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# The localization theorem for finite-dimensional compact group actions

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**Abstract:** The localization theorem is known for compact $G$-spaces, where $G$ is a compact Lie group. In this study, we show that the localization theorem remains true for finite-dimensional compact group actions, and Borel's fixed point theorem holds not only for torus actions but for arbitrary finite-dimensional compact connected abelian group actions.

**Key words:** Equivariant cohomology, fixed point, compact groups

## 1. Introduction

The abelian localization theorem of Borel, Atiyah-Segal, and Quillen was introduced in [1, 2, 13]. Furthermore, Hsiang [7] stated the general form of the localization theorem as follows:

**Theorem 1.1** *Let $G$ be a compact Lie group and $X$ be a compact or a paracompact space of finite cohomological dimension and finitely many orbit types. Let $S \subseteq H^*(B_G, \mathbb{Q})$ be a multiplicative system (i.e. multiplicative semigroup with $\{1\} \subseteq S$) and*

$$X^S = \{x \in X : \text{no element of } S \text{ maps to zero in } H^*(B_G, \mathbb{Q}) \to H^*(B_{G_x}, \mathbb{Q})\}.$$

*Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X, \mathbb{Q}) \to S^{-1}H_G^*\left(X^S, \mathbb{Q}\right)$$

*is an isomorphism.*

**Remark 1.2** *The localization theorem above was extended by Deo et al. [4] for compact Lie group actions on finitistic spaces with finitely many orbit types.*

The localization theorem is a powerful tool for cohomology theory of compact, especially compact abelian, transformation groups. One of the most profound results of the localization theorem is to determine the cohomology structure of a fixed point set by the equivariant cohomology of the space for the torus or $p$-torus actions.

The following well-known Borel's fixed point theorem is one of the important consequences of the localization theorem.

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**Theorem 1.3** *Let $G$ be a finite-dimensional torus and $X$ be a compact $G$-space. Then the fixed point set $X^G \neq \emptyset$ if and only if $H^*(B_G, \mathbb{Q}) \to H^*_G(X, \mathbb{Q})$ is injective [2].*

In this paper, the localization theorem has been proved for finite-dimensional compact group actions on compact spaces and as a corollary we extend Borel's fixed point theorem to finite-dimensional compact connected abelian transformation groups.

## 2. Preliminaries

The Borel construction is a powerful tool for cohomology theory of topological transformation groups. It is defined as follows:

For any topological group $G$, we have a universal principal $G$-bundle $E_G \to B_G$ (see Milnor [9, 10]). $B_G$ is unique up to homotopy equivalence and is called the classifying space of $G$.

Let $X$ be a $G$-space. There is the diagonal action on $E_G \times X$ and the Borel construction is defined to be the orbit space $(E_G \times X)/G$ and denoted by $X_G$. This leads to the following commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & E_G \times X & \longrightarrow & E_G \\ \downarrow & \searrow^{i_G} & \downarrow & & \downarrow \\ X/G & \xleftarrow{\pi_2} & X_G & \xrightarrow{\pi_1} & B_G \end{array}$$

where $\pi_1$ is a fiber bundle mapping with fiber $X$ and structure group $G/K$ where $K$ is the ineffective kernel of the $G$ action on $X$, $\pi_2$ is a mapping such that $\pi_2^{-1}(x^*) = B_{G_x}$, where $x^* \in X/G$, and $x \in x^*$. Moreover, $H^*(X_G, k)$ is an algebra over $H^*(B_G, k)$ by $\pi_1^* : H^*(B_G, k) \to H^*(X_G, k)$, which is called the equivariant graded cohomology algebra of $X$ with coefficient $k$ and denoted by $H^*_G(X; k)$.

**Remark 2.1** *Some results in Alexander–Spanier cohomology are in need of some compactness assumptions (for example, Proposition 3.1.). Even if $G$ is a compact group and $X$ is a compact $G$-space, $B_G$ and $X_G$ are not necessarily compact. It is often convenient to consider $N$-classifying space $B_G^N$ (compact) instead of $B_G$. Since the morphisms $H^i\left(B_G^{N+1}, \mathbb{Q}\right) \to H^i\left(B_G^N, \mathbb{Q}\right)$ are isomorphisms for all $i < N$ [6, III; Proposition 1.8], then $H^i(B_G, \mathbb{Q})$ is canonically isomorphic to $H^i\left(B_G^N, \mathbb{Q}\right)$ for all $i < N$. To prove any result for equivariant cohomology, $H^*\left(B_G^N, \mathbb{Q}\right)$ can be taken into $H^*(B_G, \mathbb{Q})$ for sufficiently large $N$ without loss of generality. Similarly, let $E_G^N$ be the inverse image of $B_G^N$ under the map $E_G \to B_G$. Let us define $X_G^N = X \times_G E_G^N$. Note that $E_G^N$ is compact and $(N-1)$-connected (i.e. $H^i\left(E_G^N, \mathbb{Q}\right) = H^i(pt, \mathbb{Q})$ for all $i < N$), and so $X_G^N$ is also compact. By Vietoris–Begle mapping theorem for sheaf cohomology [13], the inclusion $X_G^N \to X_G$ induces an isomorphism*

$$H^i(X_G, \mathbb{Q}) \to H^i\left(X_G^N, \mathbb{Q}\right)$$

*for $i < N$. One can assume $H^*\left(X_G^N, \mathbb{Q}\right)$ instead of $H^*(X_G, \mathbb{Q})$ for sufficiently large $N$ without loss of generality.*

The Mayer–Vietoris exact sequence for equivariant cohomology follows from its ordinary nonequivariant version. If $A$ and $B$ are closed invariant subspaces of the $G$-space $X$, there is a long exact sequence

$$\cdots \to H_G^i(A \cup B, k) \to H_G^i(A, k) \oplus H_G^i(B, k) \to H_G^i(A \cap B, k) \to H_G^{i+1}(A \cup B, k) \to \cdots$$

of $H^*(B_G)$-modules.

This paper deals with finite-dimensional compact groups. Therefore, we need to recall the dimension of a compact Hausdorff topological space.

**Definition 2.2** *Let $X$ be a compact Hausdorff space. We say that $X$ has dimension $n \geq 0$ if the following two conditions are satisfied:*

*1) For every finite covering of $X$ by open sets there exists a finite covering $\Sigma$ by closed sets that refines the given covering and, for each $x \in X$, the number of the sets of $\Sigma$ that contain $x$ is at most $n+1$.*

*2)There exists a finite covering of $X$ by open sets such that if $\Sigma$ is a finite closed refinement of this covering then the number of the sets of $\Sigma$ that contain $x$ exceeds $n$. If the dimension of $X$ is $n$ then we denote* $\dim X = n$.

If $G$ is a compact group (we always include the Hausdorff separation axiom) then the dimension of $G$ is the dimension of the space $G$. Of course, we say that $G$ is finite-dimensional compact group if $\dim G = n$ for some $n \geq 0$.

**Remark 2.3** *1. There are various dimension functions. Topological dimension, small inductive dimension, local large inductive dimension, cohomological dimension, and sheaf theoretical dimension are frequently used. For an arbitrary compact group $G$, all these dimensions agree, and when finite $\dim G = rank\widehat{G}$ where $\widehat{G}$ is a character group of $G$.*

*2. An important fact about finite-dimensional compact groups is that they have a totally disconnected closed normal subgroup such that the factor group is a compact Lie group with the same dimension (see for details [12, Theorem 69]).*

We need a few facts about finite-dimensional compact groups.

**Theorem 2.4** *(Nagami, [11]) Any closed subgroup of a locally compact, finite-dimensional group has a local cross section.*

According to the above, if $G$ is a finite-dimensional compact group and $N$ is a closed normal subgroup of $G$, then the quotient map $G \to G/N$ is a principal $N$-bundle.

The proof of the next theorem can also be found in the 1974 work of Bruner et al.
(https://www.math.uchicago.edu/ may/CHAR/charclasses.pdf).

**Theorem 2.5** *If $G$ is a finite-dimensional compact group and $N$ is a closed normal subgroup of $G$, then up to homotopy, the sequence $B_N \to B_G \to B_{G/N}$, induced by the inclusion $N \hookrightarrow G$ and the quotient map $G \to G/N$, is a fiber sequence.*

**Proof** Let $E_G \to B_G$ and $E_{G/N} \to B_{G/N}$ be universal bundles for $G$ and $G/N$ and take $E_G \to E_G/N = B_N$ to be the universal bundle for $N$. We may assume given a map $Ej : E_G \to E_{G/N}$ such that $(Ej)(yg) =$

$(Ej)(y)\,j(g)$. Since $j(n) = e$ for $n \in N$, $Ej$ factors through $B_N$ and we obtain a bundle map

$$\begin{array}{ccc} B_N & \longrightarrow & E_{G/N} \\ {\scriptstyle Bi}\downarrow & & \downarrow \\ B_G & \xrightarrow[Bj]{} & B_{G/N} \end{array}$$

In particular, this square is a pullback. We can replace the map $Bj$ by a fibration. The fiber of this fibration is the homotopy fiber $F(Bj)$ of $Bj$, where the homotopy fiber of $Bj$ is defined by the pullback diagram

$$\begin{array}{ccc} F(Bj) & \longrightarrow & PB_{G/N} \\ \downarrow & & \downarrow \\ B_G & \xrightarrow[Bj]{} & B_{G/N} \end{array}$$

We shall show that $F(Bj)$ is homotopy equivalence to $B_N$. Notice that $PB_{G/N}$ is homotopy equivalence to $E_{G/N}$ since $E_{G/N}$ is contractible.

We see by the universal property of pullbacks that there is a map $\theta : B_N \to F(Bj)$ such that the following diagram commutes.

$$\begin{array}{ccccc} G/N & \longrightarrow & B_N & \longrightarrow & B_G \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \theta} & & \| \\ \Omega B_{G/N} & \longrightarrow & F(Bj) & \longrightarrow & B_G \end{array}$$

By Zeeman's comparison theorem [14], since $\zeta$ is homotopy equivalance, $\theta$ induces an isomorphism on homotopy groups and is thus a homotopy equivalence. The result follows. □

Throughout the present article Alexander–Spanier cohomology with rational coefficients is used. Indeed, Alexander–Spanier cohomology, Čech cohomology, and Sheaf cohomology are naturally equivalent for paracompact spaces [3].

## 3. Main results

Since each finite-dimensional compact group has a totally disconnected (i.e. zero-dimensional) closed normal subgroup such that the factor group is a compact Lie group (see Remark 2.3.), the compact totally disconnected group actions play an important role for compact transformation group theory. We give a few well-known propositions for totally disconnected group actions.

**Proposition 3.1** *If $G$ is a totally disconnected compact group and $X$ is a compact $G$-space, then the orbit map $\pi : X \to X/G$ induces an isomorphism*

$$H^*(X/G, \mathbb{Q}) \simeq (H^*(X, \mathbb{Q}))^G$$

*where $(H^*(X, \mathbb{Q}))^G$ is the invariant subspace of $H^*(X, \mathbb{Q})$ under the induced action of $G$ on $H^*(X, \mathbb{Q})$.*

**Proposition 3.2** *Let $G$ be a compact totally disconnected group. Then $H^*(B_G, \mathbb{Q}) = \mathbb{Q}$.*

The proofs of Propositions 3.1. and 3.2. are trivial from continuity of cohomology and can be found in [8].

From now on, unless otherwise stated, $G$ is a finite-dimensional compact group and $N$ is a totally disconnected closed normal subgroup of $G$ such that the factor group $G/N$ is a compact Lie group with the same dimension as $G$. Let $X$ be a compact $G$-space and $S \subseteq H^*(B_G, \mathbb{Q})$ be a multiplicative system (i.e. multiplicative semigroup with $\{1\} \subseteq S$).

Moreover, we set, in the sense of Hsiang's book [7],

$$X^S = \{x \in X : \text{no element of S maps to zero in } H^*(B_G, \mathbb{Q}) \to H^*(B_{G_x}, \mathbb{Q})\}.$$

Note that if $X$ is a paracompact space and one is using Alexander–Spanier cohomology, then $X^S$ is a closed invariant subspace of $X$.

**Lemma 3.3** *The homomorphism $B\pi^* : H^*\left(B_{G/N}, \mathbb{Q}\right) \to H^*\left(B_G, \mathbb{Q}\right)$ is induced by the quotient map $\pi : G \to G/N$ is an isomorphism.*

**Proof** We recall by Theorem 2.5. that there is a Serre fibration $B_N \overset{Bi}{\to} B_G \overset{B\pi}{\to} B_{G/N}$, induced by the inclusion $i : N \hookrightarrow G$ and the quotient map $\pi : G \to G/N$. Moreover, there is naturally a spectral sequence with $E_2^{p,q} \cong H^p(B_{G/N}, \mathcal{H}^q)$ where $\mathcal{H}^q$ is the coefficient system $x \mapsto H^q((B\pi)^{-1}(x), \mathbb{Q})$. Since $B_N$ is acyclic, i.e. $H^*(B_N, \mathbb{Q}) = H^*(pt, \mathbb{Q})$, then the fundamental group of $B_{G/N}$ acts trivially on the cohomology of the fiber $B_N$. Therefore, from the results of Leray–Serre spectral sequences $E_2^{p,q} \cong H^p(B_{G/N}, \mathbb{Q}) \otimes H^q(B_N, \mathbb{Q})$.

Since the spectral sequence converges to $H^*(B_G, \mathbb{Q})$ and $H^p(B_N, \mathbb{Q}) = 0$ for $p \neq 0$, then

$$H^n(B_G, \mathbb{Q}) \cong \bigoplus_{r+s=n} E_\infty^{r,s} \cong \bigoplus_{r+s=n} E_2^{r,s} \cong E_2^{n,0}.$$

On the other hand, since $B\pi^n : H^n\left(B_{G/N}, \mathbb{Q}\right) \to H^n\left(B_G, \mathbb{Q}\right)$ is composite $H^n\left(B_{G/N}, \mathbb{Q}\right) = E_2^{n,0} \twoheadrightarrow E_3^{n,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{n+1}^{n,0} = E_\infty^{n,0} \cong H^n(B_G, \mathbb{Q})$ for each $n$ and then the result follows. □

Now consider the multiplicative system $R = (B\pi^*)^{-1}(S)$ in $H^*(B_{G/N}, \mathbb{Q})$, which is a copy of $S$ and induced finite-dimensional compact Lie group $G/N$ action on the orbit space $X/N$.

**Lemma 3.4** *The orbit map $p : X \to X/N$ induces the map $X^S \to (X/N)^R$.*

**Proof** Let $x \in X^S$ and assume that $x^* = N(x) \notin (X/N)^R$. In this case, $Bj^*(r) = 0$ for some $r \in R$, where

$$Bj^* : H^*(B_{G/N}, \mathbb{Q}) \to H^*(B_{(G/N)_{x^*}}, \mathbb{Q}).$$

Put $s = (B\pi^*)(r) \in S$. From the following commutative diagram,

$$\begin{array}{ccc} H^*(B_{G/N}, \mathbb{Q}) & \xrightarrow{Bj^*} & H^*(B_{(G/N)_{x^*}}, \mathbb{Q}) \\ \downarrow B\pi^* & & \downarrow \\ H^*(B_G, \mathbb{Q}) & \xrightarrow{Bi^*} & H^*(B_{G_x}, \mathbb{Q}) \end{array}$$

$s \in S$ maps to zero under $Bi^* : H^*(B_G, \mathbb{Q}) \to H^*(B_{G_x}, \mathbb{Q})$. This contradicts $x \in X^S$. □

**Remark 3.5** *The isotropy subgroups of the induced action of $G/N$ on $X/N$ are explicitly discussed in [5, Prop.10.31]. We have*

$$(G/N)_{x^*} = G_x N/N \simeq G_x/(G_x \cap N).$$

*Furthermore, since $G_x \cap N$ is a totally disconnected closed normal subgroup of $G_x$, $H^*(B_{G_x \cap N}, \mathbb{Q}) = \mathbb{Q}$. Thus, as in Lemma 3.3., the right vertical map of the above diagram is induced by the canonical map $G_x \to G_x N/N$ is also an isomorphism by considering the Leray–Serre spectral sequence of the homotopy-fiber sequence $B_{G_x \cap N} \to B_{G_x} \to B_{G_x/G_x \cap N}$.*

**Corollary 3.6** $X^S/N = (X/N)^R$.

**Proof** Consider the previous diagram.

If $X^S = \emptyset$, then there exists an $s \in S$ such that $s$ maps to zero in $H^*(B_G, \mathbb{Q}) \to H^*(B_{G_x}, \mathbb{Q})$. Set $r = (B\pi^*)^{-1}(s)$. Since $H^*\left(B_{(G/N)_{x^*}}, \mathbb{Q}\right) \to H^*(B_{G_x}, \mathbb{Q})$ is an isomorphism from Remark 3.5., then $Bj^*(r) = 0$, which implies that $(X/N)^R = \emptyset$.

Suppose $X^S \neq \emptyset$. It is easy to see that Lemma 3.4. implies $X^S/N \subseteq (X/N)^R$. We take now $N(x) \in (X/N)^R$. Assume that $x \notin X^S$. Then $Bi^*(s) = 0$ for some $s \in S$. Set $r = (B\pi^*)^{-1}(s)$. Since $H^*\left(B_{(G/N)_{x^*}}, \mathbb{Q}\right) \to H^*(B_{G_x}, \mathbb{Q})$ is an isomorphism, then $Bj^*(r) = 0$. This contradicts $N(x) \in (X/N)^R$. Hence, $x \in X^S$, which implies $N(x) \in X^S/N$. □

First we prove the localization theorem for finite-dimensional compact connected group action on a compact space.

**Theorem 3.7** *Let $G$ be a finite-dimensional compact connected group and $X$ be a compact $G$-space. Then the localized restriction homomorphism*

$$S^{-1}H^*_G(X, \mathbb{Q}) \to S^{-1}H^*_G\left(X^S, \mathbb{Q}\right)$$

*is an isomorphism.*

**Proof** Since $G$ is connected, its action (and hence that of $N$) on $H^*(X, \mathbb{Q})$ is trivial (see [3, II.10.6 cf. II.11.11]). Therefore, Proposition 3.1. implies that the orbit map $X \to X/N$ induces the isomorphism

$$H^*(X, \mathbb{Q}) \simeq H^*(X/N, \mathbb{Q}).$$

We now prove the theorem by reducing the $G$ action on $X$ to the $G/N$ action on the compact space $X/N$.

By the localization theorem for compact Lie transformation groups,

$$R^{-1}H^*_{G/N}(X/N, \mathbb{Q}) \to R^{-1}H^*_{G/N}\left((X/N)^R, \mathbb{Q}\right)$$

is an isomorphism.

On the other hand, from Zeeman's comparison theorem [14], we have that

$$H^*_{G/N}(X/N, \mathbb{Q}) \to H^*_G(X, \mathbb{Q})$$

is an isomorphism by comparing the spectral sequence of the bundles $X \to X_G \to B_G$ and $X/N \to (X/N)_{G/N} \to B_{G/N}$. Thus,

$$R^{-1}H^*_{G/N}(X/N,\mathbb{Q}) \simeq S^{-1}H^*_G(X,\mathbb{Q}).$$

Similarly, we have

$$R^{-1}H^*_{G/N}\left(X^S/N,\mathbb{Q}\right) \simeq S^{-1}H^*_G\left(X^S,\mathbb{Q}\right)$$

by considering restricted $G$-action on $X^S$. Therefore, we have

$$R^{-1}H^*_{G/N}\left((X/N)^R,\mathbb{Q}\right) \simeq S^{-1}H^*_G\left(X^S,\mathbb{Q}\right)$$

from Corollary 3.6.

Because of the following commutative diagram,

$$\begin{array}{ccc} R^{-1}H^*_{G/N}(X/N,\mathbb{Q}) & \xrightarrow{\simeq} & R^{-1}H^*_{G/N}((X/N)^R,\mathbb{Q}) \\ \simeq\downarrow & & \downarrow\simeq \\ S^{-1}H^*_G(X,\mathbb{Q}) & \longrightarrow & S^{-1}H^*_G(X^S,\mathbb{Q}) \end{array}$$

we obtain the isomorphism $S^{-1}H^*_G(X,\mathbb{Q}) \to S^{-1}H^*_G\left(X^S,\mathbb{Q}\right)$. □

The next theorem implies that the localization theorem is provided for general finite-dimensional compact group actions.

**Theorem 3.8** *Let $G$ be a finite-dimensional compact group, $X$ be a compact $G$-space, and $S \subseteq H^*(B_G,\mathbb{Q})$ be a multiplicative system. Then the localized restriction homomorphism*

$$S^{-1}H^*_G(X,\mathbb{Q}) \to S^{-1}H^*_G\left(X^S,\mathbb{Q}\right)$$

*is an isomorphism.*

**Proof** Let $G_0 \subseteq G$ denote the connected component of the identity element, which is a closed normal subgroup of $G$ and $G/G_0$ is a totally disconnected compact group (see [12, Section 22] for details).

Since $G_0$ acts freely on $E_G$, we may take for $E_{G_0} = E_G$ and $B_{G_0}$ the quotient space $E_G/G_0$. There are then obvious induced $G/G_0$-actions on $X_{G_0}$ and $B_{G_0}$. Since $G/G_0$ is totally disconnected, then, by Proposition 3.1., the orbit maps $X_{G_0} \to X_G$ and $B_{G_0} \to B_G$ induce the monomorphisms $H^*(X_G,\mathbb{Q}) \to H^*(X_{G_0},\mathbb{Q})$ and $H^*(B_G,\mathbb{Q}) \to H^*(B_{G_0},\mathbb{Q})$.

**Case 1:** First we assume that $X^S = \emptyset$. Let $T$ be the direct image of $S$ in $H^*(B_{G_0},\mathbb{Q})$. Then, by considering restricted finite-dimensional compact connected group $G_0$, action on $X$, and the following commutative diagram,

$$\begin{array}{ccc} H^*(B_G,\mathbb{Q}) & \longrightarrow & H^*(B_{G_x},\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*(B_{G_0},\mathbb{Q}) & \longrightarrow & H^*(B_{(G_0)_x},\mathbb{Q}) \end{array}$$

it is easy to see that $X^T = \emptyset$. Hence, Theorem 3.7. implies that there is an element $t$ of $T$ such that the image of $t$ under $H^*(B_{G_0},\mathbb{Q}) \to H^*(X_{G_0},\mathbb{Q})$ is zero. Set $s$ is the preimage of $t$ under the monomorphism $H^*(B_G,\mathbb{Q}) \to H^*(B_{G_0},\mathbb{Q})$. Now consider the following commutative diagram,

$$\begin{array}{ccc} H^*(B_G,\mathbb{Q}) & \longrightarrow & H^*(X_G,\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*(B_{G_0},\mathbb{Q}) & \longrightarrow & H^*(X_{G_0},\mathbb{Q}) \end{array}$$

which is induced by the following commutative diagram of the bundles:

$$\begin{array}{ccccc} X & \longrightarrow & X_{G_0} & \longrightarrow & B_{G_0} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X_G & \longrightarrow & B_G \end{array}$$

Since the orbit map $X_{G_0} \to X_G$ induces the monomorphism $H^*(X_G,\mathbb{Q}) \to H^*(X_{G_0},\mathbb{Q})$, it is easy to see that $s$ maps to zero under $H^*(B_G,\mathbb{Q}) \to H^*(X_G,\mathbb{Q})$, which means $S^{-1}H_G^*(X,\mathbb{Q}) = \{0\}$.

**Case 2:** Now assume that $X^S \neq \emptyset$. Let $B$ be a closed invariant neighborhood of $X^S$ in $X$ and $B' = X - int(B)$. Thus, consider the long exact sequence

$$\cdots \to H_G^*(X,\mathbb{Q}) \to H_G^*(B,\mathbb{Q}) \oplus H_G^*(B',\mathbb{Q}) \to H_G^*(B\cap B',\mathbb{Q}) \to \cdots$$

Hence, by exactness of localization, we have

$$\cdots \to S^{-1}H_G^*(X,\mathbb{Q}) \to S^{-1}H_G^*(B,\mathbb{Q}) \oplus S^{-1}H_G^*(B',\mathbb{Q}) \to S^{-1}H_G^*(B\cap B',\mathbb{Q}) \to \cdots$$

Since $(B')^S = \emptyset$ and $(B\cap B')^S = \emptyset$, we obtain that $S^{-1}H_G^*(B',\mathbb{Q}) = \{0\}$ and $S^{-1}H_G^*(B\cap B',\mathbb{Q}) = \{0\}$ from Case 1. Now $S^{-1}H_G^*(X,\mathbb{Q}) \to S^{-1}H_G^*(B,\mathbb{Q})$ is an isomorphism. Taking the inverse limit as $B$ run over all invariant closed neighborhoods of $X^S$, because of the continuity property of Alexander–Spanier cohomology, one has $H_G^*\left(X^S,\mathbb{Q}\right) = \varprojlim_{B\supset X^S} H_G^*(B,\mathbb{Q})$. Using the continuity property and the fact the localization commutes with the inverse limit, we obtain

$$\begin{aligned} S^{-1}H_G^*\left(X^S,\mathbb{Q}\right) &= S^{-1}(\varprojlim_{B\supset X^S} H_G^*(B,\mathbb{Q})) \\ &= \varprojlim_{B\supset X^S} S^{-1}H_G^*(B,\mathbb{Q}) \\ &\simeq \varprojlim_{B\supset X^S} S^{-1}H_G^*(X,\mathbb{Q}) = S^{-1}H_G^*(X,\mathbb{Q}). \end{aligned}$$

□

We will then shift our attention to the abelian case. In the next theorem, Hofmann and Mostert [6] give the cohomology algebra structure of the classifying space of compact (connected) abelian groups, which is irrespective of the choice of the classifying space.

**Theorem 3.9** *([6, pp.206, 212]). For any compact abelian group $G$, the homogeneous component $H^2(B_G,\mathbb{Z})$ of degree 2 is naturally isomorphic to the character group $\widehat{G}$. If $G$ is connected, the entire cohomology algebra*

*$H^*(B_G,\mathbb{Z})$ is the symmetric $\mathbb{Z}$-algebra $P(\widehat{G})$ generated by this component of homogeneous degree 2. Also, in this case, the homogeneous component $H^1(G,\mathbb{Z})$ of degree 1 of the space cohomology is also naturally isomorphic to $\widehat{G}$, and the entire cohomology algebra $H^*(G,\mathbb{Z})$ is the exterior algebra $\bigwedge(H^1(G,\mathbb{Z})$.*

*In particular, for any commutative unital ring $R$ and compact connected abelian $G$,*

$$H^*(B_G,R) \cong R\otimes P(\widehat{G}) \cong P(R\otimes \widehat{G}).$$

**Remark 3.10** *If $G$ is a compact abelian group and $N$ is any closed normal subgroup of $G$, then, by duality, the inclusion morphism $i : N \to G$ induces a quotient morphism $\widehat{i} : \widehat{G} \to \widehat{N}$ with kernel*

$$\ker \widehat{i} = N^{\perp} = \{\chi \in \widehat{G} : \chi(n) = 0 \text{ for all } n \in N\}$$

*where $N^{\perp}$ is called the annihilator of $N$ in $\widehat{G}$. Therefore, $H^2(Bi,\mathbb{Z}) : H^2(B_G,\mathbb{Z}) \to H^2(B_N,\mathbb{Z})$ is injective if and only if $N = G$, and if $G$ is connected, this conclusion persists upon tensoring with $\mathbb{Q}$; that is, $H^2(Bi,\mathbb{Q}) : H^2(B_G,\mathbb{Q}) \to H^2(B_N,\mathbb{Q})$ is injective if and only if $N = G$.*

*This observation applies to a compact connected abelian group $G$ acting on $X$ with $N = G_x$. Then one does obtain the statement that $x$ is a fixed point of the action if and only if $Bi^* : H^*(B_G,\mathbb{Q}) \to H^*(B_{G_x},\mathbb{Q})$ is injective for the inclusion map $i : G_x \to G$.*

*Thus, if a compact connected abelian group $G$ acts on $X$, then $X^S = X^G$ for the multiplicative system $S = H^*(B_G,\mathbb{Q}) - \{0\} \subset H^*(B_G,\mathbb{Q})$.*

Thus, we obtain an extension of Borel's fixed point theorem to finite-dimensional compact connected abelian group actions.

**Corollary 3.11** *Let $G$ be a finite-dimensional compact connected abelian group and $X$ be a compact $G$-space. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X,\mathbb{Q}) \to S^{-1}H_G^*\left(X^G,\mathbb{Q}\right) = H^*\left(X^G,\mathbb{Q}\right) \otimes_{\mathbb{Q}} \left(S^{-1}H^*(B_G,\mathbb{Q})\right)$$

*is an isomorphism where $S = H^*(B_G,\mathbb{Q}) - \{0\}$.*

**Corollary 3.12** *Let $G$ be a finite-dimensional compact connected abelian group and $X$ be a compact $G$-space. Then $X^G \neq \emptyset$ if and only if $H^*(B_G,\mathbb{Q}) \to H_G^*(X,\mathbb{Q})$ is injective.*

**Proof** $\Rightarrow$: Suppose that $X^G \neq \emptyset$ and $x \in X^G$. Thus, $B_G \to \{x\}_G \subseteq X_G$ is a cross-section of the fiber bundle $X_G \to B_G$ and consequently $H^*(pt,\mathbb{Q}) = H^*(B_G,\mathbb{Q}) \to H_G^*(X,\mathbb{Q})$ must be an injective map.

$\Leftarrow$: If $H^*(B_G,\mathbb{Q}) \to H_G^*(X,\mathbb{Q})$ is injective, then $1 \in H_G^*(X,\mathbb{Q})$ is torsion-free and hence $S^{-1}H_G^*(X,\mathbb{Q}) \neq \{0\}$. Therefore, it follows from Corollary 3.11. that $S^{-1}H_G^*\left(X^G,\mathbb{Q}\right) \simeq S^{-1}H_G^*(X,\mathbb{Q}) \neq \{0\}$. This clearly implies $X^G \neq \emptyset$. □

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## References

[1] Atiyah MF, Segal GB. Equivariant Cohomology and Localization. Lecture notes. Coventry, UK: University of Warwick, 1965.

[2] Borel A. Seminar on Transformation Groups. Princeton, NJ, USA: Princeton University Press, 1960.

[3] Bredon GE. Sheaf Theory. 2nd ed. Graduate Texts in Mathematics 170. New York, NY, USA: Springer, 1997.

[4] Deo S, Singh TB, Shukla RA. On an extension of localization theorem and generalized Conner conjecture. T Am Math Soc 1982; 269: 395-402.

[5] Hofmann KH, Morris SA. The Structure of Compact Groups. 3rd ed. Berlin, Germany: de Gruyter, 2013.

[6] Hofmann KH, Mostert PS. Cohomology Theories for Compact Abelian Groups. Berlin, Germany: Springer, 1973.

[7] Hsiang WY. Cohomology Theory of Transformation Groups. Berlin, Germany: Springer, 1975.

[8] Ku MC. Some topics in compact transformation groups. PhD, Tulane University, New Orleans, LA, USA, 1967.

[9] Milnor J. Construction of universal bundles I. Ann Math 1956; 63: 272-284.

[10] Milnor J. Construction of universal bundles II. Ann Math 1956; 63: 430-436.

[11] Nagami K. Cross sections in locally compact groups. J Math Soc Japan 1963; 15: 301-303.

[12] Pontryagin LS. Topological Groups. 3rd ed. New York, NY, USA: Gordon and Breach Science Publishers, 1986.

[13] Quillen D. The spectrum of an equivariant cohomology ring I. Ann Math 1971; 94: 549-572.

[14] Zeeman EC. A proof of the comparison theorem for spectral sequence. Proc Cambridge Philos Soc 1957; 53: 57-62.