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The order supergraph of the power graph of a finite group

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Abstract: The power graph $\mathcal{P}(G)$ is a graph with group elements as a vertex set and two elements are adjacent if one is a power of the other. The order supergraph $\mathcal{S}(G)$ of the power graph $\mathcal{P}(G)$ is a graph with vertex set G in which two elements $x, y \in G$ are joined if $o(x)|o(y)$ or $o(y)|o(x)$. The purpose of this paper is to study certain properties of this new graph together with the relationship between $\mathcal{P}(G)$ and $\mathcal{S}(G)$.

Key words: Power graph, order supergraph, proper order supergraph

1. Introduction

All groups in this paper are finite and we will consider only simple undirected graphs. Suppose Γ and Δ are graphs in which $V(\Gamma) \subseteq V(\Delta)$ and $E(\Gamma) \subseteq E(\Delta)$. Then Γ is called a *subgraph* of Δ and Δ a *supergraph* for Γ. Suppose x, y are vertices of Γ. The length of a minimal path connecting x and y is called the *topological distance* between x and y. The maximum topological distances between vertices of Γ is called its *diameter*. The topological distance between x and y and the diameter of Γ are denoted by $d_\Gamma(x, y)$ and $diam(\Gamma)$, respectively. The number of edges incident to a vertex x is called the *degree* of x denoted by $deg(x)$.

There are several kinds of simple undirected graphs associated with finite groups that are currently of interest in the field. For a finite group G, there are two graphs associated to the set of elements of G, the *power graph* $\mathcal{P}(G)$ and its *order supergraph*, which is denoted by $\mathcal{S}(G)$. This graph is also recorded in the literature as the main supergraph [9]. Two elements $x, y \in G$ are adjacent in the power graph if and only if one is a power of the other. They are joined to each other in $\mathcal{S}(G)$ if and only if $o(x)|o(y)$ or $o(y)|o(x)$. The aim of this paper is studying the following two questions:

1. Which graph can occur as $\mathcal{P}(G)$ or $\mathcal{S}(G)$?

2. What is the structure of G if $\mathcal{P}(G)$ or $\mathcal{S}(G)$ is given?

Suppose Γ is a simple graph with vertex set V and edge set E and R is a partition of V. Define the quotient graph $\widehat{\Gamma}$ with vertex set R. Two vertices A and B in R are adjacent if and only if there exists a vertex in A and another one in B such that they are adjacent in Γ. Bubboloni et al. [2] introduced the notion

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of the order graph of a finite group G, $\mathcal{O}(G)$, as a simple graph with the following vertex and edge sets:

$$\begin{aligned} V(\mathcal{O}(G)) &= \{o(x) \mid x \in G\}, \\ E(\mathcal{O}(G)) &= \{rs \mid \{r, s\} \subseteq V(\mathcal{O}(G)) \ \& \ (r \mid s \ or \ s \mid r)\}. \end{aligned}$$

It is easy to see that $\mathcal{O}(G)$ is isomorphic to a quotient graph of $\mathcal{S}(G)$ in which the set $V(\mathcal{S}(G))$ is partitioned into elements with the same order.

The *directed power graph* of groups and semigroups was introduced by Kelarev and Quinn in their seminal paper [12]. They defined the directed power graph of a group G to be a directed graph with vertex set G and all arcs of the form uv, where v is a power of u. One of the main results of the mentioned paper gives a very technical description of the structure of the power graphs of all finite abelian groups. The same authors [13] studied the power graph of the multiplicative subsemigroup of the ring of $n \times n$ matrices over a skew-field and a subsemigroup of the monoid of row and column-monomial $n \times n$ matrices over a group with 0. We refer the interested readers to consult papers [10, 11] for more information about the power graphs of semigroups.

The undirected power graph of finite groups was introduced by Chakrabarty et al. [6]. Chakrabarty et al. [6] proved that the undirected power graph of a finite group G is complete if and only if G is a cyclic $p-$group, for prime number p. Cameron and Ghosh [4] proved that two abelian groups with isomorphic power graphs must be isomorphic and conjectured that two finite groups with isomorphic power graphs have the same number of elements of each order. This conjecture was responded to affirmatively by Cameron [5]. Mirzargar et al. [15] investigated some combinatorial properties of the power graph of finite groups and in [17] some properties of the power graphs of finite simple groups are considered.

Suppose A is a simple graph and $\mathcal{G} = \{\Gamma_a\}_{a \in A}$ is a set of graphs labeled by vertices of A. Following Sabidussi [19, p. 396], the $A-$join of $\mathcal{G}$ is the graph Δ with the following vertex and edge sets:

$$\begin{aligned} V(\Delta) &= \{(x, y) \mid x \in V(A) \ \& \ y \in V(\Gamma_x)\}, \\ E(\Delta) &= \{(x, y)(x', y') \mid xx' \in E(A) \ or \ else \ x = x' \ \& \ yy' \in E(\Gamma_x)\}. \end{aligned}$$

It is easy to see that this graph can be constructed from A by replacing each vertex $a \in V(A)$ by the graph Γ_a and inserting either all or none of the possible edges between vertices of Γ_a and Γ_b depending on whether or not a and b are joined by an edge in A. If A is an $p-$vertex labeled graph then the $A-$join of $\Delta_1, \Delta_2, \ldots, \Delta_p$ is denoted by $A[\Delta_1, \Delta_2, \ldots, \Delta_p]$.

Suppose G is a finite group. The *exponent* of G, $Exp(G)$, is defined to be the least common multiple of its element orders. This group is called *full exponent* if there is an element $x \in G$ such that $o(x) = Exp(G)$. The set of all prime factors of $|G|$, the set of all element orders of G, and the number of elements of order i in G are denoted by $\pi(G)$, $\pi_e(G)$, and $\omega_i(G)$, respectively. The $\varphi(n)$ denotes the Euler totient function. The notation *complete graph* of order n is denoted by K_n and $K_{m,n}$ denotes the *complete bipartite graph* with parts of sizes m and n, respectively. Our other notations are standard and can be taken from [18, 22]. We encourage the interested readers to consult papers [14, 16] for more information on this topic.

2. Main results

The *proper power graph* $\mathcal{P}^*(G)$ [2, 3] and its *proper order supergraph* $\mathcal{S}^*(G)$ are defined as graphs constructed from $\mathcal{P}(G)$ and $\mathcal{S}(G)$ by removing the identity element of G, respectively. We start this section by comparing the power graph and its order supergraph. It is easy to see that every nonidentity element is adjacent with identity in $\mathcal{S}(G)$, which proves that $\mathcal{S}(G)$ is connected and its diameter is at most two. On the other hand, if G has even order then identity has odd degree and so $\mathcal{S}(G)$ is not Eulerian.

Suppose Γ and Δ are two graphs, u is a vertex of Γ, and v is a vertex in Δ. A splice of Γ and Δ at the vertices u and v is obtained by identifying the vertices u and v in the union of Γ and Δ, which is denoted by $S(\Gamma, \Delta, u, v)$ [8].

Example 2.1 *Consider the dihedral group $D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1}\rangle$. If n is a power of 2 then obviously $\mathcal{S}(D_{2n})$ is a complete graph. If n is odd then sr^i, $1 \leq i \leq n$, are all involutions of D_{2n} which forms an $n-$vertex complete subgraph of $\mathcal{S}(D_{2n})$. Thus, $\mathcal{S}(D_{2n})$ is formed from $\mathcal{S}(Z_n)$ and complete graph K_n, so that all vertices of K_n are joined to the identity element of $\mathcal{S}(Z_n)$. Finally, if n is even but not a power of 2 then $\mathcal{S}(D_{2n})$ can be constructed from $\mathcal{S}(Z_n)$ by adding an $n-$clique T such that each vertex of T is adjacent with all elements of $\langle r\rangle$ of even order.*

It is obvious that $G \cong H$ implies that $\mathcal{S}(G) \cong \mathcal{S}(H)$. If G and H are two nonisomorphic $p-$groups with the same order then $\mathcal{S}(G)$ and $\mathcal{S}(H)$ are complete and so they are isomorphic. Thus, the converse is not generally correct.

Theorem 2.2 *$\mathcal{P}(G) = \mathcal{S}(G)$ if and only if G is cyclic.*

Proof Let G be a cyclic group of order n generated by x. Since $\mathcal{P}(G)$ and $\mathcal{S}(G)$ have the same vertex set G, it is enough to prove that $E(\mathcal{S}(G)) \subseteq E(\mathcal{P}(G))$. Suppose a, b are adjacent vertices in $\mathcal{S}(G)$. Then $o(a)|o(b)$ or $o(b)|o(a)$. Set $a = x^m$ and $b = x^k$. Without loss of generality, we can assume that $o(a)|o(b)$. This implies that $\frac{n}{(n,m)}|\frac{n}{(n,k)}$ and so $(n,k)|(n,m)$. Hence $(n,k)|m$, which shows that $x^m \in \langle x^k\rangle$. Therefore, a and b are adjacent in $E(\mathcal{P}(G))$, as desired.

Conversely, we assume that $E(\mathcal{S}(G)) = E(\mathcal{P}(G))$. Then for each element $x, y \in G$ with $o(x) = o(y)$ we have $\langle x\rangle = \langle y\rangle$. Choose $x \in G$. Hence, there is a unique cyclic subgroup of order $o(x)$ and so each cyclic subgroup is normal. This implies that each subgroup is normal and so G is abelian or Hamiltonian [20]. If G is Hamiltonian, then G can be written as the direct product of the quaternion group Q_8, an elementary abelian $2-$group, and a finite abelian group A of odd order. Since Q_8 has more than one cyclic subgroup, G is abelian. However, every abelian noncyclic group has at least two subgroups of a prime order p, where $p||G|$. Therefore, G is cyclic. □

Theorem 2.3 *Let G be a finite group. $\mathcal{S}(G)$ is complete if and only if G is a $p-$group.*

Proof If G is a finite $p-$group then clearly $\mathcal{S}(G)$ is complete. Conversely, we assume that p and q are two distinct prime divisors of G. Then there are elements x and y of orders p and q, respectively. Therefore, x and y are not adjacent in $\mathcal{S}(G)$, proving the result. □

A group G is said to be periodic if and only if every element of G has finite order. It is possible to define $\mathcal{S}(G)$, when G is periodic and since the identity element is again adjacent to all elements of G, $\mathcal{S}(G)$ will be connected.

Suppose G is a group and Γ is a simple graph. The group G is called an $EPPO-$group, if all elements of G have prime power order. It is an $EPO-$group, if all elements have prime order. An independent set for Γ

is a set of vertices such that no two of which are adjacent. The cardinality of an independent set with maximum size is called the independent number of Γ, denoted by $\alpha(\Gamma)$.

Theorem 2.4 $|\pi(G)| \leq \alpha(\mathcal{S}(G)) \leq |\pi_e(G)| - 1$. *The right-hand equality is attained if and only if G is an $EPO-$group.*

Proof Suppose $|G| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $r \geq 1$ and p_i, $1 \leq i \leq r$, are prime numbers. Thus $|\pi(G)| = r$. Choose elements g_i of order p_i, $1 \leq i \leq r$ and set $A = \{g_1, \ldots, g_r\}$. Then A is an independent subset of G and so $|\pi(G)| \leq \alpha(\mathcal{S}(G))$. We now assume that $\pi_e(G) = \{a_1, \ldots, a_k\}$. Define a graph Δ such that $V(\Delta) = \pi_e(G)$ and two vertices x and y are adjacent if and only if $x|y$ or $y|x$. Define the induced subgraph R_i of $\mathcal{S}(G)$ with the set of all elements of order a_i as vertex set. Then R_i is a complete graph of order $\omega_{a_i}(G)$. Then $\mathcal{S}(G) = \Delta[R_1, \ldots, R_k] \cong \Delta[K_{\omega_{a_1}(G)}, \ldots, K_{\omega_{a_k}(G)}]$. If the order of an element in $K_{\omega_{a_i}(G)}$ divides the order of an element in $K_{\omega_{a_j}(G)}$ then all elements of $K_{\omega_{a_i}(G)}$ will be adjacent to all elements of $K_{\omega_{a_j}(G)}$. As a consequence $\alpha(\mathcal{S}(G)) \leq |\pi_e(G)| - 1$.

If G is an $EPO-$group then all elements of order p, p is prime, will be a clique. Hence, $\alpha(\mathcal{S}(G)) = |\pi_e(G)| - 1$. Conversely, we assume that $\alpha(\mathcal{S}(G)) = |\pi_e(G)| - 1$. If G has a nonidentity element of a nonprime order a then there exists an element $b \in \langle a \rangle$ of prime order. Thus, each element of order $o(a)$ will be adjacent to each element of order $o(b)$, which contradicts our assumption. Therefore, all nonidentity elements of G have prime order, as desired. $\square$

It can be easily seen that if G is an $EPPO-$group then $|\pi(G)| = \alpha(\mathcal{S}(G))$. On the other hand, $2 = |\pi(Z_6)| = \alpha(\mathcal{S}(Z_6))$, but Z_6 is not an $EPPO-$group. Thus, the following question remains open:

Question 2.5 *What is the structure of groups with $|\pi(G)| = \alpha(\mathcal{S}(G))$?*

For a given group G, the number of edges in the order supergraph $\mathcal{S}(G)$ is denoted by $e(\mathcal{S}(G))$. In the following theorem an exact expression for $e(\mathcal{S}(G)) = |E(\mathcal{S}(G))|$ is calculated.

Theorem 2.6 $e(\mathcal{S}(G)) = \frac{1}{2}\sum_{x\in G}\left(2\sum_{d|o(x)}\omega_d(G) - \omega_{o(x)}(G) - 1\right)$.

Proof Define the directed graph $\overrightarrow{\mathcal{S}}(G)$ with vertex set G and arc set $E(\overrightarrow{\mathcal{S}}(G)) = \{(x, g) \mid o(g)|o(x)\}$. Suppose $x \in G$. Then

$$Outdeg(x) = |\{g \in G, o(g)|o(x)\}| = \sum_{d|o(x)} \omega_d(G) - 1.$$

It is clear that the whole number of arcs is equal to the sum of $Outdeg(x)$ overall vertices of $\overrightarrow{\mathcal{S}}(G)$.

On the other hand, $\sum_{x\in V(\overrightarrow{\mathcal{S}}(G))} Outdeg(x) = 2e(\overrightarrow{\mathcal{S}}(G))$. To compute the number of edges in $\mathcal{S}(G)$, it is sufficient to count once the edges of $\overrightarrow{\mathcal{S}}(G)$ with two different directions. However, we have an undirected edge connecting g and x, when $o(g)|o(x)$ and $o(x)|o(g)$. In such a case, we will have $o(x) = o(g)$. Therefore, the number of edges with two different directions is $\omega_{o(x)}(G) - 1$ and so

$$e(\mathcal{S}(G)) = \frac{1}{2}\sum_{x\in G}\left(2\sum_{d|o(x)}\omega_d(G) - \omega_{o(x)}(G) - 1\right),$$

proving the result. $\square$

In the following corollary, we apply the previous theorem to present a new proof for [6, Corollary 4.3].

Corollary 2.7 $2e(\mathcal{P}(Z_n)) = \sum_{d|n} [2d - \varphi(d) - 1] \varphi(d)$.

Proof It is easy to check the following formulas:

$$\begin{aligned} \sum_{x \in Z_n} \sum_{d | o(x)} \omega_d(Z_n) &= \sum_{d|n} d\varphi(n), \\ \sum_{x \in Z_n} \left[\omega_{o(x)}(Z_n) - 1\right] &= \sum_{d|n} \varphi^2(d) - \sum_{d|n} \varphi(d). \end{aligned}$$

Therefore, by Theorem 2.6 and the fact that $\mathcal{P}(Z_n) = \mathcal{S}(Z_n)$, $2e(\mathcal{P}(Z_n)) = 2e(\mathcal{S}(Z_n)) = \sum_{d|n} [2d - \varphi(d) - 1] \varphi(d)$, as desired. □

Theorem 2.8 *Let G be a finite group of order $p_1^{n_1} \cdots p_k^{n_k}$ and*

$$V_i = \{g \in G \mid g \neq 1, |g| \, |p_i^{n_i} \}.$$

G is an $EPPO-$group if and only if $\mathcal{S}(G) = K_1 + (\bigcup_{i=1}^{k} K_{|V_i|})$.

Proof Suppose G is an $EPPO-$group; then by the structure of the order supergraph of G, each V_i is a clique in $\mathcal{S}(G)$ and there is no edge connecting V_i and V_j, $i \neq j$. Hence $\mathcal{S}(G) = K_1 + (\bigcup_{i=1}^{k} K_{|V_i|})$.

Conversely, we assume that $\mathcal{S}(G) = K_1 + (\bigcup_{i=1}^{k} K_{|V_i|})$, $e \neq g \in G$ and $p_i p_j || G|$. We also assume that g_i and g_j are two elements in G such that $o(g_i) = p_i$ and $o(g_j) = p_j$. Obviously, $g_i \in V_i$ and $g_j \in V_j$ and so $g_i \in K_{|V_i|}$ and $g_j \in K_{|V_j|}$. Since $|g_i|||g|$ and $|g_j|||g|$, g is adjacent to g_i and g_j, which is impossible. Therefore, G is an $EPPO-$group, as desired. □

Corollary 2.9 *Suppose G is an $EPPO-$group that is not a $p-$group. Then the number of components in $\mathcal{S}^*(G)$ is equal to $|\pi(G)|$.*

Following Williams [23], we assume that G is a finite group and construct its prime graph as follows: the vertices are the primes dividing the order of the group, and two vertices p and q are joined by an edge if and only if G contains an element of order pq.

Corollary 2.10 *If the prime graph of a group G is totally disconnected then $\mathcal{S}^*(G)$ is disconnected.*

Proof It is clear that G is an $EPPO-$group if and only if its prime graph is totally disconnected. We now apply Corollary 2.9 to deduce the result. □

The vertex connectivity of a graph Γ is the minimum number of vertices, $\kappa(\Gamma)$, whose deletion from Γ disconnects it.

Theorem 2.11 *The vertex connectivity of $\mathcal{S}(G)$ can be computed as follows:*

- $\kappa(\mathcal{S}(Z_n)) = n - 1$, *where $n = p^m$, p is prime and m is a nonnegative integer.*

- *Suppose n is not a prime power. Then $\kappa(\mathcal{S}(Z_n)) \geq \varphi(n)+1$. The equality is satisfied if and only if $n = pq$, p and q are distinct prime numbers.*

Proof

If $n = p^m$ then the graph $\mathcal{S}(G)$ is an $n-$vertex complete graph and so $\kappa(\mathcal{S}(Z_n)) = n-1$. To prove the second part, we have to note that if an $n-$vertex graph Γ has exactly s elements of degree $n-1$ then $\kappa(G) \geq s$. In fact, we have to delete all such elements to find a disconnected graph. Since n is not a prime power and the cyclic group Z_n has exactly $\varphi(n)$ generators, it has at least $\varphi(n)+1$ elements of degree $n-1$. Therefore, $\kappa(\mathcal{S}(Z_n)) \geq \varphi(n)+1$. If the equality is satisfied and n is divisible by at least three different primes then by deleting the identity and all elements of order n the resulting graph H will be connected. To prove this, we first assume that n is not a square-free integer with exactly three prime factors. Choose two elements g and h in H such that $o(g) = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, $r \geq 3$, and $o(h) = q_1^{\beta_1}q_2^{\beta_2}\cdots q_t^{\beta_t}$, where p_i's and q_j's are prime numbers; α_i's and β_j's are positive integers. If u is an element of order p_1q_1 then $(o(g), o(u)) = p_1$ and $(o(h), o(u)) = q_1$, and there are paths connecting u and g as well as u and h. This implies that there exists a path connecting g and h and so $\kappa(\mathcal{S}(Z_n)) > \varphi(n)+1$. Thus, it is sufficient to check the case that $n = pqr$. Since Z_n has elements of orders pq, pr, and qr, all elements of the graph obtained from $\mathcal{S}(Z_n)$ by deleting the identity and all elements of order pqr will be again connected and so $\kappa(\mathcal{S}(Z_n)) > \varphi(n)+1$. Thus, by our assumption, n has exactly two prime factors, say $n = p^kq^l$, where p, q are distinct primes and k, l are positive integers. If $k \geq 2$ or $l \geq 2$ then by choosing an element of order pq and applying a similar argument as above, we can see again the resulting graph will be connected. Therefore, $n = pq$, where p and q are distinct prime numbers. Conversely, if $n = pq$ then clearly $\kappa(\mathcal{S}(Z_n)) = \varphi(n)+1$, which completes the proof. □

Kuratowski's theorem states that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$. Here a subdivision of a graph Γ is a graph resulting from the subdivision of edges in Γ. In what follows, we apply this theorem to give a classification of the planar order supergraph of a finite group.

Theorem 2.12 *The order supergraph of a finite group G is planar if and only if $G \cong 1, Z_2, Z_3, Z_4, Z_2 \times Z_2$ or S_3.*

Proof Suppose the order supergraph of G is planar, $n = Max\{o(x) \mid x \in G\}$, and $x \in G$ is an element of order n. If $5|n$ or n is a prime number ≥ 5 then $\mathcal{S}(G)$ has a subgraph isomorphic to K_5. On the other hand, $\langle x \rangle$ has exactly $\varphi(n)$ generators and so $\mathcal{S}(G)$ has a subgraph isomorphic to $K_{\varphi(n)+1}$. Apply Kuratowski's theorem, we deduce that $n = 2^\alpha 3^\beta$. We claim that $\alpha \leq 2$ and $\beta \leq 1$. Otherwise, $\mathcal{S}(G)$ has an induced subgraph isomorphic to K_5, a contradiction. Thus, $n \in \{1, 2, 3, 4, 6, 12\}$. If G has an element of order 6 or 12 then we will have again a subgraph isomorphic to K_5 and so $\pi_e(G) \subseteq \{1, 2, 3, 4\}$. Since elements of the same order in G constitute a clique in $\mathcal{S}(G)$, $\omega_i(G) < 4$, when $i \in \{2, 3, 4\}$. Again since all elements of order 2 are adjacent to all elements of order 4, $\omega_2(G) + \omega_4(G) < 4$. Since $\pi_e(G) \subseteq \{1, 2, 3, 4\}$ and $\omega_i(G) < 4$, $2 \leq i \leq 4$, by counting the number of elements of each order we have $|G| \leq 1+3+3+3 = 10$ and a simple calculation by small group library of GAP [21] proves that G is isomorphic to $1, Z_2, Z_3, Z_4, Z_2 \times Z_2$ or S_3. The converse is clear. □

By the previous Theorem, the graph $\mathcal{S}(Z_n)$ is planar if and only if $n < 5$. Suppose Γ is a finite graph. The clique number of Γ, $\omega(\Gamma)$, is the size of a maximal clique in Γ and the chromatic number of Γ, $\chi(\Gamma)$, is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. It is clear that $\chi(\Gamma) \geq \omega(\Gamma)$.

Corollary 2.13 *If the order supergraph of a finite group G is planar then $\chi(\mathcal{S}(G)) = \omega(\mathcal{S}(G))$.*

Theorem 2.14 *The proper order supergraph of a finite group G is planar if and only if $G \cong 1, Z_2, Z_3, Z_4, Z_2 \times Z_2, Z_5, Z_6$ or S_3.*

Proof Suppose $\mathcal{S}^\star(G)$ is planar, $n = Max\{o(g) \mid g \in G\}$, and $x \in G$ has order n. Since $\langle x \rangle$ has exactly $\varphi(n)$ elements of order n and these elements constitute a clique in $\mathcal{S}^\star(G)$, $\varphi(n) < 5$. On the other hand, if n is a prime power and $n \geq 7$ then $\mathcal{S}^\star(G)$ has a clique isomorphic to K_5, which is impossible. Thus $\pi(G) \subseteq \{2, 3, 5\}$ and $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6, 10, 12\}$. If G has an element x of order 10 or 12 then the set A containing all generators of $\langle x \rangle$ and x^2 will be a clique of order 5, which leads us to another contradiction. This shows that $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\}$. Since elements of the same order are a clique in the planar graph $\mathcal{S}^\star(G)$, $\omega_i(G) \leq 4$, for $i \leq 6$. Finally, the elements of orders $2, 4$, the elements of orders $2, 6$, and the elements of orders $3, 6$ constitute three cliques in $\mathcal{S}^\star(G)$ and so $\omega_2(G) + \omega_4(G) \leq 4$, $\omega_2(G) + \omega_6(G) \leq 4$, and $\omega_3(G) + \omega_6(G) \leq 4$. Since $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\}$ and $\omega_i(G) \leq 4$, $2 \leq i \leq 6$, by counting the number of elements of each order $|G| \leq 1 + 4 + 4 + 4 + 4 + 4 = 21$ and a simple calculation by small group library of GAP proves that $G \cong 1, Z_2, Z_3, Z_4, Z_2 \times Z_2, Z_5, Z_6$ or S_3. The converse is obvious. □

Suppose G is a finite group and $S(G)$ is the set of all elements $x \in G$ such that x is adjacent to all elements of $G \setminus \{x\}$ in $\mathcal{S}(G)$. In the following theorem, the group G with $|S(G)| > 1$ is characterized.

Theorem 2.15 *$|S(G)| > 1$ if and only if G is a nontrivial full exponent finite group.*

Proof Suppose $|S(G)| > 1$ and $e \neq g \in S(G)$. If G is a $p-$group then G is a full exponent, as desired. We assume that G does not have prime power order and p is a prime factor of $|G|$. Then obviously $p|o(g)$ and so $o(g)$ is divisible by all prime factors of $|G|$. Choose an element $x \in G$ of order p^r, where p is a prime number and r is a nonnegative integer. Since $deg(g) = |G| - 1$, g and x are adjacent. Thus, $o(x)|o(g)$ or $o(g)|o(x)$. By our assumption, G does not have prime power order and hence $o(x)|o(g)$. This shows that $o(g) = Exp(G)$ and therefore G is a full exponent.

Conversely, we assume that G is a full exponent and a is an element of G such that $o(a) = Exp(G)$. Thus a is adjacent to all elements of $G \setminus \{a\}$. This proves that $|S(G)| > 1$, proving the result. □

Corollary 2.16 *If G is an isomorphic nilpotent group, a dihedral group of order $2n$ with even n, $H \times Z_n$ such that H is an arbitrary finite group and $n = Exp(H)$ or H^n with $n = |\pi(H)|$ then $|S(G)| > 1$.*

Theorem 2.17 *The order supergraph of a finite group is bipartite if and only if $G \cong 1, Z_2$.*

Proof If $|G|$ has an odd prime factor p and $x \in G$ has order p, then $\langle x \rangle$ is a subgraph isomorphic to K_p, which is impossible. Thus G is a $2-$group and, by Theorem 2.3, G is complete. This shows that $n \leq 2$, as desired. □

Corollary 2.18 *The order supergraph of a finite group is a tree if and only if $G \cong 1, Z_2$.*

Theorem 2.19 *Let G be a nontrivial finite group. Then $\mathcal{S}^*(G)$ is bipartite if and only if $G \cong Z_2$ or Z_3.*

Proof Suppose $\mathcal{S}^*(G)$ is bipartite and G has an element x of order ≥ 4. It is well known that if $n \geq 4$ then $\varphi(n) \geq 2$. Hence the cyclic group $\langle x \rangle$ has at least two generators x, x^t, where $t > 1$. Choose an element x^j different from x and x^t and so $j \neq 1, t$. Since $o(x^j)|o(x) = o(x^t)$, x, x^j, x^t, x is a cycle of length 3, contradicted by bipartivity of $\mathcal{S}^*(G)$. Thus $\pi_e(G) \subseteq \{1, 2, 3\}$. On the other hand, since all element of a given order constitute a clique in $\mathcal{S}^*(G)$, $\omega_i(G) \leq 2$, $i = 2, 3$. This implies that $|G| \leq 5$. Therefore, $G \cong Z_2$ or Z_3. □

Corollary 2.20 *The graph $\mathcal{S}^*(G)$ is a tree if and only if $G \cong Z_2$ or Z_3.*

Suppose Γ is a connected graph with exactly n vertices and m edges. If $c = m - n + 1$ then Γ is called $c-$cyclic. It is easy to see that $c = 0$ if and only if Γ is a tree. In the case that $c = 1, 2, 3, 4$, and 5, we call Γ unicyclic, bicyclic, tricyclic, tetracyclic, and pentacyclic, respectively.

Lemma 2.21 *The order supergraph $\mathcal{S}(G)$ has a pendant vertex if and only if $|G| \leq 2$.*

Proof Suppose $|G| > 1$ and x is a nonidentity element such that $deg_{\mathcal{S}(G)}(x) = 1$. If $o(x) > 2$ then there exists $y \in \langle x \rangle$ such that $xy, xe \in E(\mathcal{S}(G))$, which is impossible. Hence G is elementary abelian and in such a case all involutions are adjacent. This shows that $G \cong Z_2$, proving the lemma. □

Theorem 2.22 *Let G be a finite group. The following hold:*

1. *The order supergraph $\mathcal{S}(G)$ is unicyclic if and only if $G \cong Z_3$.*
2. *The order supergraph $\mathcal{S}(G)$ cannot be bicyclic and pentacyclic.*
3. *The order supergraph $\mathcal{S}(G)$ is tricyclic if and only if $G \cong Z_4$ or $Z_2 \times Z_2$.*
4. *The order supergraph $\mathcal{S}(G)$ is tetracyclic if and only if $G \cong D_6$.*

Proof Suppose $|G| = n$. Our main proof will have some separate cases as follows:

1. Assume that $\mathcal{S}(G)$ is unicyclic. Thus $m = n$. Since $deg(e) = n - 1$ the edges in $\mathcal{S}(G)$ are those adjacent to e and only a further edge. Thus $\mathcal{S}(G)$ is a triangle or it contains some pendant vertex. In that last case, by Lemma 2.21 we have $G \cong Z_2$ and the graph is a tree, a contradiction. Thus $\mathcal{S}(G)$ is necessarily a triangle. This gives $|G| = 3$ and thus $G \cong Z_3$. The converse is clear.

2. Since $\mathcal{S}(G)$ is bicyclic, $\mathcal{S}^*(G)$ has exactly two edges. By Lemma 2.21, if these edges are incident then $|G| = 4$ and in another case we have $|G| = 5$. In both cases, the resulting order supergraph will not be bicyclic, a contradiction. Assume next that $\mathcal{S}(G)$ is pentacyclic. Then $G \ncong 1, Z_2$ and thus, by Lemma 2.21, $\mathcal{S}(G)$ has no pendant vertex. The graph $\mathcal{S}^*(G)$ contains five edges. Those edges can be incident or not. If they are not incident two by two then $\mathcal{S}(G)$ is a friendship graph on 11 vertices. On the other hand, $\mathcal{S}^*(G)$ must contain at least four vertices because if there are only three vertices there are at most three edges. It follows that $5 \leq |G| \leq 11$. A case by case investigation of all groups with orders $5 \leq |G| \leq 11$ shows that there is no group G with a pentacyclic order supergraph.

3. Since $\mathcal{S}(G)$ is tricyclic, $\mathcal{S}^*(G)$ has exactly three edges. A case by case investigation shows that we have five possible graph structures for $\mathcal{S}(G)$ depending on whether those edges can be incident or not. On the other hand, a similar argument as case (2) shows that $\mathcal{S}(G)$ must contain at least four and at most seven vertices, i.e. $4 \leq |G| \leq 7$. By considering all groups of these orders, one can easily prove that $|G| = 4$, as desired.

4. Since $\mathcal{S}(G)$ is tetracyclic, $\mathcal{S}^*(G)$ has exactly four edges. A similar argument like other cases shows that $5 \leq |G| \leq 9$ and a case by case investigation leads us to the result.

This completes the proof. $\square$

Theorem 2.23 *Let G be a finite group and $e = xy \in E(\mathcal{S}^*(G))$. e is a cut edge if and only if $\{x, y\}$ is a component of $\mathcal{S}(G)$.*

Proof Suppose $e = xy$ is a cut edge for $\mathcal{S}^*(G)$. Without loss of generality, we assume that $o(y)|o(x)$. Suppose $o(x) = 2$. Then $o(y) = 2$ and obviously G does not have another element of even order. This shows that $\{x, y\}$ is a component of $\mathcal{S}^*(G)$. If $o(x) = 3$ then $o(y) = 3$ and the same argument shows that $\{x, y\}$ is a component of $\mathcal{S}^*(G)$. Thus, we can assume $o(x) \geq 4$. In this case, since $\varphi(o(x)) \geq 2$, the cyclic subgroup $\langle x \rangle$ has at least two generators and one more element z such that $o(z)|o(x)$. This implies that e is an edge of a cycle, which is impossible. $\square$

The graph Γ is called perfect, if $\chi(\Gamma) = \omega(\Gamma)$, where $\chi(\Gamma)$ and $\omega(\Gamma)$ denote the chromatic and clique numbers of G, respectively.

Theorem 2.24 (*Chudnovsky et al.* [7]) *The graph Γ is perfect if and only if Γ and $\overline{\Gamma}$ do not have an induced odd cycle of length at least five.*

Theorem 2.25 *The order supergraph of a finite group is perfect.*

Proof Suppose $\mathcal{S}(G)$ has an induced odd cycle C of length at least five and $\vec{C}$ is its associated directed cycle in the directed order supergraph of G. Choose a directed path $x \to y \to z$ in $\vec{C}$. Then $o(y)|o(x)$ and $o(z)|o(y)$, which implies that $xz \in E(\mathcal{S}(G))$. Therefore, $\mathcal{S}(G)$ has an odd cycle of length 3, contradicted by Theorem 2.24.

We now assume that $\overline{C}$ is an induced odd cycle of length at least five in $\overline{\mathcal{S}(G)}$. If $l(\overline{C}) = 5$ then C is an induced cycle of length 5, contradicted by the first part of the proof. Hence, we can assume that $l(\overline{C}) \geq 7$. Then C contains a triangle. On the other hand, for each $x \in V(\vec{C})$, $Indeg(x) = 0$ or $Outdeg(x) = 0$, which is not correct for a triangle. This leads us to another contradiction. Hence by Theorem 2.24 the order supergraph $\mathcal{S}(G)$ is perfect. $\square$

In what follows, we apply results of [2, 3] to compute the number of connected components of $\mathcal{S}^*(A_n)$ and $\mathcal{S}^*(S_n)$, where A_n and S_n denote as usual the alternating and symmetric groups on n symbols. In our next results, P stands for the set of all prime numbers and $aP + b = \{ap + b \mid p \in P\}$.

Lemma 2.26 *If $n \in \{p, p+1\}$, $p \in P$, then elements of order p is a component of $\mathcal{S}^*(S_n)$. If $n \geq 4$ and $n \in \{p, p+1, p+2\}$, $p \in P$, then elements of order p are a component of $\mathcal{S}^*(A_n)$.*

Proof The proof follows from the fact that the groups A_n and S_n do not have elements of order kp, $k \geq 2$. $\square$

Lemma 2.27 (Bubboloni et al. [2, Page 24]) *Suppose* $B_1(n) = P \cap \{n, n-1\}$ *and* $\Sigma_n = \{g \in S_n \mid o(g) \notin B_1(n)\}$, $n \geq 8$. *Then* Σ_n *is a component of* $\mathcal{S}^*(S_n)$.

Set $A = P \cup (P+1) \cup (P+2) \cup 2P \cup (2P+1)$. For the sake of completeness, we mention here a result of [3].

Theorem 2.28 (Bubboloni et al. [3, Corollary C]) *The proper power graph* $\mathcal{P}^*(A_n)$ *is connected if and only if* $n = 3$ *or* $n \notin A$.

Corollary 2.29 *If* $n = 3$ *or* $n \notin A$ *then* $\mathcal{S}^*(A_n)$ *is connected.*

Proof If $n = 3$ or $n \notin A$ then by Theorem 2.28, the proper power graph $\mathcal{P}^*(A_n)$ is connected. The result now follows from this fact that $\mathcal{P}^*(A_n)$ is a subgraph of $\mathcal{S}^*(A_n)$. $\square$

Define $B_2(n) = \{n \mid n \in P, n-1 \in P, n-2 \in P, \frac{n}{2} \in P or \frac{n-1}{2} \in P\}$, $\theta_n = (\pi_e(A_n) \setminus \{1\}) \setminus B_2(n)$ and $\overline{\xi}_n = \{g \in A_n \mid o(g) \in \theta_n\}$. In what follows, the number of connected components of a graph Γ is denoted by $c(\Gamma)$. The elements of $B_2(n)$ are called the critical orders of n.

Corollary 2.30 (Bubboloni et al. [3, Page 15]) *There exists a component in* $\mathcal{S}^*(A_n)$ *containing* $\overline{\xi}_n$.

Theorem 2.31 *The proper order supergraph* $\mathcal{S}^*(A_n)$ *is connected if and only if* $n = 3$ *or* $n, n-1$ *and* $n-2$ *are not prime integers. The maximum number of connected component is* 3 *and the graph* $\mathcal{S}^*(A_n)$ *attained the maximum possible of component if and only if* $n \in P \cap (P+2)$.

Proof We first assume that $n \leq 10$. The graph $\mathcal{S}^*(A_2)$ is empty and obviously $c(\mathcal{S}^*(A_3)) = 1$. Our calculations by GAP [21] show that

$$\begin{aligned}
\pi_e(A_4) &= \{1,2,3\}, \\
\pi_e(A_5) &= \{1,2,3,5\}, \\
\pi_e(A_6) &= \{1,2,3,4,5\}, \\
\pi_e(A_7) &= \{1,2,3,4,5,6,7\}, \\
\pi_e(A_8) &= \{1,2,3,4,5,6,7,15\}, \\
\pi_e(A_9) &= \{1,2,3,4,5,6,7,9,10,12,15\}, \\
\pi_e(A_{10}) &= \{1,2,3,4,5,6,7,8,9,10,12,15,21\}.
\end{aligned}$$

From the set of element orders one can see that $c(\mathcal{S}^*(A_4)) = 2$, $c(\mathcal{S}^*(A_5)) = c(\mathcal{S}^*(A_6)) = c(\mathcal{S}^*(A_7)) = 3$, $c(\mathcal{S}^*(A_8)) = c(\mathcal{S}^*(A_9)) = 2$, and $c(\mathcal{S}^*(A_{10})) = 1$. If $n \geq 11$ then a similar argument as the proof of [3, Theorem A and Corollary C] will complete the proof. $\square$

Theorem 2.32 *The number of connected components of* $\mathcal{S}^*(S_n)$ *can be computed as follows:*

1. $c(\mathcal{S}^*(S_2)) = 1$ *and* $c(\mathcal{S}^*(S_3)) = c(\mathcal{S}^*(S_4)) = c(\mathcal{S}^*(S_5)) = c(\mathcal{S}^*(S_6)) = c(\mathcal{S}^*(S_7)) = 2$.

2. *If* $n \geq 8$ *then* $c(\mathcal{S}^*(S_n)) = \begin{cases} 1 & n \notin P \cup (P+1) \\ 2 & otherwise \end{cases}$.

Proof The proof is completely similar to the proof of [2, Theorem B and Corollary C] and so it is omitted. □

Corollary 2.33 *(See Bubboloni et al. [2, Corollary C]) If* $n \geq 2$ *then* $\mathcal{S}^*(S_n)$ *is* $2-$*connected if and only if* $n = 2$ *or* $n \notin P \cup (P+1)$.

Since the power graph of a group G is a spanning subgraph of the order supergraph of G, it is easy to see that $2-$connectedness of the power graph implies the same property for the order supergraph. Akbari and Ashrafi [1] conjectured that the power graph of a nonabelian finite simple group G is $2-$connected if and only if $G \cong A_n$, where $n = 3$ or $n \notin P \cup (P+1) \cup (P+2) \cup 2P \cup (2P+1)$. They verified this conjecture in some classes of simple groups. We now apply the order supergraph to prove that the power graphs of some other classes of simple groups are not $2-$connected.

Theorem 2.34 *The order supergraphs of sporadic groups are not* $2-$ *connected.*

Proof It is easy to see that if we can partition $\pi_e(G)$ into subsets $L(G)$ and $K(G)$ such that any element of $L(G)$ is co-prime with any element of $K(G)$ then order supergraph of G is not $2-$connected. In the Table, the set $L(G)$ for a sporadic group G is computed. By this table, one can see that there is no sporadic simple group with $2-$connected order supergraph. □

Table. The set $L(G)$, for all sporadic groups G.

$L(M_{11}) = \{11, 5\}$	$L(M_{12}) = \{11\}$	$L(McL) = \{11\}$
$L(M_{23}) = \{11, 23\}$	$L(M_{24}) = \{11, 23\}$	$L(Co_2) = \{11, 23\}$
$L(HS) = \{7, 11\}$	$L(J_2) = \{7\}$	$L(Co_1) = \{23\}$
$L(Co_2) = \{23\}$	$L(Suz) = \{11, 13\}$	$L(He) = \{17\}$
$L(HN) = \{19\}$	$L(Th) = \{13, 19, 31\}$	$L(Fi_{22}) = \{13\}$
$L(Fi_{23}) = \{17, 23\}$	$L(Fi'_{24}) = \{17, 23, 29\}$	$L(B) = \{31, 47\}$
$L(M) = \{41, 71, 59\}$	$L(J_1) = \{7, 11, 19\}$	$L(O'N) = \{11, 19, 31\}$
$L(J_3) = \{17, 19\}$	$L(Ru) = \{29\}$	$L(J_4) = \{23, 31, 29, 37, 43\}$
$L(M_{22}) = \{5, 7, 11\}$	$L(Ly) = \{31, 37, 67\}$	

If two elements x and y in a finite group G have co-prime order then obviously they cannot be adjacent in $\mathcal{P}(G)$. In [1], the authors applied this simple observation to prove that the power graphs of some classes of simple groups are not connected. The following result is an immediate consequence of [1, Theorems 2–10].

Theorem 2.35 *The order supergraphs of the following simple groups are not* $2-$ *connected:*

1. ${}^2F_4(q)$, *where* $q = 2^{2m+1}$ *and* $m \geq 1$;
2. ${}^2G_2(q)$, *where* $q = 3^{2m+1}$ *and* $m \geq 0$;
3. $A_1(q)$, $A_2(q)$, $B_2(q)$, $C_2(q)$, *and* $S_4(q)$, *where* q *is an odd prime power;*
4. $F_4(2^m)$, $m \geq 1$, *and* $U_3(q)$, *where* q *is a prime power.*

At the end of this paper, we have to say that it remains an open question to classify all simple groups with $2-$connected order supergraph.

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