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Coframe bundle and problems of lifts on its cross-sections

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Abstract: The main purpose of this paper is to study the complete and horizontal lifts of vector and tensor fields of type (1,1) on cross-sections in the coframe bundle. Explicit formulas of these lifts are obtained. Finally, complete lifts of almost complex structures restricted to almost analytic cross-sections are investigated.

Key words: Coframe bundle, cross-section, Tachibana operator, Nijenhuis–Shirokov tensor, almost complex structure

1. Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞ and F^*M its coframe bundle. The differential geometry of the cotangent bundle has been studied by many authors (see, for example, [2, 3, 9–11]).

When a field of global coframes is given on M, its defines a cross-section $\sigma : M \to F^*M$ in the coframe bundle. In this paper, we study the behavior on this cross-section of lifts of tensor fields from M to F^*M .

In 2 we briefly describe the definitions and results that are needed later, after which the complete and horizontal lifts of affinor fields (tensor fields of type $(1,1)$) are constructed in 3. In 4 and 5 we consider, respectively, the complete and horizontal lifts of the vector and affinor fields along the $n-$dimensional submanifold $\sigma(M)$ of F^*M defined by cross-section σ. In 6 we study the particular case of an almost complex structure on $\sigma(M)$.

All results in this paper can be closely compared with those of the corresponding theory for cross-sections in the cotangent bundle [12]. A similar approach was applied in [1], when studying lifts on cross-sections of the bundle of frames by means of the tangent bundle.

2. Preliminaries

Manifolds, tensor fields, and linear connections under consideration are all assumed to be differentiable and of class C^∞. Indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$ and indices $A, B, C, ...$ run from 1 to $n + n^2$. We put $h_\alpha = \alpha \cdot n + h$. Summation over repeated indices is always implied. Entries of matrices are written as A^i_j, A_{ij} or A^{ij}, and in all cases i is the row index while j is the column index.

Let M be an $n-$dimensional differentiable manifold of class C^∞. Coordinate systems in M are denoted by (U, x^i), where U is the coordinate neighborhood and x^i are the coordinate functions. We denote the Lie derivative by L_X, and by $\Im^r_s(M)$ the set of all differentiable tensor fields of type (r, s) on M.

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Let T_x^*M be the cotangent space at a point $x \in M$, $(X^\alpha) = (X^1, ..., X^n)$ a coframe at x and F^*M the coframe bundle over M, that is, the set of all coframes at all points of M (see [4]). Let $\pi : F^*M \to M$ be the canonical projection of F^*M onto M. For the coordinate system (U, x^i) in M we put $F^*U = \pi^{-1}(U)$. A coframe (X^α) at x can be expressed uniquely in the form $X^\alpha = X_i^\alpha (dx^i)_x$. The induced coordinate system in F^*U is $\left\{ F^*U, (x^i, X_i^\alpha) \right\}$. We shall denote $\frac{\partial}{\partial x^i}$ by ∂_i and $\frac{\partial}{\partial X_i^\alpha}$ by ∂_{i_α}. The matrix (X_i^α) is nonsingular and its inverse will be written as (X_α^i). We denote by ∇ the linear connection on M with components Γ_{ij}^k.

Let V be a vector field on M, and let V^i be its components in U. Then the complete lift ${}^CV$ and horizontal lift ${}^HV$ of V to F^*M are given by (see [4])

$$ {}^CV = V^i \partial_i - X_j^\alpha (\partial_i V^j) \partial_{i_\alpha}, \tag{2.1}$$

$$ {}^HV = V^i \partial_i + X_j^\alpha \Gamma_{ki}^j V^k \partial_{i_\alpha}, \tag{2.2}$$

respectively.

3. Lifts of affinor fields to the coframe bundle

Let φ be an affinor field on M and let φ_i^j be its local components in U.

The following Theorem 1 holds.

Theorem 1 *If we put*

$$\begin{cases} \tilde{\varphi}_j^i = \varphi_j^i, & \tilde{\varphi}_{j_\beta}^i = 0, \\ \tilde{\varphi}_j^{i_\alpha} = X_k^\alpha (\partial_j \varphi_i^k - \partial_i \varphi_j^k), & \tilde{\varphi}_{j_\beta}^{i_\alpha} = \delta_\beta^\alpha \varphi_i^j, \end{cases} \tag{3.1}$$

then we get an affinor field $\tilde{\varphi}$ *on* F^*M *whose components are* $\tilde{\varphi}_J^I$ *with respect to the coordinate system* $\left\{F^*U, (x^i, X_i^\alpha)\right\}$, *where* δ_β^α *is the Kronecker delta.*

Proof We shall show that under the coordinate transformation

$$\begin{cases} x^{i'} = x^{i'}(x^1, ..., x^n), \\ X_{i'}^\alpha = A_{i'}^i X_i^\alpha \end{cases} \tag{3.2}$$

on $F^*U \bigcap F^*U'$, the equation

$$\tilde{\varphi}_{J'}^{I'} = A_I^{I'} A_{J'}^J \tilde{\varphi}_J^I \tag{3.3}$$

holds good, where $A_{i'}^i = \frac{\partial x^i}{\partial x^{i'}}$ are elements of the Jacobian matrix of the inverse transformation $x^i = x^i(x^{1'}, ..., x^{n'})$, and $A_I^{I'}$ are elements of the Jacobian matrix of the transformation (3.2), i.e.

$$(A_I^{I'}) = \begin{pmatrix} A_i^{i'} & 0 \\ X_j^\alpha \partial_i A_{i'}^j & A_{i'}^i \delta_\beta^\alpha \end{pmatrix}. \tag{3.4}$$

On the other hand, the Jacobian matrix $(A_{J'}^J)$ of the inverse transformation has the structure

$$(A_{J'}^J) = \begin{pmatrix} A_{j'}^j & 0 \\ X_{k'}^\alpha \partial_{j'} A_j^{k'} & A_j^{j'} \delta_\beta^\alpha \end{pmatrix}. \tag{3.5}$$

In the case where $I' = i', J' = j'$, we can easily verify that the right-hand side of (3.3) reduces to

$$A_I^{i'} A_{j'}^J \tilde{\varphi}_J^I = A_i^{i'} A_{j'}^j \tilde{\varphi}_j^i + A_{i_\gamma}^{i'} A_{j'}^j \tilde{\varphi}_j^{i_\gamma} + A_i^{i'} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^i$$

$$+ A_{i_\gamma}^{i'} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^{i_\gamma} = A_i^{i'} A_{j'}^j \varphi_j^i = \varphi_{j'}^{i'} = \tilde{\varphi}_{j'}^{i'}.$$

In the case where $I' = i', J' = j'_\beta$ or $I' = i'_\alpha, J' = j'_\beta$, it follows that (3.3) holds good by the same manner as before. In the case where $I' = i'_\alpha, J' = j'$, the left-hand side of (3.3) reduces to

$$\tilde{\varphi}_{j'}^{i'_\alpha} = X_{k'}^\alpha (\partial_{j'} \varphi_{i'}^{k'} - \partial_{i'} \varphi_{j'}^{k'}),$$

which is the sum of the following six terms $a_1, a_2, ..., a_6$:

$$a_1 = X_{k'}^\alpha (\partial_{j'} A_m^{k'}) A_{i'}^i \varphi_i^m, a_2 = X_{k'}^\alpha A_m^{k'} (\partial_{j'} A_{i'}^i) \varphi_i^m,$$

$$a_3 = X_{k'}^\alpha A_m^{k'} A_{i'}^i (\partial_{j'} \varphi_i^m), a_4 = -X_{k'}^\alpha (\partial_{i'} A_m^{k'}) A_{j'}^j \varphi_j^m,$$

$$a_5 = -X_{k'}^\alpha A_m^{k'} (\partial_{i'} A_{j'}^j) \varphi_j^m, a_6 = -X_{k'}^\alpha A_m^{k'} A_{j'}^j (\partial_{i'} \varphi_j^m).$$

On the other hand, the right-hand side of (3.3) can be written as

$$A_I^{i'_\alpha} A_{j'}^J \tilde{\varphi}_J^I = A_i^{i'_\alpha} A_{j'}^j \tilde{\varphi}_j^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^j \tilde{\varphi}_j^{i_\gamma} + A_i^{i'_\alpha} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^i + A_{i_\gamma}^{i'_\alpha} A_{j'}^{j_\lambda} \tilde{\varphi}_{j_\lambda}^{i_\gamma}.$$

The last expression is the sum of the following four terms $b_1, ..., b_4$:

$$b_1 = X_k^\alpha (\partial_i A_{i'}^k) A_{j'}^j \varphi_j^i, b_2 = X_k^\alpha A_{i'}^i A_{j'}^j (\partial_j \varphi_i^k),$$

$$b_3 = -X_k^\alpha A_{i'}^i A_{j'}^j (\partial_i \varphi_j^k), b_4 = X_{k'}^\alpha A_{i'}^i (\partial_{j'} A_j^{k'}) \varphi_i^j.$$

After some calculations we get the following relations:

$$a_1 = b_4, \quad a_3 = b_2, \quad a_4 = b_1, \quad a_2 + a_5 = 0, \quad a_6 = b_3. \tag{3.6}$$

Hence, by virtue of (3.6), we see that (3.3) holds good. Consequently, $\tilde{\varphi}$ is an affinor field on F^*M. An affinor field $\tilde{\varphi}$ is called a complete lift of φ to F^*M. □

Theorem 2 *If we put*

$$\begin{cases} \bar{\varphi}_j^i = \varphi_j^i, & \bar{\varphi}_{j_\beta}^i = 0, \\ \bar{\varphi}_j^{i_\alpha} = X_k^\alpha (\varphi_j^m \Gamma_{mi}^k - \varphi_i^m \Gamma_{jm}^k), & \bar{\varphi}_{j_\beta}^{i_\alpha} = \delta_\beta^\alpha \varphi_i^j, \end{cases} \tag{3.7}$$

then we get an affinor field $\bar{\varphi}$ *on* F^*M *whose components are* $\bar{\varphi}_J^I$ *with respect to the coordinate system* $\{F^*U, (x^i, X_i^\alpha)\}$.

Proof We shall show that under the coordinate transformation (3.2) the equation

$$\bar{\varphi}_{J'}^{I'} = A_I^{I'} A_{J'}^J \bar{\varphi}_J^I \tag{3.8}$$

holds good.

In the case $I' = i', J' = j'$, we can easily verify that the right-hand side of (3.8) reduces to

$$A^{i'}_I A^J_{j'} \bar{\varphi}^I_J = A^{i'}_i A^j_{j'} \bar{\varphi}^i_j + A^{i'}_{i_\gamma} A^j_{j'} \bar{\varphi}^{i_\gamma}_j + A^{i'}_i A^{j_\lambda}_{j'} \bar{\varphi}^i_{j_\lambda}$$

$$+ A^{i'}_{i_\gamma} A^{j_\lambda}_{j'} \bar{\varphi}^{i_\gamma}_{j_\lambda} = A^{i'}_i A^j_{j'} \varphi^i_j = \varphi^{i'}_{j'} = \bar{\varphi}^{i'}_{j'}.$$

In the cases $I' = i', J' = j'_\beta$ and $I' = i'_\alpha, J' = j'_\beta$, it follows that (3.8) holds good by the same manner as before. In the case where $I' = i'_\alpha, J' = j'$, the left-hand side of (3.8) reduces to

$$\bar{\varphi}^{i'_\alpha}_{j'} = X^\alpha_{k'} (\varphi^{m'}_{j''} \Gamma^{k'}_{m'i'} - \varphi^{m'}_{i'} \Gamma^{k'}_{j'm'}),$$

which is the sum of the following four terms $c_1, ..., c_4$:

$$c_1 = X^\alpha_{k'} \varphi^{m'}_{j'} A^{k'}_k A^m_{m'} A^i_{i'} \Gamma^k_{mi}, c_2 = X^\alpha_{k'} \varphi^{m'}_{j'} A^{k'}_k (\partial_{m'} A^k_{i'}),$$

$$c_3 = -X^\alpha_{k'} \varphi^{m'}_{i'} A^{k'}_k A^m_{m'} A^j_{j'} \Gamma^k_{jm}, c_4 = -X^\alpha_{k'} \varphi^{m'}_{i'} A^{k'}_k (\partial_{j'} A^k_{m'}).$$

On the other hand, the right-hand side of (3.8) can be written as

$$A^{i'_\alpha}_I A^J_{j'} \bar{\varphi}^I_J = A^{i'_\alpha}_i A^j_{j'} \bar{\varphi}^i_j + A^{i'_\alpha}_{i_\gamma} A^j_{j'} \bar{\varphi}^{i_\gamma}_j + A^{i'_\alpha}_i A^{j_\lambda}_{j'} \bar{\varphi}^i_{j_\lambda} + A^{i'_\alpha}_{i_\gamma} A^{j_\lambda}_{j'} \bar{\varphi}^{i_\gamma}_{j_\lambda}.$$

The last expression is the sum of the following four terms $d_1, ..., d_4$:

$$d_1 = X^\alpha_k (\partial_i A^k_{i'}) A^j_{j'} \varphi^i_j, d_2 = X^\alpha_k A^i_{i'} A^j_{j'} \varphi^m_j \Gamma^k_{mi},$$

$$d_3 = -X^\alpha_k A^i_{i'} A^j_{j'} \varphi^m_i \Gamma^k_{jm}, d_4 = X^\alpha_{k'} A^i_{i'} (\partial_{j'} A^{k'}_j) \varphi^j_i.$$

After some calculations we get the following relations:

$$c_1 = d_2, \quad c_2 = d_1, \quad c_3 = d_3, \quad c_4 = d_4. \tag{3.9}$$

Hence, by virtue of (3.9), we see that (3.8) holds good. It means that $\bar{\varphi}$ is an affinor field on F^*M. An affinor field $\bar{\varphi}$ is called a horizontal lift of φ to F^*M. □

4. Lifts of vector fields on cross-sections

Let σ be a cross-section of the coframe bundle F^*M, that is $\sigma : M \to F^*M$ a mapping of class C^∞ such that $\pi \circ \sigma = Id_M$. Then σ defines a field of global coframes on M, that is, at each point $x \in M, \sigma(x) = (\sigma^1(x), ..., \sigma^n(x))$ is a linear coframe at x. If we put $\sigma = (\sigma^1, ..., \sigma^n)$ then each σ^α is a covector field globally defined on M. Assume that σ^α has local components $\sigma^\alpha_h(x)$ with respect to a coordinate system (U, x^i) in M, that is $\sigma^\alpha = \sigma^\alpha_h(x) dx^h$ in U. Then $\sigma(M)$, which will be called a cross-section determined by σ, is the $n-$dimensional submanifold of F^*M locally expressed in F^*U by

$$\begin{cases} x^h = x^h, \\ X^\alpha_h = \sigma^\alpha_h(x). \end{cases} \tag{4.1}$$

Thus tangent vectors $B_i^H = \partial_i x^H$ to the cross-section $\sigma(M)$ have components

$$B_i^H = \left(\frac{\partial x^H}{\partial x^i}\right) = \begin{pmatrix} \delta_i^h \\ \partial_i \sigma_h^\alpha \end{pmatrix}. \tag{4.2}$$

On the other hand, the fiber being represented by

$$\begin{cases} x^h = const, \\ X_h^\alpha = X_h^\alpha, \end{cases} \tag{4.3}$$

the tangent vectors $C_{i_\beta}^H = \partial_{i_\beta} x^H$ to the fiber have components

$$C_{i_\beta}^H = C^{i_\beta H} = \begin{pmatrix} 0 \\ \delta_h^i \delta_\beta^\alpha \end{pmatrix}. \tag{4.4}$$

The vectors B_i^H and $C_{i_\beta}^H$, being linearly independent, form a frame $E_I^H = (B_i^H, C_{i_\beta}^H)$ along the cross-section $\sigma(M)$. We call this the frame (B, C) along the cross-section. The coframe $\tilde{E}_H^J = (\tilde{B}_H^j, \tilde{C}_H^{j_\gamma})$ corresponding to this frame is given by

$$\tilde{B}_H^j = (\delta_h^j, 0), \tilde{C}_H^{j_\gamma} = (-\partial_h \sigma_j^\gamma, \delta_j^h \delta_\alpha^\gamma). \tag{4.5}$$

Let V be a vector field on M and ${}^C V$ its complete lift to F^*M, which is locally given by (2.1):

$${}^C V = {}^C V^h \partial_h + {}^C V^{h_\alpha} \partial_{h_\alpha} = V^h \partial_h - X_j^\alpha (\partial_h V^j) \partial_{h_\alpha}. \tag{4.6}$$

On the other hand, the complete lift ${}^C V$ has the following decomposition with respect to the (B, C)-frame along the cross-section $\sigma(M)$:

$${}^C V = \tilde{V}^i B_i + \tilde{V}^{i_\beta} C_{i_\beta}. \tag{4.7}$$

Thus, from (4.6) and (4.7) we have

$$\begin{aligned} {}^C V^h \partial_h + {}^C V^{h_\alpha} \partial_{h_\alpha} &= \tilde{V}^i B_i + \tilde{V}^{i_\beta} C_{i_\beta} = \tilde{V}^i B_i^h \partial_h + \tilde{V}^i B_i^{h_\alpha} \partial_{h_\alpha} \\ +\tilde{V}^{i_\beta} C_{i_\beta}^h \partial_h + \tilde{V}^{i_\beta} C_{i_\beta}^{h_\alpha} \partial_{h_\alpha} &= \left(\tilde{V}^i B_i^h + \tilde{V}^{i_\beta} C^{\ h}_{i_\beta}\right) \partial_h \\ &+ \left(\tilde{V}^i B_i^{h_\alpha} + \tilde{V}^{i_\beta} C^{\ h_\alpha}_{i_\beta}\right) \partial_{h_\alpha}. \end{aligned} \tag{4.8}$$

By using (4.2) and (4.4), from (4.8) we obtain:

$${}^C V^h = \tilde{V}^i B_i^h + \tilde{V}^{i_\beta} C_{i_\beta}^h = \tilde{V}^i \delta_i^h = \tilde{V}^h,$$

$${}^C V^{h_\alpha} = -\sigma_j^\alpha \partial_h V^j = \tilde{V}^i \partial_i \sigma_h^\alpha + \tilde{V}^{i_\beta} C_{i_\beta}^{h_\alpha} = V^i \partial_i \sigma_h^\alpha + \tilde{V}^{i_\beta} \delta_h^i \delta_\beta^\alpha.$$

Thus the complete lift ${}^C V$ of a vector field V in M to F^*M, having components (2.1) with respect to the natural frame, has components

$$\begin{pmatrix} V^h \\ -L_V \sigma_h^\alpha \end{pmatrix}$$

with respect to the frame (B,C) along the cross-section $\sigma(M)$.

This means that

$$^{C}V = V^{h}B_{h}^{A} - (L_{V}\sigma_{h}^{\alpha})\, C_{h_{\alpha}}^{A}.$$

From here follows

Theorem 3 *The complete lift ^{C}V of a vector field V in M to $F^{*}M$ is tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^{1},...,\sigma^{n})$ if and only if the Lie derivative of each σ^{α} with respect to V vanishes, i.e. $L_{V}\sigma^{\alpha} = 0,\ 1 \le \alpha \le n$.*

By analogy, the horizontal lift ^{H}V of a vector field V in M to $F^{*}M$, having components (2.2) with respect to the natural frame, has components

$$\begin{pmatrix} V^{h} \\ -\nabla_{V}\sigma_{h}^{\alpha} \end{pmatrix}$$

with respect to the frame (B,C) along the cross-section $\sigma(M)$, where ∇_{V} is a covariant derivative along a vector field V in an affine connection ∇. Therefore

$$^{H}V = V^{h}B_{h}^{A} - (\nabla_{V}\sigma_{h}^{\alpha})\, C_{h_{\alpha}}^{A},$$

from which follows

Theorem 4 *The horizontal lift ^{H}V of a vector field V in M to $F^{*}M$ is tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^{1},...,\sigma^{n})$ if and only if the covariant derivative of each σ^{α} with respect to V vanishes, i.e. $\nabla_{V}\sigma^{\alpha} = 0,\ 1 \le \alpha \le n$.*

5. Lifts of affinor fields on cross-sections

Let φ be an affinor field on M and $^{C}\varphi$ its complete lift to $F^{*}M$, which is locally given by (3.1) with respect to the natural frame, i.e.

$$^{C}\varphi = \begin{pmatrix} \varphi_{i}^{h} & 0 \\ X_{k}^{\alpha}(\partial_{i}\varphi_{h}^{k} - \partial_{h}\varphi_{i}^{k}) & \varphi_{h}^{i}\delta_{\beta}^{\alpha} \end{pmatrix}. \tag{5.1}$$

If $^{C}\tilde{\varphi}_{J}^{I}$ are components of the complete lift $^{C}\varphi$ with respect to the (B,C)-frame along the cross-section $\sigma(M)$, then we have

$$^{C}\varphi_{I}^{J} = {}^{C}\tilde{\varphi}_{H}^{A}E_{A}^{J}\tilde{E}_{I}^{H}. \tag{5.2}$$

By using (4.2), (4.4), (4.5), and (5.1) we have

$$1)^{C}\varphi_{i}^{j} = \varphi_{i}^{j} = {}^{C}\tilde{\varphi}_{h}^{a}\delta_{a}^{j}\delta_{i}^{h} + {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a}\delta_{a}^{j}(-\partial_{i}\sigma_{h}^{\alpha}) = {}^{C}\tilde{\varphi}_{i}^{j} - {}^{C}\tilde{\varphi}_{h_{\alpha}}^{j}(\partial_{i}\sigma_{h}^{\alpha}), \tag{5.3}$$

$$2)^{C}\varphi_{i_{\beta}}^{j} = 0 = {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a}E_{a}^{j}\tilde{E}_{i_{\beta}}^{h_{\alpha}} = {}^{C}\tilde{\varphi}_{h_{\alpha}}^{a}\delta_{a}^{j}\delta_{h}^{i}\delta_{\beta}^{\alpha},$$

from which it follows that

$$^{C}\tilde{\varphi}_{h_{\alpha}}^{a} = 0. \tag{5.4}$$

Using (5.4), from (5.3) we get

$$^{C}\tilde{\varphi}_{h}^{a} = \varphi_{h}^{a}.$$

3) ${}^{C}\varphi_{i_\beta}^{j_\gamma} = \varphi_j^i\delta_\beta^\gamma = {}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_{i_\beta}^{h_\alpha} = {}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_h^i\delta_\beta^\alpha$, consequently

$$ {}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau} = \varphi_a^h\delta_\alpha^\tau. $$

$$ 4)\,{}^{C}\varphi_i^{j_\gamma} = \sigma_k^\gamma\partial_i\varphi_j^k - \sigma_k^\gamma\partial_j\varphi_i^k = {}^{C}\tilde{\varphi}_h^aE_a^{j_\gamma}\tilde{E}_i^h + {}^{C}\tilde{\varphi}_h^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^h + {}^{C}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^{h_\alpha} $$

$$ = \varphi_h^a\partial_a\sigma_j^\gamma\delta_i^h + {}^{C}\tilde{\varphi}_h^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_i^h + \varphi_a^h\delta_\alpha^\tau\delta_j^a\delta_\tau^\gamma(-\partial_i\sigma_h^\alpha) $$

or

$$ {}^{C}\tilde{\varphi}_h^{a_\sigma}\delta_j^a\delta_\sigma^\gamma\delta_i^h = \sigma_k^\gamma\partial_i\varphi_j^k - \sigma_k^\gamma\partial_j\varphi_i^k - \varphi_i^k\partial_k\sigma_j^\gamma + \varphi_j^h\partial_i\sigma_h^\gamma, $$

from which we obtain

$$ {}^{C}\tilde{\varphi}_h^{a_\tau} = \sigma_k^\tau\partial_h\varphi_a^k - \sigma_k^\tau\partial_a\varphi_h^k - \varphi_h^k\partial_k\sigma_a^\tau + \varphi_a^k\partial_h\sigma_k^\tau = -(\varphi_h^k\partial_k\sigma_a^\tau $$

$$ -\varphi_a^k\partial_h\sigma_k^\tau - \sigma_k^\tau\partial_h\varphi_a^k + \sigma_k^\tau\partial_a\varphi_h^k) = -(\Phi_\varphi\sigma^\tau)_{ha}, $$

where $\Phi_\varphi\sigma^\tau$ is the Tachibana operator applied to σ^τ (see [7]).

Thus we have

Theorem 5 *The complete lift ${}^{C}\varphi$ having components (5.1) with respect to the natural frame has the nonzero components*

$$ {}^{C}\tilde{\varphi}_h^a = \varphi_h^a, \quad {}^{C}\tilde{\varphi}_h^{a_\tau} = -(\Phi_\varphi\sigma^\tau)_{ha}, \quad {}^{!}\tilde{\varphi}_{h_\alpha}^{a_\tau} = \varphi_a^h\delta_\alpha^\tau $$

with respect to the frame (B, C) along the cross-section $\sigma(M)$.

Now we assume that ${}^{H}\varphi$ is the horizontal lift of the affinor field φ to F^*M, given by (3.7) with respect to the natural frame, i.e.

$$ {}^{H}\varphi = \begin{pmatrix} \varphi_i^h & 0 \\ X_k^\alpha(\varphi_i^m\Gamma_{mh}^k - \varphi_h^m\Gamma_{im}^k) & \varphi_h^i\delta_\beta^\alpha \end{pmatrix}. \tag{5.5} $$

On the other hand, the horizontal lift ${}^{H}\varphi$ has the following decomposition with respect to the (B, C)-frame along the cross-section $\sigma(M)$:

$$ {}^{H}\varphi_I^J = {}^{H}\tilde{\varphi}_H^AE_A^J\tilde{E}_I^H. \tag{5.6} $$

Using (3.7), (3.2), (3.4), and (5.5) we find

$$ 1)\,{}^{H}\varphi_i^j = \varphi_i^j = {}^{H}\tilde{\varphi}_h^a\delta_a^j\delta_i^h + {}^{H}\tilde{\varphi}_{h_\alpha}^a\delta_a^j(-\partial_i\sigma_h^\alpha) = {}^{H}\tilde{\varphi}_i^j - {}^{H}\tilde{\varphi}_{h_\alpha}^j(\partial_i\sigma_h^\alpha). \tag{5.7} $$

2) ${}^{H}\varphi_{i_\beta}^j = 0 = {}^{H}\tilde{\varphi}_{h_\alpha}^aE_a^j\tilde{E}_{i_\beta}^{h_\alpha} = {}^{H}\tilde{\varphi}_{h_\alpha}^a\delta_a^j\delta_h^i\delta_\beta^\alpha$, consequently

$$ {}^{H}\tilde{\varphi}_{h_\alpha}^a = 0. \tag{5.8} $$

Based on equality (5.8), from (5.7) we get

$$ {}^{H}\tilde{\varphi}_h^a = \varphi_h^a. $$

$$ 3)\,{}^{H}\varphi_{i_\beta}^{j_\gamma} = \varphi_j^i\delta_\beta^\gamma = {}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_{i_\beta}^{h_\alpha} = {}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_h^i\delta_\beta^\alpha, $$

from which it follows that

$$^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}=\varphi_a^h\delta_\alpha^\tau.$$

$$4)^{H}\varphi_i^{j_\gamma}=\sigma_k^\gamma\varphi_i^m\Gamma_{mj}^k-\sigma_k^\gamma\varphi_j^m\Gamma_{mi}^k={}^{H}\tilde{\varphi}_h^a E_a^{j_\gamma}\tilde{E}_i^h+{}^{H}\tilde{\varphi}_h^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^h$$

$$+{}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}E_{a_\tau}^{j_\gamma}\tilde{E}_i^{h_\alpha}=\varphi_h^a\partial_a\sigma_j^\gamma\delta_i^h+{}^{H}\tilde{\varphi}_h^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_i^h+\varphi_a^h\delta_\alpha^\tau\delta_j^a\delta_\tau^\gamma(-\partial_i\sigma_h^\alpha)$$

or

$$^{H}\tilde{\varphi}_h^{a_\tau}\delta_j^a\delta_\tau^\gamma\delta_i^h=\sigma_k^\gamma\varphi_i^m\Gamma_{mj}^k-\sigma_k^\gamma\varphi_j^m\Gamma_{mi}^k-\varphi_i^k\partial_k\sigma_j^\gamma+\varphi_j^h\partial_i\sigma_h^\gamma,$$

from which we obtain

$$^{H}\tilde{\varphi}_h^{a_\tau}=\sigma_k^\tau\varphi_h^m\Gamma_{ma}^k-\sigma_k^\tau\varphi_a^m\Gamma_{mh}^k-\varphi_h^k\partial_k\sigma_a^\tau+\varphi_a^k\partial_h\sigma_k^\tau$$

$$=-\varphi_h^k(\partial_k\sigma_a^\tau-\Gamma_{ka}^m\sigma_m^\tau)+\varphi_a^k(\partial_h\sigma_k^\tau-\Gamma_{kh}^m\sigma_m^\tau)=-\varphi_h^k\nabla_k\sigma_a^\tau+\varphi_a^k\nabla_h\sigma_k^\tau$$

$$=-(\varphi_h^k\nabla_k\sigma_a^\tau-\varphi_a^k\nabla_h\sigma_k^\tau)=-(\tilde{\Phi}_\varphi\sigma^\tau)_{ha},$$

where $\tilde{\Phi}_\varphi\sigma^\tau$ is the Vishnevskii operator applied to σ^τ (see [7]).
Thus we have

Theorem 6 *The horizontal lift $^{H}\varphi$ having the nonzero components (5.5) with respect to the natural frame has the nonzero components*

$$^{H}\tilde{\varphi}_h^a=\varphi_h^a,\quad {}^{H}\tilde{\varphi}_h^{a_\tau}=-(\tilde{\Phi}_\varphi\sigma^\tau)_{ha},\quad {}^{H}\tilde{\varphi}_{h_\alpha}^{a_\tau}=\varphi_a^h\delta_\alpha^\tau$$

with respect to the frame (B,C) along the cross-section $\sigma(M)$.

6. Complete lift of almost complex structure on cross-sections

Suppose that the manifold M has an almost complex structure F. Its mean that $F^2=-I$. We have

Theorem 7 *Let M be a differentiable manifold with an almost complex structure F. Then the complete lift ^{C}F of F to F^*M is an almost complex structure if and only if $X_k^\beta Q(F,F)_{ij}^k=0$, where $Q(F,F)-$ the Nijenhuis–Shirokov tensor of F (see [5]).*

Proof From (5.1) we have

$$1)^{C}F_i^{H\,C}F_H^j={}^{C}F_i^{h\,C}F_h^j+{}^{C}F_i^{h_\gamma\,C}F_{h_\gamma}^j=F_i^hF_h^j=-\,\delta_i^j=-{}^{C}I_i^j,$$

$$2)^{C}F_{i_\alpha}^{H\,C}F_H^j={}^{C}F_{i_\alpha}^{h\,C}F_h^j+{}^{C}F_{i_\alpha}^{h_\gamma\,C}F_{h_\gamma}^j=0=-{}^{C}I_{i_\alpha}^j,$$

$$3)^{C}F_{i_\alpha}^{H\,C}F_H^{j_\beta}={}^{C}F_{i_\alpha}^{h\,C}F_h^{j_\beta}+{}^{C}F_{i_\alpha}^{h_\gamma\,C}F_{h_\gamma}^{j_\beta}=F_h^i\delta_\alpha^\gamma F_j^h\delta_\gamma^\beta=-\delta_j^i\delta_\alpha^\beta=$$

$$=-{}^{C}I_{i_\alpha}^{j_\beta}, \tag{6.1}$$

$$4)^{C}F_i^{H\,C}F_H^{j_\beta}={}^{C}F_i^{h\,C}F_h^{j_\beta}+{}^{C}F_i^{h_\gamma\,C}F_{h_\gamma}^{j_\beta}=F_i^hX_k^\beta(\partial_hF_j^k-\partial_jF_h^k)$$

$$+X_k^\gamma(\partial_iF_h^k-\partial_hF_i^k)F_j^h\delta_\gamma^\beta=X_k^\beta(F_i^h\partial_hF_j^k-F_i^h\partial_jF_h^k+F_j^h\partial_iF_h^k$$

$$-F_j^h\partial_h F_i^k) = X_k^\beta(\partial_i(F_j^h F_h^k) - \partial_j(F_i^h F_h^k)) + X_k^\beta(F_i^h\partial_h F_j^k$$

$$-F_j^h\partial_h F_i^k - F_h^k\partial_i F_j^h + F_h^k\partial_j F_i^h) = -{}^C I_i^{j\beta} + X_k^\beta Q(F,F)_{ij}^k.$$

From (6.1) we obtain

$$({}^C F)^2 = {}^C(F^2) + \gamma(X \circ Q(F,F)), \tag{6.2}$$

where

$$\gamma(X\circ Q(F,F)) = \begin{pmatrix} 0 & 0 \\ X_k^\beta Q(F,F)_{ij}^k & 0 \end{pmatrix}.$$

Equation (6.2) completes the proof of Theorem 7. □

The complete lift ${}^C F$ having the components (5.1) with respect to the natural frame has the components

$$\begin{pmatrix} F_i^h & 0 \\ \sigma_k^\alpha(\partial_i F_h^k - \partial_h F_i^k) - F_i^k\partial_k\sigma_h^\alpha + F_h^k\partial_k\sigma_i^\alpha & F_h^i\delta_\beta^\alpha \end{pmatrix} \tag{6.3}$$

with respect to the frame (B, C) along the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^1, ..., \sigma^n)$.

It is well known that for an arbitrary almost analytic 1-form (or almost analytic covector field) σ on a differentiable manifold M with an almost complex structure F, we have the relation

$$\sigma \circ N_F = 0$$

(see [8]), where N_F is the Nijenhuis tensor for F ([6, p. 38]).

Now by using (6.3) along the cross-section $\sigma(M)$ determined by $\sigma = (\sigma^1, ..., \sigma^n)$ on M, similarly to (6.1) we obtain

$$({}^C F)^2 = {}^C(F^2) + \gamma(\sigma^\beta \circ N_F), \tag{6.4}$$

where

$$\gamma(\sigma^\beta \circ N_\varphi) = \begin{pmatrix} 0 & 0 \\ \sigma_k^\beta N_{ij}^k & 0 \end{pmatrix}.$$

Thus from (6.4) we have

Theorem 8 *Let M be a differentiable manifold with an almost complex structure F. Then the complete lift ${}^C F \in \Im_1^1(F^*M)$ of F, which is restricted to the cross-section $\sigma(M)$ determined by an almost analytic covector field $\sigma^1, ..., \sigma^n$ on M, is an almost complex structure.*

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