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Bounded solutions and asymptotic stability of nonlinear second-order neutral difference equations with quasi-differences

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Abstract: This work is devoted to the study of the nonlinear second-order neutral difference equations with quasi-differences of the form

$$\Delta\left(r_n\Delta\left(x_n+q_nx_{n-\tau}\right)\right)=a_nf(x_{n-\sigma})+b_n$$

with respect to (q_n). For $q_n\to 1$, $q_n\in(0,1)$ the standard fixed point approach is insufficient to get the existence of the bounded solution, so we combine this method with an approximation technique to achieve our goal. Moreover, for $p\geq 1$ and $\sup|q_n|<2^{1-p}$, using Krasnoselskii's fixed point theorem we obtain sufficient conditions for the existence of the solution that belongs to l^p space.

Key words: Nonlinear neutral difference equation, Krasnoselskii's fixed point theorem, approximation

1. Introduction

Difference equations are used in mathematical models in diverse areas such as economy, biology, and computer science; see, for example, [1, 7]. In the past thirty years, oscillation, nonoscillation, and the asymptotic behavior and existence of bounded solutions to many types of second-order difference equations have been widely examined; see, for example, [2, 4, 6, 9–11, 13, 14, 17–20, 26–31], and references therein.

The second-order difference equation with quasi-difference of the form

$$\Delta\left(r_n\Delta\left(x_n+q_nx_{n-\tau}\right)\right)=F(n,x_{n-\sigma})$$

is studied in the literature with respect to a sequence (q_n). Fixed point theory is the standard technique to prove the existence of the bounded solution to the considered problem with constant (q_n) and sequences (q_n) for which absolute values are lower or grater than 1. Let us present a short overview of papers that deal with this problem. By using Banach's fixed point theorem, Jinfa [12] and Liu et al. [15] investigated the nonoscillatory solution to the second-order neutral delay difference equation of the following form:

$$\Delta\left(r_n\Delta\left(x_n+qx_{n-\tau}\right)\right)+f(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}})=c_n.$$

Liu et al. [15] proved the existence of uncountably many bounded nonoscillatory solutions for the above problem under the Lipschitz continuity condition. With the Leray–Schauder type of condensing operators Agarwal et

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al. [2] examined the existence of a nonoscillatory solution to the problem

$$\Delta\left(r_n\Delta\left(x_n + qx_{n-\tau}\right)\right) + F(n+1, x_{n+1-\sigma}) = 0,$$

where $q \in \mathbb{R}\setminus\{\pm 1\}$. Liu et al. [16] discussed the existence of uncountably many bounded positive solutions to

$$\Delta\left(r_n\Delta\left(x_n + b_n x_{n-\tau} - c_n\right)\right) + f(n, x(f_1(n)), \dots, x(f_k(n))) = d_n,$$

where $\sup_{n\in\mathbb{N}} b_n = b^\star$, $b^\star \neq 1$ or $\inf_{n\in\mathbb{N}} b_n = b_\star$, $b_\star \neq -1$ by Krasnoselskii's fixed point theorem.

On the other hand, Petropoulous and Siafarikas considered different types of difference equations in Hilbert space; see [21–23]. Moreover, the functional-analytical method for general nonautonomous difference equations of the form $x_{k+1} = f_k(x_k, x_{k+1})$ was considered by Ey and Pötzsche [8] and Pötzsche [24]. This approach allows us to better characterize solutions to difference equations.

In this paper we study the following second-order neutral difference equation with quasi-difference

$$\Delta\left(r_n\Delta\left(x_n + q_n x_{n-\tau}\right)\right) = a_n f(x_{n-\sigma}) + b_n, \tag{1}$$

where $\tau \in \mathbb{N}_0$, $\sigma \in \mathbb{Z}$, $a, b, q : \mathbb{N}_0 \to \mathbb{R}$, $r : \mathbb{N}_0 \to \mathbb{R}\setminus\{0\}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. If the sequence (q_n) is convergent to 1, then the fixed point approach can not be applied to solve the studied problem, because Krasnoselskii's fixed point theorem need two operators, one of which is a contraction. To overcome the limitation of this method we combine this approach with the approximation technique under the additional assumption $q_n \in (0,1)$. The approximation approach in this type of difference equations and the case when (q_n) is convergent to 1 has not been discussed so far, to our knowledge. Moreover, in the case $p \geq 1$ and $\sup|q_n| < 2^{1-p}$ we establish sufficient conditions of the existence of the solution to (1), which belongs to l^p space. To get our result Krasnoselskii's fixed point theorem is used.

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator, $\mathbb{N}_k := \{k, k+1, \dots\}$, where k is a given nonnegative integer and $\mathbb{R}$ is a set of all real numbers.

Let $k \in \mathbb{N}_0$. We consider the Banach space l_k^∞ of all real bounded sequences $x\colon \mathbb{N}_k \to \mathbb{R}$ equipped with the standard supremum norm, i.e.

$$\|x\| = \sup_{n\in\mathbb{N}_k} |x_n|, \text{ for } x = (x_n)_{n\geq k} \in\ l_k^\infty.$$

Definition 1 *[5] A subset A of l_k^∞ is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_k$ such that $|x_i - x_j| < \varepsilon$ for any $i, j \geq n_0$ and $x = (x_n) \in A$.*

Theorem 1 *[5] A bounded, uniformly Cauchy subset of l_k^∞ is relatively compact.*

For a real $p \geq 1$ we define l_k^p as the Banach space of p-summable sequences as follows:

$$l_k^p := \{x\colon \mathbb{N}_k \to \mathbb{R} : \sum_{n=k}^{\infty} |x(n)|^p < \infty\},$$

with the standard norm, i.e.

$$\|x\|_{l^p_k} = \left(\sum_{n=k}^{\infty} |x(n)|^p\right)^{1/p}.$$

The relative compactness criterion in l^p is given in the following theorem:

Theorem 2 *([3], p.106) Let $p \in [1,\infty)$, $k \in \mathbb{N}_0$. A subset A of l^p_k is relatively compact if and only if A is bounded and*

$$\lim_{l\to\infty} \sup_{x=(x_n)\in A} \sum_{n=l}^{\infty} |x_n|^p = 0.$$

To get the main results of this paper we use Krasnoselskii's fixed point theorem of the following form.

Theorem 3 *([32], 11.B p. 501) Let X be a Banach space; B be a bounded, closed, convex subset of X; and $S, G : B \to X$ be mappings such that $Sx + Gy \in B$ for any $x, y \in B$. If S is a contraction and G is compact, then the equation*

$$Sx + Gx = x$$

has a solution in B.

To use the approximation technique we need the following Banach–Alaoglu theorem.

Theorem 4 *[25] If X is a Banach space and $S^\star = \{x^\star \in X^\star : ||x^\star|| \le 1\}$, then $S^\star$ is weak*-compact.*

Let us close the preliminaries paragraph with definitions of different types of solutions to (1). By a solution to equation (1) we mean a sequence $x : \mathbb{N}_{k-\max\{\tau,\sigma\}} \to \mathbb{R}$ that satisfies (1) for every $n \in \mathbb{N}_k$ for some $k \ge \max\{\tau,\sigma\}$. By a full solution to equation (1) we mean a sequence $x : \mathbb{N}_0 \to \mathbb{R}$ that satisfies (1) for every $n \ge \max\{\tau,\sigma\}$. For $p \ge 1$, a solution x to (1) is said to be an l^p solution if $x \in l^p_k$ for some $k \in \mathbb{N}_0$.

3. Dependence of existence of bounded solutions on the sequence (q_n)

Sufficient conditions for the existence of a bounded solution to equation (1) with respect to values of sequence (q_n) are derived. At the beginning of this section we formulate and prove the theorem in which values of sequences $(|q_n|)$ are less than 1. Based on this result we prove the existence of the bounded full solution for (q_n) convergent to 1.

In this section, unless otherwise noted, we assume $\tau \in \mathbb{N}_0$, $\sigma \in \mathbb{Z}$, $a, b, q : \mathbb{N}_0 \to \mathbb{R}$, $r : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \to \mathbb{R}$.

Theorem 5 *Assume that*

(H_f) *f is a continuous function;*

(H_s) $\sum_{s=0}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} |a_t| < +\infty, \quad \sum_{s=0}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} |b_t| < +\infty;$

(H_q) $\sup_{n\in\mathbb{N}_0} |q_n| = q^\star < 1.$

Then equation (1) possesses a bounded solution.

Proof Let $M > 0$. From the continuity of f on $[-M, M]$ we get the existence of $Q > 0$ such that

$$|f(x)| \le Q, \text{ for } x \in [-M, M].$$

By (H_s) there exists $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$\sum_{s=n_0}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < (1 - q^{\star})M. \tag{2}$$

We consider the Banach space l_0^{∞} and its subset

$$A_{n_0} = \{x = (x_n)_{n\in\mathbb{N}_0} \in l_0^{\infty} : x_0 = \ldots = x_{n_0+\beta-1} = 0,\ |x_n| \le M, n \ge n_0 + \beta\}.$$

Observe that A_{n_0} is a nonempty, bounded, convex, and closed subset of l_0^{∞}.
Define two mappings $T_1, T_2 \colon l_0^{\infty} \to l_0^{\infty}$ as follows:

$$(T_1x)_n = \begin{cases} 0, & \text{for } 0 \le n < n_0 + \beta \\ -q_n x_{n-\tau}, & \text{for } n \ge n_0 + \beta, \end{cases}$$

$$(T_2x)_n = \begin{cases} 0, & \text{for } 0 \le n < n_0 + \beta \\ \sum\limits_{s=n}^{\infty} \frac{1}{r_s} \sum\limits_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t), & \text{for } n \ge n_0 + \beta. \end{cases}$$

Our next goal is to check assumptions of Theorem 3, Krasnoselskii's fixed point.
First, we show that $T_1x + T_2y \in A_{n_0}$ for $x, y \in A_{n_0}$. Let $x, y \in A_{n_0}$. For $n < n_0 + \beta$ $(T_1x + T_2y)_n = 0$. For $n \ge n_0 + \beta$ from assumption (H_q) and (2) we get

$$|(T_1x + T_2y)_n| \le |q_n x_{n-\tau}| + \sum_{s=n}^{\infty} \tfrac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) \le q^{\star}M + (1 - q^{\star})M = M.$$

It is easy to see that

$$||T_1x - T_1y|| \le q^{\star}||x - y||, \text{ for } x, y \in A_{n_0},$$

so T_1 is a contraction.
Now we prove the continuity of T_2. Let $x \in A_{n_0}$, $\varepsilon > 0$. The continuity of f implies that f is a uniformly continuous function on compact set $[-M, M]$. Hence, there exists $\delta > 0$ such that

$$|f(u) - f(v)| < \frac{\varepsilon}{\left(1 + \sum_{s=0}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} |a_t|\right)}, \text{ for } |u - v| < \delta,\ u, v \in [-M, M]. \tag{3}$$

Let $y \in A_{n_0}$ such that $||x - y|| < \delta$. Then for any $n \ge n_0 + \beta$ we have that $|x_n - y_n| < \delta$, $x_n, y_n \in [-M, M]$, and from (3)

$$|(T_2x - T_2y)_n| \le \sum_{s=n}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| \cdot |f(x_{t-\sigma}) - f(y_{t-\sigma})|) < \varepsilon.$$

Hence, $||T_2x - T_2y|| < \varepsilon$ for $y \in A_{n_0}$, $||x-y|| < \delta$, which proves the continuity of T_2 in any $x \in A_{n_0}$.
Now we show that $T_2(A_{n_0})$ is uniformly Cauchy. Let $\varepsilon > 0$. From (H_s) we get the existence of $n_\varepsilon \geq \beta + n_0$ such that

$$2\sum_{s=n_\varepsilon}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < \varepsilon.$$

For $m > n \geq n_\varepsilon$ and for $x \in A_{n_0}$ we have

$$|(T_2x)_n - (T_2x)_m| = \left| \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) - \sum_{s=m}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t) \right|$$

$$\leq 2\sum_{s=n_\varepsilon}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < \varepsilon.$$

Since $T_2(A_{n_0})$ is uniformly Cauchy and bounded, then by Theorem 1 $T_2(A_{n_0})$ is relatively compact in l_0^∞, which means that T_2 is a compact operator.

From Krasnosielskii's theorem we get that there exists $x = (x_n)_{n\in\mathbb{N}_0}$, the fixed point of $T_1 + T_2$ on A_{n_0}. Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ a second time for $n \geq n_0 + 2\beta$, we get $x = (x_n)_{n\in\mathbb{N}_{n_0+\beta}}$ as the solution to (1). □

To achieve the main result of this section we need to have a full solution to (1), which means a solution defined for $n \in \mathbb{N}_0$. In the case of $\sup|q_n| < 1$ we obtain the full solution to (1) under one additional assumption in Theorem 5.

Corollary 1 *Let the assumptions of Theorem 5 be fulfilled. If we additionally assume*

(H_0') $\tau, \sigma \in \mathbb{N}_0$, $\tau > \sigma$ *and* $q_n \neq 0$ *for* $n \in \mathbb{N}_0$,

then equation (1) possesses a bounded full solution.

Proof We find previous $n_0 + \max\{\tau, \sigma\}$ terms of sequence x by the following formula:

$$x_{n-\tau} = \frac{1}{q_n}\left(-x_n + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} (a_t f(x_{t-\sigma}) + b_t)\right),$$

starting with putting $n := n_0 + 2\tau - 1$. □

Using the same technique, we get the following result.

Corollary 2 *Assume that:*

(H_0) $\tau, \sigma \in \mathbb{N}_0$, $\tau > \sigma$;

(H_f) f *is a continuous function;*

(H_s) $\sum\limits_{s=0}^{\infty} \frac{1}{|r_s|} \sum\limits_{t=s}^{\infty} |a_t| < +\infty, \quad \sum\limits_{s=0}^{\infty} \frac{1}{|r_s|} \sum\limits_{t=s}^{\infty} |b_t| < +\infty;$

(H_q^1) $\inf\limits_{n\in\mathbb{N}_0} q_n = q^\star > 1$.

Then there exists a bounded full solution to (1).

Proof The proof is similar to the proof of Theorem 5 with operators

$$(T_1x)_n = \begin{cases} 0, & \text{for } 0 \le n < n_0 \\ -\frac{1}{q_{n+\tau}} x_{n+\tau}, & \text{for } n \ge n_0, \end{cases}$$

$$(T_2x)_n = \begin{cases} 0, & \text{for } 0 \le n < n_0 \\ \frac{1}{q_{n+\tau}} \sum\limits_{s=n+\tau}^{\infty} \frac{1}{r_s} \sum\limits_{t=s}^{\infty} \left(a_t f(x_{t-\sigma}) + b_t\right), & \text{for } n \ge n_0, \end{cases}$$

where for any $M > 0$ there exist $Q > 0$ and $n_0 > \beta := \max\{\tau, \sigma\}$ such that

$$\sum_{s=n_0}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|Q + |b_t|) < (1 - \tfrac{1}{q^\star})M.$$

□

Now we are in a position to formulate and prove the main result of this section using an approximation technique.

Theorem 6 *Assume that:*

(H_0) $\tau, \sigma \in \mathbb{N}_0\,,\ \tau > \sigma\,;$

(H_f) f *is a continuous function;*

(H_{sb}) *there exist* $D > 0\,,\ C \in (0,1)\,,\ k_0 \in \mathbb{N}$ *and increasing sequence* $(w_k)_{k\in\mathbb{N}} \subset (0,1)$ *with* $\sum_{k=0}^{\infty}(1-w_k) < \infty$ *such that*

$$\sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t|\, P + |b_t|) \le D(1-w_k)\,(Cw_k)^k\,,\ \textit{for } k \ge k_0,$$

where $\max\limits_{|x|\le D} |f(x)| \le P\,;$

$(H_{q=1})$ $q_n \in (0,1),\ n \in \mathbb{N}_0\,,\ \lim\limits_{n\to\infty} q_n = 1\,,\ \inf\limits_{n\in\mathbb{N}_0} q_n > 0\,.$

Then there exists a bounded full solution to equation (1)*.*

Proof For any $k \in \mathbb{N}$, let us consider an auxiliary problem,

$$\Delta\left(r_n \Delta\left(x_n + w_k q_n x_{n-\tau}\right)\right) = a_n f(x_{n-\sigma}) + b_n, \tag{4}$$

where (w_k) is the sequence satisfying (H_{sb}). It is obvious that

$$\sup\{w_k q_n : n \in \mathbb{N}_0\} = w_k < 1.$$

Without loss of generality we can assume that

$$\inf\{q_n : n \in \mathbb{N}_0\} > C,$$

where C is the constant from assumption (H_{sb}). By (H_{sb}) there exist $k_0 \in \mathbb{N}$, $D > 0$ such that for any $k \geq k_0$

$$\sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \leq D(1 - w_k) \left(C w_k\right)^k . \tag{5}$$

From Theorem 5, (4) possesses a bounded solution $x^k = (x_n^k)_{n \in \mathbb{N}_{n_k+\tau}}$ for some $n_k \in \mathbb{N}$. Moreover, by the proof of Theorem 5 we see that (5) implies (2) with $M_k = D(Cw_k)^k \leq D$. Hence, $x^k = (x_n^k)_{n \in \mathbb{N}_{n_k+\tau}} \in l^{\infty}_{n_k+\tau}$ is the fixed point of $T_1 + T_2$ on A_{n_k} with $n_k := k$ for $k \geq k_0$. This means that for $k \geq k_0$, $x^k = (x_n^k)_{n \geq k+\tau}$ solves (4) for $n \geq k + 2\tau$ and $|x_n^k| \leq D(Cw_k)^k$ for $n \geq k + \tau$. We find previous $k + \tau$ terms of sequence $(x_n^k)_{n \geq 0}$ and estimate them by using the following formula:

$$x_{n-\tau}^k = \frac{1}{w_k q_n} \left(-x_n^k + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{t=s}^{\infty} \left(a_t f(x_{t-\sigma}) + b_t \right) \right).$$

Let $k \geq k_0$. Putting $n := n_k + 2\tau - 1 = k + 2\tau - 1$ above we obtain

$$\begin{aligned} |x_{n_k+\tau-1}^k| = |x_{k+\tau-1}^k| &\leq \frac{1}{Cw_k} \left(D(Cw_k)^k + \sum_{s=k+2\tau-1}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \right) \\ &\leq \frac{1}{Cw_k} \left(D(Cw_k)^k + \sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \right) \leq D(1 + (1 - w_k))(Cw_k)^{k-1}. \end{aligned}$$

Moreover,

$$\begin{aligned} |x_{n_k+\tau-2}^k| = |x_{k+\tau-2}^k| &\leq \frac{1}{Cw_k} \left(|x_{k+2\tau-2}| + \sum_{s=k+2\tau-2}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \right) \\ &\leq \begin{cases} \frac{1}{Cw_k} \left(D(1 + (1 - w_k))(Cw_k)^{k-1} + \sum_{s=k}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \right), & \text{for } \tau = 1 \\ \frac{1}{Cw_k} \left(D(Cw_k)^k + D(1 - w_k)(Cw_k)^k \right), & \text{for } \tau \geq 2 \end{cases} \\ &\leq \begin{cases} D(1 + 2(1 - w_k))(Cw_k)^{k-2}, & \text{for } \tau = 1 \\ D(1 + (1 - w_k))(Cw_k)^{k-1}, & \text{for } \tau \geq 2 \end{cases}. \end{aligned}$$

We give the estimation of $|x_n^k|$ for the case $\tau = 1$, $\sigma = 0$. The other cases are analogous and are left to the reader. Indeed, for $k \geq k_0 + 1$,

$$\begin{aligned} |x_{k-2}^k| &\leq (Cw_k)^{-1} \left(|x_{k-1}^k| + \sum_{s=k-1}^{\infty} \frac{1}{|r_s|} \sum_{t=s}^{\infty} (|a_t| P + |b_t|) \right) \\ &\leq (Cw_k)^{k-3} \left(D + 2D(1 - w_k) \right) + D \tfrac{(Cw_{k-1})^{k-1}}{Cw_k} (1 - w_{k-1}) \\ &\leq (Cw_k)^{k-3} (D + 2D(1 - w_k)) + D \tfrac{(Cw_k)^{k-1}}{Cw_k} (1 - w_{k-1}) \\ &\leq (Cw_k)^{k-3} (D + 2D(1 - w_k) + D(1 - w_{k-1})) \leq D + 2D(1 - w_k) + D(1 - w_{k-1}). \end{aligned}$$

By induction for $i=3,\dots,k-k_0+1$,

$$|x^k_{k-i}| \le D(Cw_k)^{k-i-1}\left(1+2(1-w_k)+\sum_{j=k-i+1}^{k-1}(1-w_j)\right)$$

$$\le 2D + D\sum_{i=0}^{\infty}(1-w_i).$$

Moreover,

$$|x^k_{k_0-2}| \le \tfrac{1}{Cw_0}\left(2D + D\sum_{i=0}^{\infty}(1-w_i) + \sum_{s=k_0-1}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|P+|b_t|)\right),$$

and by induction

$$|x^k_0| \le \tfrac{1}{Cw_0}\left(2D + D\sum_{i=0}^{\infty}(1-w_i) + \sum_{j=1}^{k_0-1}\sum_{s=j}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|P+|b_t|)\right).$$

Hence, for $n\in\mathbb{N}_\tau$, $k\ge k_0$,

$$|x^k_n| \le \tfrac{1}{Cw_0}\left(2D + D\sum_{i=0}^{\infty}(1-w_i) + \sum_{j=1}^{k_0-1}\sum_{s=j}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|P+|b_t|)\right).$$

This means that the sequence $(x^k)_{k\ge k_0}$ is bounded in l_0^∞. Since $(l_0^1)^\star = l_0^\infty$, then from Theorem 4, the Banach–Alaoglu theorem, we get that there exists $(x^{k_l})_{l\in\mathbb{N}} \subset (x^k)_{k\ge k_0}$, which is convergent on its coordinates. This means that there exists $\overline{x}=(\overline{x}_n)_{n\in\mathbb{N}_0}\in l_0^\infty$ such that

$$\lim_{l\to\infty} x^{k_l}_n = \overline{x}_n, \text{ for } n\in\mathbb{N}_0$$

and

$$|\overline{x}_n| \le \tfrac{1}{Cw_0}\left(2D + D\sum_{i=0}^{\infty}(1-w_i) + \sum_{j=1}^{k_0-1}\sum_{s=j}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|P+|b_t|)\right), \tag{6}$$

for $n\in\mathbb{N}_0$. To get our results we pass with $l\to\infty$ in

$$x^{k_l}_n + w_{k_l}q_n x^{k_l}_{n-\tau} = \sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t f(x^{k_l}_{t-\sigma}) + b_t\right), \text{ for } n\ge\tau. \tag{7}$$

It is easy to see that for $n\ge\tau$ we have

$$\lim_{l\to\infty}\left(x^{k_l}_n + w^{k_l}q_n x^{k_l}_{n-\tau}\right) = \overline{x}_n + q_n\overline{x}_{n-\tau}.$$

From Lebesgue's dominated convergence theorem and the continuity of f we get

$$\lim_{l\to\infty}\left(\sum_{s=n}^{\infty}\tfrac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t f(x^{k_l}_{t-\sigma})+b_t\right)\right) = \sum_{s=n}^{\infty}\tfrac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t\left(\lim_{l\to\infty} f(x^{k_l}_{t-\sigma})\right)+b_t\right) =$$

$$\sum_{s=n}^{\infty}\tfrac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t f(\overline{x}_{t-\sigma})+b_t\right).$$

From (7) we get that

$$\overline{x}_n + q_n\overline{x}_{n-\tau} = \sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t f(\overline{x}_{t-\sigma}) + b_t\right),$$

for $n \geq \tau$. Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ a second time we get

$$\Delta\left(r_n\Delta(\overline{x}_n + q_n\overline{x}_{n-\tau})\right) = a_n f(\overline{x}_{n-\sigma}) + b_n, \text{ for } n \geq \tau.$$

From (6) we get that $(\overline{x}_n)_{n\in\mathbb{N}_0}$ is the bounded full solution to (1). □

Now we present an example of an equation that can be considered by our method.

Example 1 *The following problem,*

$$\Delta\left((-1)^n\Delta\left(x_n + \left(1 - \tfrac{1}{2^n}\right)x_{n-3}\right)\right) = \tfrac{3}{4}\tfrac{1}{2^n}(x_{n-1})^6, \; n \geq 3, \tag{8}$$

with $\tau = 3$, $\sigma = 1$, $r_n = (-1)^n$, $q_n = 1 - \frac{1}{2^n}$, $a_n = \frac{3}{4}\frac{1}{2^n}$, $b_n = 0$, $n \geq 1$, and $f(x) = x^6$ fulfills the assumptions of Theorem 6. We have to check only (H_{sb}). For $C = 9/10$ and $w_k = 1 - (5/8)^k$, $k \geq 1$ and any $D > 0$ (with $P = D^6$), we get the existence k_0 such that for any $k \geq k_0$

$$3D^5\left(\tfrac{8}{9}\right)^k < \left(1 - \left(\tfrac{5}{8}\right)^k\right)^k.$$

Thus, for any $k \geq k_0$,

$$D^6\sum_{s=k}^{\infty}\sum_{t=s}^{\infty}|a_t| = D^6\sum_{s=k}^{\infty}\sum_{t=s}^{\infty}\tfrac{3}{4}\tfrac{1}{2^t} = 3D^6\tfrac{1}{2^k} < D\left(\tfrac{9}{10}\right)^k\left(1 - \left(\tfrac{5}{8}\right)^k\right)^k\left(\tfrac{5}{8}\right)^k.$$

It is easy to see that $x_n = (-1)^n$ is the bounded solution to (8).

Using the same technique, we get the following result.

Theorem 7 *Assume that:*

(H_0) *$\tau, \sigma \in \mathbb{N}_0$, $\tau > \sigma$;*

(H_f) *f is a continuous function;*

(H_{sb}) *there exist $D > 0$, $k_0 \in \mathbb{N}$ and decreasing sequence $(w_k)_{k\in\mathbb{N}} \subset (1, \infty)$ with $\sum_{k=0}^{\infty}(w_k - 1) < \infty$ such that*

$$\sum_{s=k}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|P + |b_t|) \leq D\,(w_k - 1)\,w_k^{-k}, \text{ for } k \geq k_0$$

where $\max_{|x|\leq D}|f(x)| \leq P$;

$(H^1_{q=1})$ *$q_n > 1$, $n \in \mathbb{N}_0$, $\lim_{n\to\infty} q_n = 1$.*

Then there exists a bounded full solution to (1).

Remark 1 *We obtain analogous theorems if we change the assumption* $(H_{q=1})$ *or* $(H^1_{q=1})$ *to one of the following assumptions:*

$(H_{q=-1})$ $q_n \in (-1,0),\ n \in \mathbb{N}_0, \quad \lim_{n\to\infty} q_n = -1\,,\ \sup_{n\in\mathbb{N}_0} q_n > 0\,.$

$(H^1_{q=-1})$ $q_n < -1,\ n \in \mathbb{N}_0, \quad \lim_{n\to\infty} q_n = -1.$

4. The existence of l^p-solution

To get a better characterization of solutions to (1), we formulate sufficient conditions for the existence of an l^p solution to (1).

In this section, we also assume $\tau \in \mathbb{N}_0\,,\ \sigma \in \mathbb{Z}\,,\ a,b,q : \mathbb{N}_0 \to \mathbb{R}\,,\ r : \mathbb{N}_0 \to \mathbb{R}\setminus\{0\}\,,$ and $f : \mathbb{R} \to \mathbb{R}\,.$

Theorem 8 *Assume that:*

(H_p) $p \geq 1\,;$

(H_f) f *is a continuous function;*

(H_{sp}) $\sum_{n=0}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p < +\infty, \quad \sum_{n=0}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p < +\infty;$

(H_{qp}) $\sup_{n\in\mathbb{N}_0}|q_n| = q^\star \in (0, 2^{1-p})\,.$

Then equation (1) *possesses an* l^p*-solution.*

Proof From the continuity of f on $[-1,1]$ we get the existence of $W > 0$ such that

$$|f(x)| \leq W, \text{ for } x \in [-1,1].$$

The assumption (H_{sp}) implies there exists $n_0 > \beta := \max\{\tau,\sigma\}$ such that

$$4^{p-1}\left[W^p\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p + \sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\right] < 1 - 2^{p-1}q^\star. \tag{9}$$

We consider the Banach space l^p_0 and its subset

$$A_{n_0} = \{x = (x_n)_{n\in\mathbb{N}_0} \in l^p_0 : x_0 = \ldots = x_{n_0+\beta-1} = 0,\ ||x||_{l^p} \leq 1\}\,.$$

Observe that A_{n_0} is a nonempty, bounded, convex, and closed subset of l^p_0.
Define two mappings $T_1, T_2 \colon l^p_0 \to l^p_0$ as follows:

$$(T_1x)_n = \begin{cases} 0, & \text{for } 0 \leq n < n_0+\beta \\ -q_n x_{n-\tau}, & \text{for } n \geq n_0+\beta, \end{cases}$$

$$(T_2x)_n = \begin{cases} 0, & \text{for } 0 \leq n < n_0+\beta \\ \sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{t=s}^{\infty}\left(a_t f(x_{t-\sigma}) + b_t\right), & \text{for } n \geq n_0+\beta. \end{cases}$$

Now we prove the assumptions of Theorem 3 on Krasnoselskii's fixed point.
First, we show that $T_1x + T_2y \in A_{n_0}$ for $x, y \in A_{n_0}$. Let $x, y \in A_{n_0}$, for $n < n_0 + \beta$ $(T_1x + T_2y)_n = 0$. Using twice the classical inequality

$$(x+y)^p \leq 2^{p-1}(x^p + y^p) \text{ for } x, y \geq 0,\ p \geq 1,$$

for $n \geq n_0 + \beta$ we get

$$\begin{aligned}
|(T_1x + T_2y)_n|^p &\leq 2^{p-1}\left[(q^\star)^p\,|x_{n-\tau}|^p + \left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}(|a_t|W + |b_t|)\right)^p\right], \\
&\leq 2^{p-1}\left[q^\star\,|x_{n-\tau}|^p + 2^{p-1}\left(W^p\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p + \left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\right)\right] \\
&\leq 2^{p-1}q^\star\,|x_{n-\tau}|^p + 4^{p-1}\left[W^p\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p + \left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\right].
\end{aligned}$$

By (9) we obtain that

$$\begin{aligned}
||T_1x + T_2y||_{l^p}^p \leq 2^{p-1}q^\star||x||_{l^p}^p + 4^{p-1}\Bigg[W^p\sum_{n=n_0+\beta}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p \\
+ \sum_{n=n_0+\beta}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\Bigg] \leq 1.
\end{aligned}$$

It is easy to see that

$$||T_1x - T_1y||_{l^p} \leq q^\star||x-y||_{l^p},\ \text{for } x, y \in A_{n_0},$$

so T_1 is a contraction.
Now we prove the continuity of T_2. Let $x \in A_{n_0}$, $\varepsilon > 0$. The continuity of f implies that f is a uniformly continuous function on $[-1,1]$, which means there exists $\delta > 0$ such that

$$|f(u) - f(v)| < \left(\frac{\varepsilon}{\left(1 + \sum\limits_{n=0}^{\infty}\left(\sum\limits_{s=n}^{\infty}\frac{1}{|r_s|}\sum\limits_{t=s}^{\infty}|a_t|\right)^p\right)}\right)^{1/p},\ \text{for } |u-v| < \delta,\ u, v \in [-1,1].$$

Hence, for $y \in A_{n_0}$ and $n \geq n_0 + \beta$,

$$\begin{aligned}
|(T_2x - T_2y)_n|^p &\leq \left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t||f(x_{t-\sigma}) - f(y_{t-\sigma})|\right)^p \\
&\leq \left(\sum_{s=n}^{\infty}\frac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p \frac{\varepsilon}{\left(1 + \sum\limits_{n=0}^{\infty}\left(\sum\limits_{s=n}^{\infty}\frac{1}{|r_s|}\sum\limits_{t=s}^{\infty}|a_t|\right)^p\right)}.
\end{aligned}$$

Hence, we get that

$$||T_2x - T_2y||_{l_p} < \varepsilon,\ \text{for } ||x-y||_{l_p} < \delta,\ y \in A_{n_0},$$

which proves the continuity of T_2 on A_{n_0}. In an analogous way we get for $x \in A_{n_0}$ and $n \geq n_0 + \beta$

$$|(T_2 x)_n|^p \leq 2^{p-1}\left[W^p\left(\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p + \left(\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\right]$$

and hence by (H_{sp})

$$\lim_{l\to\infty}\sup_{x\in A_{n_0}}\sum_{n=l}^{\infty}|(T_2 x)_n|^p \leq$$
$$\lim_{l\to\infty}\sum_{n=l}^{\infty}2^{p-1}\left[W^p\left(\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t|\right)^p + \left(\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t|\right)^p\right] = 0,$$

which means that $T_2(A_{n_0})$ is a relatively compact subset of l^p.
From Theorem 3 we get that there exists $x = (x_n)_{n\in\mathbb{N}_1}$, the fixed point of $T_1 + T_2$ on A_{n_0}. Applying operator Δ to both sides of the above equation and multiplying by r_n and applying operator Δ a second time for $n \geq n_0 + 2\beta$ we get that $x = (x_n)_{n\in\mathbb{N}_{n_0+\beta}}$ is the $l^p_{n_0+\beta}$-solution to (1). □

Remark 2 *It is worth mentioning that for $p = 1$ assumption (H_{sb}) implies assumption (H_{sp}).*

Now we present an example of an equation for which our method can be applied.

Example 2 *Let us consider the following problem:*

$$\Delta\left((-1)^n \Delta\left(x_n + q_n x_{n-3}\right)\right) = 2^{-n} f(x_{n-1}) + \tfrac{1}{n(n+1)(n+2)(n+3)}, \ n \geq 3 \tag{10}$$

with $\tau = 3$, $\sigma = 1$, $r_n = (-1)^n$, $a_n = 2^{-n}$, $b_n = \frac{1}{n(n+1)(n+2)(n+3)}$, $n \geq 1$, and any (q_n), $\sup|q_n| < 1$, $f \in C^0(\mathbb{R})$. Note

$$\sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|a_t| = 8,$$
$$\sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\tfrac{1}{|r_s|}\sum_{t=s}^{\infty}|b_t| = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\tfrac{1}{t(t+1)(t+2)(t+3)}$$
$$= \sum_{n=0}^{\infty}\sum_{s=n}^{\infty}\tfrac{1}{4s(s+1)(s+2)} = \sum_{n=0}^{\infty}\tfrac{1}{12n(n+1)} < \infty,$$

which means that assumptions of Theorem 6 are fulfilled with $p = 1$. Hence, (10) has an l^1-solution. It is obvious that this l^1-solution is an l^p-solution for any $p > 1$.

Corollary 3 *If in Theorem 8 we additionally assume*

(H_0') $\tau, \sigma \in \mathbb{N}_0$, $\tau > \sigma$ *and* $q_n \neq 0$, $n \in \mathbb{N}_0$,

then equation (1) possesses a full l^p-solution.

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