$ for each $t \in \mathbb{R}$, then

$$\Psi \in C(\mathbb{R}), \quad \varphi \in C(\mathbb{C}).$$

Now we consider (2.2) for given $\Psi \in C(\mathbb{R})$ and $\varphi \in C(\mathbb{C})$ and show that the space of bounded functions in $z \in \mathbb{C}$ for each $t \in \mathbb{R}$ is $B_{\mathbb{C}}(\mathbb{C} \times \mathbb{R})$.

Lemma 2.1 *Let $\Psi \in C(\mathbb{R})$, $\varphi \in C(\mathbb{C})$, and the coefficients a, b satisfy the inequality*

$$\int_{\mathbb{C}} (\|a(\zeta)\| + \|b(\zeta)\|) \frac{d\xi d\eta}{|\zeta - z|} \leq \alpha < 1. \tag{2.4}$$

Then equation (2.2) has a unique solution in $B_{\mathbb{C}}(\mathbb{C} \times \mathbb{R})$, where the standard norm of a matrix $M = (m_{ij})$ is given by

$$\|M\|^2 = \sum_{i,j} |m_{ij}|^2.$$

Proof Suppose that two solutions of Eq. (2.2) are w_1 and w_2, which have the same initial data and the same asymptotic behavior when $z \to \infty$. In that case, the difference $\omega := w_1 - w_2$ is a solution of the homogeneous equation

$$\omega = \mathbb{T}\omega := -J\left(a(z)\omega(z,t) + b(z)\overline{\omega(z,t)}\right) - J\left(\int_0^t \left(c(z)w(z,\tau) + d(z)\overline{w(z,\tau)}\right) d\tau\right).$$

In addition to the inequality provided by the coefficients a and b, we find an upper bound of

$$\int_{\mathbb{C}} (\|c(\zeta)\| + \|d(\zeta)\|) \frac{d\xi d\eta}{|\zeta - z|}$$

in terms of β and if we define

$$\|\omega\|_1 := \sup_{z\in\mathbb{C},|t|\le 1} \|\omega(z,t)\|,$$

then after some simple calculations, we get

$$\|\mathbb{T}\omega\|_1 \le (\alpha + \beta|t|)\|\omega\|_1.$$

Thus, as

$$|t| < \min\{1, \frac{1-\alpha}{\beta}\},$$

we conclude that $\mathbb{T}$ is a contractive operator. Therefore, $\omega = \mathbb{T}\omega$ has only the trivial solution

$$\omega(z,t) \equiv 0 \quad for \quad z \in \mathbb{C}, \quad |t| \le t_0 := \min\{1, \frac{1-\alpha}{\beta}\}.$$

The above statement is true as well for $|t| = t_0$ by virtue of continuity. This conclusion can be extended to read

$$\omega(z,t) \equiv 0 \quad (z \in \mathbb{C},\ t \in \mathbb{R})$$

since the equation (1.3) is an autonomous differential equation with respect to t. □

Let us denote the class of functions having real derivatives up to order p, which are continuous and bounded with $B^p(\mathbb{C})$.

Corollary 2.2 *Let $w(\infty,t) \in C(\mathbb{R})$, and $w(z,0) \in B^0(\mathbb{C})$. If the inequality (2.4) is fulfilled, then equation (1.3) has a unique solution in $B_{\mathbb{C}}(\mathbb{C}\times\mathbb{R})$.*

Lemma 2.3 *Let $\Psi \in C^1(\mathbb{R})$, $\varphi \in B^1(\mathbb{C})$, and let the inequality (2.4) hold. Then integral equation (2.2) is solvable.*

Proof This lemma will be proved using iteration. We can rewrite (2.2) in the following form:

$$w - \mathbb{T}w = \Psi + \varphi.$$

To solve this equation, by the help of iteration method, let us put

$$\begin{aligned} w_0 &:= \Psi + \varphi \\ w_k &= \Psi + \varphi + \mathbb{T}w_{k-1}, \quad k \in \mathbb{N}_0. \end{aligned}$$

Then the solution of (2.2) can be obtained as

$$w = \lim_{k\to\infty} w_k = \sum_{k=0}^{\infty} \mathbb{T}^k (\Psi + \varphi).$$

The series converges because of the estimates

$$\begin{aligned} \left\|\left(\mathbb{T}^k w\right)(z,t)\right\| &\le \sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l}\beta^l \frac{|t|^l}{l!} \|w_0\|_t, \\ (\ \|w_0\|_t &= \sup_{z\in\mathbb{C},|\tau|\le|t|} \|w_0(z,\tau)\|\ ), \end{aligned}$$

and

$$\|w(z,\tau)\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\beta |t|}{1-\alpha} \right)^k \frac{\|w_0\|_t}{1-\alpha}, (z \in \mathbb{C}, t \in \mathbb{R}).$$

□

Next we deal with special solutions of (1.3) that are bounded. If the function φ given by (2.3) is a bounded Q-holomorphic function in $\mathbb{C}$, then φ must be a constant, namely

$$\varphi(z) \equiv w(\infty, 0),$$

so that

$$\Psi(t) + \varphi(z) \equiv w(\infty, t).$$

Therefore, we can consider the equation

$$w - \mathbb{T}w = \psi, \tag{2.5}$$

where ψ is a differentiable function of t in $\mathbb{R}$.

(i) ψ is a real matrix function: the unique solution of (2.5) is represented by

$$\sum_{k=0}^{\infty} \mathbb{T}^k \psi = \sum_{k=0}^{\infty} \alpha_k \psi_k$$

where α_k is an $m \times m$ complex matrix valued function of z only, and ψ_k are defined by means of

$$\psi_0 := \psi, \ \psi_k(t) := \int_0^t \psi_{k-1}(\tau) \, d\tau \quad (k \in \mathbb{N}).$$

The functions α_k in solutions of (2.5) are determined using the coefficient functions $a,\ b,\ c,\ d$ of the operator L. They do not depend on ψ. Since $a,\ b,\ c,\ d \in L_{p,2}(\mathbb{C})$, $p > 2$, J is a compact operator (see [8], p. 445), and the following recursive system can be solved for $k = 1,\ 2,\ \ldots$:

$$\begin{cases} \alpha_0(z) + J\left(a(z)\alpha_0(z) + b(z)\overline{\alpha_0(z)}\right) = I \\ \alpha_k(z) + J\left(a(z)\alpha_k(z) + b(z)\overline{\alpha_k(z)}\right) = -J\left(c(z)\alpha_{k-1}(z) + d(z)\overline{\alpha_{k-1}(z)}\right). \end{cases} \tag{2.6}$$

Let (F, G) be the generating pair corresponding to the functions (a, b), and then α_0 has to be the function F on the left ([9], p. 944), and α_ks, denoted by F_k, are obtained recursively by solving the system. For a given real matrix ψ, it can be seen that the solution of (2.5) is given by the following:

$$F_0 := F, \quad (\mathbb{F}\psi)(z,t) := \sum_{k=0}^{\infty} F_k(z)\psi_k(t) \quad (z \in \mathbb{C}, t \in \mathbb{R}). \tag{2.7}$$

(ii) ψ is a pure imaginary matrix function: this case is similar to the case (i). A solution of (2.5) can be obtained as

$$G_0 := G, \quad (\mathbb{G}\psi)(z,t) := \sum_{k=0}^{\infty} G_k(z)\psi_k(t), \quad (z \in \mathbb{C}, t \in \mathbb{R}) \tag{2.8}$$

where G is the function on the right-hand side of the generating pair (F, G) and the G_ks are the solutions of the following system:

$$\begin{cases} G_0(z) + J\left(a(z)\,G_0(z) + b(z)\,\overline{G_0(z)}\right) = iI \\ G_k(z) + J\left(a(z)\,G_k(z) + b(z)\,\overline{G_k(z)}\right) = -J\left(c(z)\,G_{k-1}(z) + d(z)\,\overline{G_{k-1}(z)}\right). \end{cases} \tag{2.9}$$

The coefficients F_k and G_k, which appeared in (2.7) and (2.8), respectively, are normalized at ∞ by

$$F_0(\infty) = I, \quad G_0(\infty) = iI, \quad F_k(\infty) = 0, \quad G_k(\infty) = 0, \quad (k \in \mathbb{N}).$$

(iii) $\psi = \psi_1 + i\psi_2$ is a complex matrix function: let w be a solution of

$$w - \mathbb{T}w = \psi_1 + i\psi_2$$

where ψ_1 and ψ_2 are real, differentiable functions of t. Then

$$\tilde{w} := w - \mathbb{F}\psi_1 - \mathbb{G}\psi_2$$

is a solution of the homogeneous equation

$$\tilde{w} - \mathbb{T}\tilde{w} = 0.$$

This is evident from the iterated integration of ψ at initial data

$$(\mathbb{F}\psi_1 + \mathbb{G}\psi_2)(z, 0) = F_0(z)\,\psi_1(0) + G_0(z)\,\psi_2(0)$$

and because of the properties of F_k and G_k, $k \in \mathbb{N}_0$, we have

$$(\mathbb{F}\psi_1 + \mathbb{G}\psi_2)(\infty, t) = \psi_1(t) + i\psi_2(t) = \psi(t)$$

when $z \to \infty$. By using the first equations in (2.6) and (2.9) and the corresponding function φ given by (2.3), we have

$$\varphi(z) \equiv \psi(0).$$

As a conclusion, $\tilde{w}$ is identically zero. Hence, we have the following representation for the required solution in the case of ψ being a complex matrix function as

$$w = \mathbb{F}\psi_1 + \mathbb{G}\psi_2.$$

Let Γ be a rectifiable boundary of bounded domain D and denote the closure of D with $\hat{D}$.

Remark 2.4 *We introduce a fundamental system $\chi^{(k)}(z, t; \zeta, \tau)$ commuting with Q as a pair of solutions of* (1.3)*. This system has the power series representation*

$$\chi^{(k)}(z, t; \zeta, \tau) = \sum_{v=0}^{\infty} \chi_v^{(k)}(z, \zeta)\,\frac{(t-\tau)^{v+1}}{(v+1)!}, \quad k = 1, 2, \tag{2.10}$$

such that the pair $\left\{\chi_0^{(1)}(z,\zeta), \chi_0^{(2)}(z,\zeta)\right\}$ is a fundamental system for the equation $E[w] = 0$ obtained in a similar way to the Vekua system as in [1, 5, 10], where

$$\chi_0^{(k)}(z,\zeta) = \frac{1}{2}(-i)^{k-1}(\phi(\zeta) - \phi(z))^{-1}\exp[\omega^{(k)}(z) - \omega^{(k)}(\zeta)].$$

Here, $\chi_0^{(k)} \in B^\alpha$, $\alpha = (p-2)/p$, *and* $\omega^{(k)}(z) = O(|z|^\alpha)$ *as* $|z| \to \infty$, *k=1,2 (see [7]). Substituting this series into equation (1.3) and equating powers of* $(t-\tau)$, *we obtain the recursive system*

$$E[\chi_0^{(k)}(z,\zeta)] = 0$$
$$E[\chi_{v+1}^{(k)}(z,\zeta)] + A[\chi_v^{(k)}(z,\zeta)] = 0, (k = 1,2, v = 0,1,2,...).$$

From the system of integral equations obtained above, by utilizing the Pompeiu operator we can compute the following expression for $\chi_{v+1}^{(k)}$, $(v = 0,1,2,...)$*:*

$$\chi_{v+1}^{(k)}(z,\zeta) + J(a\chi_{v+1}^{(k)} + b\overline{\chi_{v+1}^{(k)}}) = -J(c\chi_v^{(k)} + d\overline{\chi_v^{(k)}}) + \Upsilon_{v+1}^{(k)}(z,\zeta), \quad v = 0,1,2,...,$$

where the $\Upsilon_{v+1}^{(k)}$ *are arbitrary Q-holomorphic functions. We normalize the* $\chi_{v+1}^{(k)}$ *by setting* $\Upsilon_{v+1}^{(k)} \equiv 0$.

Definition 2.5 *The fundamental kernels* $\Omega^{(k)}$ *of* L *are*

$$\Omega^{(k)}(z,t;\zeta,\tau) = \chi^{(1)}(z,t;\zeta,\tau) + (-1)^{k-1} i\chi^{(2)}(z,t;\zeta,\tau), \; k = 1,2, \tag{2.11}$$

where $\chi^{(1)}$ *and* $\chi^{(2)}$ *are the fundamental solutions. Also, from the properties of the matrix norm, the local behavior of* $\Omega^{(k)}$ *can be obtained similarly to [7] as*

$$\begin{cases} \left\|\Omega^{(1)}(z,t;\zeta,\tau) - (t-\tau)(\phi(\zeta) - \phi(z))^{-1}\right\| = O\left(|z-\zeta|^{-\frac{2}{p}}|t-\tau|\right), \\ for\ \zeta - z \to 0, t-\tau \to 0, \beta = \frac{2}{p} < 1, \\ \left\|\Omega^{(2)}(z,t;\zeta,\tau)\right\| = O\left(|z-\zeta|^{-\frac{2}{p}}|t-\tau|\right), (\ \zeta - z \to 0, t-\tau \to 0, \beta = \frac{2}{p} < 1), \\ \left\|\Omega^{(k)}(z,t;\zeta,\tau)\right\| = O\left(|z|^{-1}|t-\tau|\right), \quad (z \to \infty,\ k = 1,2). \end{cases} \tag{2.12}$$

Definition 2.6 *The associated operator, corresponding to operator* L *given in (1.3), is defined by*

$$\tilde{L}v = \frac{\partial}{\partial t}[v_{\bar{\phi}} - av - b^*\overline{bv}] + cv + b^*\overline{dv}$$

where b^* *is given by* $b^* = \phi_z^{-1}\overline{\phi_z}$.

Theorem 2.7 *Let* w *be a solution of (1.3) in* $\hat{D} \times \mathbb{R}$ *and have zero initial data* $w(z,0) \equiv 0$. *If* $\tilde{\Omega}^{(1)}$ *and* $\tilde{\Omega}^{(2)}$ *are the fundamental kernels for the associated equation* $\tilde{L}v = 0$, *then*

$$P^{-1}\int_\Gamma \left[d\phi(\zeta)\,\tilde{\Omega}_\tau^{(1)}(\zeta,\tau;z,t)\,w_\tau(\zeta,\tau) - \overline{d\phi(\zeta)\,\tilde{\Omega}_\tau^{(2)}(\zeta,\tau;z,t)\,w_\tau(\zeta,\tau)}\right]d\tau = \begin{cases} -w_t(z,t), z \in D \\ 0 \qquad\quad , z \notin \hat{D} \end{cases}, t \in \mathbb{R}.$$

Proof The proof is based on a variant of Morera's theorem. Let w and v be the solutions of (1.3) and associated equation $\tilde{L}v = 0$, respectively. If for $t = 0$, we have $w = 0$ and $v = 0$, then

$$Re\left(\frac{1}{2i}\int_\Gamma\int_0^t d\phi(\zeta)\,v_\tau(\zeta,\tau)\,w_\tau(\zeta,\tau)\,d\tau\right) = 0.$$

The remaining part of the proof can be obtained as in ([7], Theorem 2.7), so we omit the remaining part of the proof. □

Now, if we use the property between the fundamental kernels of (1.3) and the fundamental kernels of the associated equation, which can be shown by using Definition 2.5 above and Theorem 2.8 given in [7], we have

$$\Omega^{(1)}(z,t;\zeta,\tau) = -\tilde{\Omega}^{(1)}_{\tau}(\zeta,\tau;z,t),$$
$$\Omega^{(2)}(z,t;\zeta,\tau) = -\overline{\tilde{\Omega}^{(2)}_{\tau}}(\zeta,\tau;z,t).$$

Thus, we obtain the following theorem.

Theorem 2.8 *A solution of (1.3) in $\hat{D}\times\mathbb{R}$, which vanishes identically at initial data, has the following form:*

$$P^{-1}\int_0^t\int_\Gamma\left[d\phi(\zeta)\,\Omega^{(1)}_{\tau}(z,t;\zeta,\tau)\,w_\tau(\zeta,\tau) - \overline{d\phi(\zeta)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau)\,\overline{w_\tau(\zeta,\tau)}\right]d\tau = \begin{cases} w(z,t), & z\in D \\ 0 & ,z\notin\hat{D}\end{cases}, t\in\mathbb{R}.$$

Let us denote the Riemann sphere by $\hat{\mathbb{C}}$ and give the following theorem:

Theorem 2.9 *If $w(z,t)$ is Q-holomorphic in $\hat{\mathbb{C}}\backslash\hat{D}$ for each $t\in\mathbb{R}$, w vanishes identically at infinity, and $w_t(z,t)$ is continuous in $\mathbb{C}\backslash D\times\mathbb{R}, w(z,0)\equiv 0$, and $a=b=c=d=0$ in $\hat{\mathbb{C}}\backslash\hat{D}$, then*

$$P^{-1}\int_0^t\int_\Gamma\left[d\phi(\zeta)\,\Omega^{(1)}_{\tau}(z,t;\ \zeta,\ \tau)\,w_\tau(\zeta,\ \tau) - \overline{d\phi(\zeta)}\Omega^{(2)}_{\tau}(z,t;\ \zeta,\ \tau)\,\overline{w_\tau(\zeta,\tau)}\right]d\tau = \begin{cases} -w(z,t), & z\notin\hat{D} \\ 0 & ,\ z\in D\end{cases},\ t\in\mathbb{R}.$$

Proof Let $G_R := \{\zeta : |\zeta| < R\}$ such that $2|z| < R$ and $\hat{D}\subset G_R$. Let us take $z\in\mathbb{C}\backslash\hat{D}$, which belongs to G_R. By hypothesis, the coefficients a, b, c, d vanish outside $\hat{D}$. w is a solution of (1.3) in $(G_R\backslash\hat{D})\times\mathbb{R}$. Since the conditions of Theorem 2.8 are fulfilled, the solution w can be written as

$$w(z,t) = P^{-1}\int_0^t\int_{\partial(G_R-\hat{D})}\left\{d\phi(\zeta)\,\Omega^{(1)}_{\tau}(z,t;\zeta,\tau)\,w_\tau(\zeta,\tau) - \overline{d\phi(\zeta)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau)\,\overline{w_\tau(\zeta,\tau)}\right\}d\tau.$$

In the case of $z\to\infty$, $w(z,t) = O\left(|z|^{-1}\right)$, $w_t(z,t) = O\left(|z|^{-1}\right)$. If we consider the above integral in two parts, the part of the integral taken over $|\zeta| = R$, i.e. on the ∂G_R, tends to zero as asymptotic behavior of fundamental kernels (2.12) as R tends to infinity. If $z\in D$, then the left-hand side of the last equation has to be replaced by 0 by Theorem 2.8. □

Theorem 2.10 *Let $w(z,0)\equiv 0$ and the coefficients a, b, c, d are equal to zero outside $\hat{D}$. Then a continuous solution of (1.3) in $\hat{D}\times\mathbb{R}$ may be represented in the form*

$$w(z,t) = P^{-1}\int_0^t\int_\Gamma\left\{d\phi(\zeta)\,\Omega^{(1)}_{\tau}(z,t;\zeta,\tau)\,\Psi_\tau(\zeta,\tau) - \overline{d\phi(\zeta)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau)\,\overline{\Psi_\tau(\zeta,\tau)}\right\}d\tau, \tag{2.13}$$

where $z\in D$, $t\in\mathbb{R}$ and

$$\Psi(z,t) := P^{-1}\int_\Gamma d\phi(\zeta)\,(\phi(\zeta)-\phi(z))^{-1}\,w(\zeta,t) \tag{2.14}$$

is a Q-holomorphic function of z in D, continuous in $\hat{D}$, and a continuously differentiable function of t in $\mathbb{R}$.

Proof Let us rewrite the equation (2.1) in the subsequent form:

$$w = \mathbb{T}w + \Psi, \tag{2.15}$$

and take into consideration that $\mathbb{T}w$ is a Q-holomorphic function of z in $\hat{\mathbb{C}}\backslash\hat{D}$ and continuously differentiable with respect to t in $\mathbb{R}$. By the help of the following calculations,

$$(\mathbb{T}w)(\infty, t) \equiv 0, \quad (\mathbb{T}w)(z, 0) \equiv 0,$$

it is shown that the hypothesis of Theorem 2.9 is fulfilled and as a conclusion of this theorem we have

$$P^{-1}\int_0^t \int_\Gamma \Big\{ d\phi\,(\zeta)\,\Omega_\tau^{(1)}\,(z,t;\zeta,\tau)\,(\mathbb{T}w)_\tau\,(\zeta,\tau) \\ - d\overline{\phi\,(\zeta)}\Omega_\tau^{(2)}\,(z,t;\zeta,\tau)\,\overline{(\mathbb{T}w)_\tau\,(\zeta,\tau)}\Big\} d\tau = 0.$$

Ψ is continuous in $\hat{D}\times\mathbb{R}$ since w and $\mathbb{T}w$ are continuous functions there and furthermore Ψ is continuously differentiable with respect to $t\in\mathbb{R}$ and Q-holomorphic in $z\in D$.

Let us substitute $\mathbb{T}w+\Psi$ into w in (2.13) in Theorem 2.8 and use the last equality to obtain (2.13).

Additionally, (2.14) can be found from the Cauchy integral formula for Q-holomorphic functions by substituting $w-\mathbb{T}w$ into Ψ where $\mathbb{T}w$ is considered to be Q-holomorphic in $\hat{\mathbb{C}}\backslash\hat{D}$ for every $t\in\mathbb{R}$ and $(\mathbb{T}w)(\infty,t)\equiv 0$. □

Now let us give another representation of solutions of (1.3) using (2.13) and (2.14).

Theorem 2.11 *Every solution of (1.3) with $a=b=c=d=0$ outside $\hat{D}$, vanishing identically at $t=0$, may be represented as*

$$w\,(z,t) \quad = \quad \Psi\,(z,t) + \int_0^t \int_D d\phi(\zeta)\overline{d\phi(\zeta)}\Big\{\Gamma_\tau^{(1)}\,(z,t;\zeta,\tau)\,\Psi_\tau\,(\zeta,\tau) \\ + \Gamma_\tau^{(2)}\,(z,t;\zeta,\tau)\,\overline{\Psi_\tau\,(\zeta,\tau)}\Big\}d\tau$$

where Ψ is given in (2.14) and $\Gamma^{(k)}$, $k=1,2$, are given with the help of fundamental kernels as

$$\Gamma^{(1)}\,(z,t;\zeta,\tau) \quad := \quad -P^{-1}\Omega_{\bar{\phi}}^{(1)}\,(z,t;\zeta,\tau)\,,$$
$$\Gamma^{(2)}\,(z,t;\zeta,\tau) \quad := \quad -P^{-1}\Omega_{\phi}^{(2)}\,(z,t;\zeta,\tau)\,.$$

Proof Applying Green's identity for Q-holomorphic functions (see [6], p. 113) to the right-hand side of the relation (2.13), we may write

$$\begin{aligned} w(z,t) = \quad & \lim_{\varepsilon\to 0}\Big\{ -P^{-1}\int_0^t\int_{D_\varepsilon} d\phi(\zeta)\overline{d\phi(\zeta)}\Big(\Omega^{(1)}_{\bar\phi\tau}(z,t;\zeta,\tau)\,\Psi_\tau(\zeta,\tau) \\ & \qquad +\Omega^{(2)}_{\phi\tau}(z,t;\zeta,\tau)\,\overline{\Psi_\tau(\zeta,\tau)}\Big)d\tau\Big\} \\ & +\lim_{\varepsilon\to 0}\Big\{P^{-1}\int_0^t\int_{|\zeta-z|=\varepsilon}\Big(d\phi(\zeta)\,\Omega^{(1)}_\tau(z,t;\zeta,\tau)\,\Psi_\tau(\zeta,\tau) \\ & \qquad -\overline{d\phi(\zeta)}\Omega^{(2)}_\tau(z,t;\zeta,\tau)\,\overline{\Psi_\tau(\zeta,\tau)}\Big)d\tau\Big\}, \end{aligned}$$

where D_ε is the intersection of the domains D and $|\zeta-z|>\varepsilon$. Taking into account the formulae (2.12), the proof is completed. □

Since the $\chi^{(k)}$ for $k=1,\ 2$, which are functions of (ζ,τ), for a fixed (z,t), are solutions of the pseudoparabolic Eq. (1.3), it is clear that $\Omega^{(k)}$ are solutions of the following equations:

$$\begin{aligned} \Omega^{(1)}_{\bar\phi\tau}(z,t;\zeta,\tau)-a(\zeta)\,\Omega^{(1)}_\tau(z,t;\zeta,\tau)-b^*(\zeta)\overline{b(\zeta)}\Omega^{(2)}_\tau(z,t;\zeta,\tau) & \\ +c(\zeta)\,\Omega^{(1)}(z,t;\zeta,\tau)+b^*(\zeta)\overline{d(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau) &= 0, \\ \Omega^{(2)}_{\phi\tau}(z,t;\zeta,\tau)-\overline{a(\zeta)}\Omega^{(2)}_\tau(z,t;\zeta,\tau)-\overline{b^*(\zeta)}b(\zeta)\,\Omega^{(1)}_\tau(z,t;\zeta,\tau) & \\ +\overline{c(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau)+\overline{b^*(\zeta)}d(\zeta)\,\Omega^{(1)}(z,t;\zeta,\tau) &= 0, \end{aligned}$$

and one can rewrite the above equations as

$$\begin{cases} P\Gamma^{(1)}_\tau(z,t;\zeta,\tau)=-a(\zeta)\,\Omega^{(1)}_\tau(z,t;\zeta,\tau)-b^*(\zeta)\overline{b(\zeta)}\Omega^{(2)}_\tau(z,t;\zeta,\tau) \\ \qquad +c(\zeta)\,\Omega^{(1)}(z,t;\zeta,\tau)+b^*(\zeta)\overline{d(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau), \\ P\Gamma^{(2)}_\tau(z,t;\zeta,\tau)=-\overline{a(\zeta)}\Omega^{(2)}_\tau(z,t;\zeta,\tau)-\overline{b^*(\zeta)}b(\zeta)\,\Omega^{(1)}_\tau(z,t;\zeta,\tau) \\ \qquad +\overline{c(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau)+\overline{b^*(\zeta)}d(\zeta)\,\Omega^{(1)}(z,t;\zeta,\tau), \end{cases} \tag{2.16}$$

where $b^*=\phi_z^{-1}\overline{\phi_z}$.

3. Integral representation of the second-kind solutions

In the previous section, we supposed that the inequality (2.4) holds and got the result under this assumption. However, unlike preceding considerations, without this restriction, a similar approach can be done. To obtain another integral formulation, the fundamental kernels $\Omega^{(k)}(z,\zeta)\;(k=1,2)$ of the equation

$$w_{\bar\phi}+aw+b\overline{w}=0$$

can be used as given in [7]. Similarly to the complex case (see [10], p. 187), taking into account the equations (2.16), it is easily seen that the special solutions of (1.3) satisfy the following integral equation:

$$\begin{aligned} w(z,t)-2iP^{-1}\int_0^t\int_{\mathbb{C}} d\phi(\zeta)\overline{d\phi(\zeta)}\Big\{\Omega^{(1)}(z,\zeta)\Big[c(\zeta)\,w(\zeta,\tau)+d(\zeta)\,\overline{w(\zeta,\tau)}\Big] & \\ +\Omega^{(2)}(z,\zeta)\Big[\overline{c(\zeta)w(\zeta,\tau)+\overline{d(\zeta)}w(\zeta,\tau)}\Big]\Big\}d\tau = \Psi(z,t), & \end{aligned} \tag{3.1}$$

where Ψ is a solution of

$$\Psi_{\bar{\phi}t} + a\Psi_t + b\overline{\Psi}_t = 0.$$

To show this, we split the integral over D into two integrals such that one is over D_ε and the other one is over $|\zeta - z| < \varepsilon$ as above. (3.1) can be differentiated with respect to t and $\bar{\phi}$.

We deal with bounded solutions of (1.3) again in $\mathbb{C} \times I$ where I is an interval in $\mathbb{R}$. Let w be a bounded solution satisfying (3.1) for some Ψ. Thus, for each t in $\mathbb{R}$, $\Psi(z,t)$ must be a bounded solution of

$$\omega_{\overline{\phi}} + a\omega + b\overline{\omega} = 0$$

in $\mathbb{C}$. As before, provided that (F_0, G_0) is a generating pair of the last equation (see Section 2), every bounded solution can be written as

$$F_0\lambda + G_0\mu$$

where λ and μ are real constant matrices. Thus, $\Psi(z,t)$ is of the form below:

$$\Psi(z,t) = F_0(z)\lambda(t) + G_0(z)\mu(t),$$

where λ and μ are real differentiable matrix functions of t in $\mathbb{R}$.

Now let us investigate the general solution according to λ and μ.

(i) Let $\lambda = \mu = 0$. Let us show the integral operator in (3.1) by $\mathbb{P}$. The problem to be solved is

$$w - \mathbb{P}w = 0. \tag{3.2}$$

Let us consider the following bound:

$$\|2iP^{-1}\| \int_{\mathbb{C}} \left\| d\phi(\zeta)\overline{d\phi(\zeta)} \right\| (\|c\| + \|d\|) \left(\left\|\Omega^{(1)}(z,\zeta)\right\| + \left\|\Omega^{(2)}(z,\zeta)\right\| \right) d\xi d\eta \le \kappa < \infty,$$

and define

$$\|w\|_\kappa := \sup_{z \in \mathbb{C}, \kappa|t| \le 1} \|w(z,t)\|.$$

Thus, for $z \in \mathbb{C}$ and $\kappa|t| \le 1$, the inequality

$$\|w(z,t)\| \le \kappa \|w\|_\kappa |t|$$

holds by (3.2). It can easily be seen that w vanishes identically in $z \in \mathbb{C}$ and $\kappa|t| \le 1$. As above, by virtue of autonomity of L, w must vanish identically in $\mathbb{C} \times \mathbb{R}$, so (3.2) has only a trivial solution.

(ii) Let $\mu = 0$. In this case we search for the solution of

$$w - \mathbb{P}w = F_0\lambda.$$

To obtain the solution, we use the iteration method, i.e.

$$w_0 := \lambda F_0, \quad w_k := w_0 + \mathbb{P}w_{k-1} \quad (k \in \mathbb{N}), \quad w := \lim_{k\to\infty} w_k = \sum_{k=0}^{\infty} \mathbb{P}^k w_0.$$

If the series

$$\sum_{k}^{\infty} \mathbb{P}^k F_0 \lambda = \sum_{k}^{\infty} F_k \lambda_k, \quad (k \in \mathbb{N}_0)$$

is convergent where

$$\lambda_0 := \lambda, \quad \lambda_k(t) = \int_0^t \lambda_{k-1}(\tau) d\tau \quad (k \in \mathbb{N}, t \in \mathbb{R}),$$

$$F_k := 2iP^{-1} \int_{\mathbb{C}} d\phi(\zeta) \overline{d\phi(\zeta)} \Big\{ \Omega^{(1)}(z,\zeta) \left(c(\zeta) F_{k-1}(\zeta) + d(\zeta) \overline{F_{k-1}(\zeta)} \right)$$
$$+ \Omega^{(2)}(z,\zeta) \left(\overline{c(\zeta) F_{k-1}(\zeta)} + \overline{d(\zeta)} F_{k-1}(\zeta) \right) \Big\}, \quad (k \in \mathbb{N}),$$

then the function w is uniquely defined. Taking into consideration the following estimates,

$$\|F_k\| \leq \kappa^k \|F_0\|, \quad \|\lambda_k\|_t \leq \frac{|t|^k}{k!} \|\lambda_0\|_t \quad (k \in \mathbb{N}_0),$$

where

$$\|F_k\| = \sup_{z \in \mathbb{C}} \|F_k(z)\|, \quad \|\lambda_k\|_t := \sup_{|\tau \leq |t|} \|\lambda_k(\tau)\|,$$

the convergence is obtained such that

$$(\mathbb{F}\lambda)(z,t) := \sum_{k=0}^{\infty} F_k(z) \lambda_k(t), \qquad (z \in \mathbb{C}, t \in \mathbb{R}) \tag{3.3}$$

is the solution and λ_k and F_k have the properties

$$\lambda_k(0) = 0, \quad F_k(\infty) = 0 \quad (k \in \mathbb{N}), \; F_0(\infty) = I.$$

(iii) Let $\lambda = 0$. If μ_k and G_k are defined in the same way as in (ii) like λ_k and F_k, respectively, the unique solution of

$$w - \mathbb{P}w = G_0 \mu$$

is

$$(\mathbb{G}\mu)(z,t) := \sum_{k=0}^{\infty} G_k(z) \mu_k(t), \qquad (z \in \mathbb{C}, t \in \mathbb{R}). \tag{3.4}$$

Moreover, μ_k and G_k have the following properties:

$$\mu_k(0) = 0, \quad G_k(\infty) = 0 \quad (k \in \mathbb{N}), \; G_0(\infty) = iI.$$

(iv) Let λ and μ be arbitrary. The equation is

$$w - \mathbb{P}w = F_0 \lambda + G_0 \mu$$

and the solution of this equation is given by

$$w = \mathbb{F}\lambda + \mathbb{G}\mu.$$

It is clear that the operators $\mathbb{F}$ and $\mathbb{G}$ are the same as in (2.7) and (2.8) since the F_ks in (3.3) are special solutions of the system (2.6) and similarly the G_ks in (3.4) are special solutions of the system (2.9). Let us take the difference of two solutions of the kth equation of (2.6) or (2.9). This difference is a bounded solution of

$$\omega_{\bar{\phi}} + a\omega + b\bar{\omega} = 0$$

and vanishes at infinity. Since this only can be the zero solution, it is obtained that w is a trivial solution. Because of these considerations, without the restriction (2.4), a similar approach can be done for a and b, as well as the coefficients c and d, which only have to be in $L_{p,2}(\mathbb{C})$.

(v) To illustrate that the equation (3.1) is not homogeneous, let us take $\Psi(z,t) = f(z)t$. In this case, equation (3.1) has the form below:

$$(w - \mathbb{P}w)(z,t) = f(z)t, \quad (f \in L_{p,2}(\mathbb{C}), p > 2),$$

where f is a function of z only. After the same calculations as above, the uniquely defined solution can be given in the form

$$\begin{aligned} w\left(z,t\right) &= \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!} f_k\left(z\right), \\ f_0 &:= f, \\ f_k\left(z\right) &:= 2iP^{-1} \int\limits_{\mathbb{C}} d\phi(\zeta)\overline{d\phi(\zeta)} \Big\{ \Omega^{(1)}\left(z,\zeta\right) \left(c\left(\zeta\right) f_{k-1}\left(\zeta\right) + d\left(\zeta\right) \overline{f_{k-1}\left(\zeta\right)} \right) \\ &\quad + \Omega^{(2)}\left(z,\zeta\right) \left(\overline{c\left(\zeta\right) f_{k-1}\left(\zeta\right)} + \overline{d\left(\zeta\right)} f_{k-1}\left(\zeta\right) \right) \Big\}. \end{aligned}$$

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