On the $J$-reflexive operators

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Received: 05.07.2017 • Accepted/Published Online: 28.03.2018 • Final Version: 24.07.2018

Abstract: A bounded linear operator $T$ on a Banach space $X$ is $J$-reflexive if every bounded operator on $X$ that leaves invariant the sets $J(T, x)$ for all $x$ is contained in the closure of $\text{orb}(T)$ in the strong operator topology. We discuss some properties of $J$-reflexive operators. We also give and prove some necessary and sufficient conditions under which an operator is $J$-reflexive. We show that isomorphisms preserve $J$-reflexivity and some examples are considered. Finally, we extend the $J$-reflexive property in terms of subsets.

Key words: Orbit of an operator, $J$-sets, $J^{\text{mix}}$-sets, $J$-class operator, reflexive operator

1. Introduction

One of the most challenging problems in operator theory is the “invariant subspace problem”, which asks whether every operator on a Hilbert space (more generally, a Banach space) admits a nontrivial invariant subspace. Here, “operator” means “continuous linear transformation” and “invariant subspace” means “closed linear manifold such that the operator maps it to itself”. A subspace is nontrivial if it is neither the zero subspace nor the whole space. An example constructed by Enflo [7] shows that for some Banach spaces there do exist operators with only trivial invariant subspaces. For a Hilbert space, however, the invariant subspace problem remains open. There is a deep connection between invariant subspaces of an operator and its reflexivity. Reflexive operators are those that can be identified by their nontrivial invariant subspaces and they have been studied for a few decades. For a good source on reflexivity, see [12] by Halmos. An operator $T$ is reflexive if any operator that leaves invariant, $T$-invariant subspaces belongs to the closure of $\{P(T) : P \text{ is a polynomial}\}$ in the weak operator topology [12, 17]. In [11], the authors introduced orbit-reflexivity: an operator $T$ on a Hilbert space is orbit-reflexive if the only operators that leave invariant every norm closed $T$-invariant subset are contained in the closure of $\text{orb}(T)$ in the strong operator topology. For example, compact operators, normal operators, contractions, and weighted shifts on the Hilbert spaces are orbit-reflexive [11]. Hadwin et al. in [10] also introduced and studied the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. They also proved that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive. For further references on these topics, see [8–11, 13, 14].

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2010 AMS Mathematics Subject Classification: Primary 47A65; Secondary 47B99
In this paper, our purpose is to characterize operators on Banach spaces that can be identified by their \( J \)-sets. First, we give some preliminaries that we need to give our results.

Let \( X \) be a complex Banach space and denote by \( B(X) \) the space of all bounded linear operators on \( X \). For \( T \in B(X) \), the set \( \text{orb}(T) = \{ T^n : n \geq 0 \} \) is called the orbit of \( T \).

If \( X \) is a separable Banach space and \( \text{orb}(T, x) = \{ T^n x : n \geq 0 \} \) is dense in \( X \) for some \( x \), then \( T \) is called a hypercyclic operator and we call \( x \) a hypercyclic vector.

The \( J \)-set of \( T \) under \( x \), \( J(T, x) \), is defined by:

\[
J(T, x) = \{ y : \text{there exists a strictly increasing sequence of positive integers} \quad (k_n) \text{ and a sequence } (x_n) \subset X \text{ such that} \quad x_n \to x, \quad T^{k_n} x_n \to y \}.
\]

The set

\[
J^{\text{mix}}(T, x) = \{ y : \text{there exists a sequence } (x_n) \subset X \text{ such that} \quad x_n \to x \text{ and} \quad T^n x_n \to y \}
\]

is called the \( J^{\text{mix}} \)-set of \( T \) under \( x \). The sets \( J(T, x) \) and \( J^{\text{mix}}(T, x) \) are closed \( T \)-invariant subsets of \( X \). If \( T \) is invertible, then all \( J \)-sets are \( T^{-1} \)-invariant. For more details, see [6].

Recall that we say \( T \in B(X) \) is power bounded with a power bound \( M > 0 \) whenever \( \| T^n \| \leq M \) for all positive integers \( n \). In [6, Proposition 2.10] it was shown that if \( T \) is power bounded we have the following:

\[
J(T, x) = \{ y : \text{there exists a strictly increasing sequence of positive integers} \quad (k_n) \text{ such that} \quad T^{k_n} x \to y \}.
\]

An operator \( T \in B(X) \) is called a \( J \)-class \( (J^{\text{mix}} \)-class) operator if \( J(T, x) = X \ (J^{\text{mix}}(T, x) = X) \) for some \( x \in X \setminus \{0\} \). The set of all \( x \in X \) satisfying \( J(T, x) = X \ (J^{\text{mix}}(T, x) = X) \) is denoted by \( A_T (A_T^{\text{mix}}) \) and its elements are called the \( J \)-vectors (\( J^{\text{mix}} \)-vectors) for \( T \). It is well known that \( A_T \) is a closed subset of \( X \) and \( A_T^{\text{mix}} \) is a closed subspace of \( X \). We know that \( T \in B(X) \) is hypercyclic if and only if \( A_T = X \). For a good source on these topics we refer the reader to [1, 3-6, 15, 16, 18].

Recall that \( x \in X \setminus \{0\} \) is called a periodic point for \( T \in B(X) \) if \( T^n x = x \) for some positive integer \( n \). The smallest such number \( n \) is called the period of \( x \).

2. Some properties of \( J \)-sets

Here we state and prove some properties of \( J \)-sets that will be used in the proof of our main results. The proof of the following lemma is straightforward and so we omit it.

**Lemma 2.1** Suppose that \( T \in B(X) \). Then for all \( x, y \in X \) we have:

- a) \( J(T, x) \subseteq J(T, Tx) \subseteq J(T, T^2 x) \subseteq \cdots \),
- b) \( J^{\text{mix}}(T, x) \subseteq J^{\text{mix}}(T, Tx) \subseteq J^{\text{mix}}(T, T^2 x) \subseteq \cdots \),
and equality holds if \( T \) has a bounded inverse.
- c) \( J(T^m, x) \subseteq J(T, x) \) for all \( m \in \mathbb{N} \).

**Lemma 2.2** Suppose that \( T \in B(X) \). We have:

- a) If \( (T^n)_n \) is a Cauchy sequence and it has a subsequence that converges to a bounded operator \( U \) in the
strong operator topology, then for every $x$, both sets $J(T,x)$ and $J^{\text{mix}}(T,x)$ are equal to the singleton $\{Ux\}$.

(b) If $(T^n)_n$ converges to an operator $U$ in the strong operator topology, then

$$2J(T,x_1 + x_2) - U(x_1 + x_2) \subseteq J(T,x_1) + J(T,x_2)$$

for all $x_1, x_2 \in X$. The same result holds for $J^{\text{mix}}$-sets.

**Proof** a) Note that $T$ is power bounded. Since for every $x \in X$, $(T^n x)_n$ is a Cauchy sequence that has a subsequence converging to $Ux$, $T^n x \to Ux$. Hence, $J^{\text{mix}}(T,x) = J(T,x) = \{Ux\}$.

b) Suppose that $y \in J(T,x_1 + x_2)$. Then there exists a sequence $(w_n)_n$ in $X$ and a strictly increasing sequence of positive integers $(k_n)_n$ such that $w_n \to x_1 + x_2$ and $T^{k_n} w_n \to y$. Now we have

$$w_n - x_1 \to x_2, \quad T^{k_n}(w_n - x_1) \to y - Ux_1$$

and

$$w_n - x_2 \to x_1, \quad T^{k_n}(w_n - x_2) \to y - Ux_2.$$

Hence, we get

$$y - Ux_1 \in J(T,x_2), \quad y - Ux_2 \in J(T,x_1),$$

and so

$$2y - U(x_1 + x_2) \in J(T,x_1) + J(T,x_2),$$

which gives the result. Putting $k_n = n$, by the same method we conclude (b) for $J^{\text{mix}}$-sets. \qed

Recall that for $T \in B(X)$, by $\{T\}'$ we mean the collection of all bounded operators on $X$, which commutes with $T$.

**Lemma 2.3** Suppose that $T \in B(X)$ and $U \in \{T\}'$. For all positive integers $m$ and $x \in X$, we have:

a) $U^m(J(T,x)) \subseteq J(T,U^m x)$,

b) $U^m(J^{\text{mix}}(T,x)) \subseteq J^{\text{mix}}(T,U^m x)$,

and equality holds if $U$ has a bounded inverse. Moreover, if $U$ is surjective, then $A_T$ is a $U$-invariant subset and $A_T^{\text{mix}}$ is a $U$-invariant subspace of $X$.

**Proof** See Lemma 2.6 in [2] for parts (a) and (b). If $U$ has a bounded inverse, then for $z \in J(T,U^m x)$, there exists a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(u_n)_n$ in $X$ such that $u_n \to U^m x$ and $T^{k_n} u_n \to z$. We therefore get

$$U^{-m} u_n \to x, \quad T^{k_n}(U^{-m} u_n) \to U^{-m} z.$$ 

This yields that $U^{-m} z \in J(T,x)$ and hence $z \in U^m(J(T,x))$. Putting $k_n = n$, we can prove it for $J^{\text{mix}}$-sets. Now if $x \in A_T$, then $J(T,x) = X$, and so we have

$$X = U(X) = U(J(T,x)) \subseteq J(T,Ux) \subseteq X.$$

Hence, $U(A_T) \subseteq A_T$. Similarly, we can prove that $A_T^{\text{mix}}$ is a $U$-invariant subspace. \qed

A simple consequence of Lemma 2.11 in [6] yields that if $x$ is a $J$-vector ($J^{\text{mix}}$-vector) for $T$, then for any nonzero scalar $\lambda$, $\lambda x$ is a $J$-vector ($J^{\text{mix}}$-vector) for $T$. If $A_T(A_T^{\text{mix}})$ is nonempty, then it is an infinite set.

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3. On the property of \( J \)-reflexive operators

In this section, we first define the \( J \)-reflexive property, and then some properties of \( J \)-reflexive operators are investigated. We also state and prove necessary and sufficient conditions for an operator to be \( J \)-reflexive. Finally, we express \( J \)-reflexivity in terms of subsets. From now on, for simplicity, we denote the closure of a subset \( A \) in the strong operator topology by \( \overline{A}^{\text{SOT}} \).

**Definition 3.1** We call \( T \in B(X) \) a \( J \)-reflexive operator if every bounded operator that leaves invariant \( J(T, x) \) (for all \( x \) in \( X \)) is contained in the \( \overline{\text{orb}(T)}^{\text{SOT}} \). \( J \)-mix-reflexivity can also be defined in a similar way.

It is clear that every \( J \)-reflexive operator on a Hilbert space is orbit-reflexive, but there exists an orbit-reflexive operator that is not \( J \)-reflexive. See the following example.

**Example 3.2** Suppose that \( U \) is a bilateral shift on \( l^2(\mathbb{Z}) \), i.e. \( Uf(n) = f(n-1) \) where \( f \in l^2(\mathbb{Z}) \). Then \( U \) is orbit-reflexive, since \( U \) is a normal operator [11]. We know that \( U^{-1}f(n) = f(n+1) \), so \( U^{-1} \) does not belong to \( \overline{\text{orb}(U)}^{\text{SOT}} \). By Proposition 2.8 in [6], \( U^{-1} \) leaves invariant \( J \)-sets of \( U \), so \( U \) is not \( J \)-reflexive.

It is easy to see that if \( T \in B(X) \), and for all \( x \in X \), \( J(T, x) \) is an empty set or the whole space \( X \) or the singleton \( \{0\} \), then \( T \) is not \( J \)-reflexive. For example, hypercyclic operators are not \( J \)-reflexive, since their \( J \)-sets are the whole space. However, the identity operator \( I \) is the only bounded operator for which \( J(I, x) = \{x\} \) for all \( x \). Obviously, \( I \) is \( J \)-reflexive.

Recall that if \( T \) is an operator, then the set \( \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible} \} \) is called the spectrum of \( T \) and it is denoted by \( \sigma(T) \). Now we investigate some properties of \( J \)-reflexive operators.

**Theorem 3.3** Let \( T \in B(X) \) be a \( J \)-reflexive operator. Then we have the following:

(a) If \( T \) is invertible, then \( (T^n)_n \) has a subsequence that converges to \( I \) in the strong operator topology.

(b) If \( T \) is not identity and \( (T^n)_n \) is a Cauchy sequence, then \( 0 \in \sigma(T) \).

(c) If \( T \) is not identity and is invertible, then \( (T^n)_n \) diverges.

**Proof**

a) Since by Proposition 2.8 in [6], \( T^{-1} \) leaves invariant \( J(T, x) \) for all \( x \), by the hypothesis there exists a sequence \( (n_k)_k \) of positive integers such that \( T^{-1}x = \lim_k T^{n_k}x \) for all \( x \). Thus, \( T^{n_k+1} \to I \) in the strong operator topology.

b) On the contrary, suppose that \( 0 \notin \sigma(T) \), so \( T \) is invertible, and by part (a), there exists a sequence of positive integers \( (n_k)_k \) such that \( T^{n_k} \to I \) in the strong operator topology. Since \( B(X) \) is a Banach space and \( (T^n)_n \) is a Cauchy sequence, it follows that \( T^n \to I \) in the strong operator topology. Similarly, since \( T^{n_k+1} \to T \) in the strong operator topology, we have \( T^n \to T \) in the strong operator topology and so \( T = I \), which is a contradiction.

c) If \( (T^n)_n \) converges, then by part (b), \( 0 \in \sigma(T) \), which is a contradiction. \( \square \)

**Lemma 3.4** If \( T \in B(X) \) is a \( J \)-reflexive operator, then \( \|T\| \geq 1 \).

**Proof** Suppose that \( \|T\| < 1 \). Then, since \( \|T^n x\| \leq \|T\|^n \|x\| \), we have

\[
J^{mix}(T, x) = J(T, x) = \{0\}
\]

for all \( x \). Hence, \( T \) can not be \( J \)-reflexive, which is a contradiction. \( \square \)
Isomorphisms preserve the $J$-reflexive property, as follows.

**Theorem 3.5** Suppose that $X$ and $Y$ are Banach spaces and $S : X \to Y$ is an isomorphism. If $T$ is a $J$-reflexive operator on $X$, then $STS^{-1}$ is also a $J$-reflexive operator on $Y$.

**Proof** First, we show that $J(STS^{-1}, y) = S(J(T, S^{-1}y))$. Note that since $S$ is an isomorphism, $S$ and $S^{-1}$ are bounded. If $z \in J(STS^{-1}, y)$, then there exist $(y_n)_n$ in $Y$ and a strictly increasing sequence of positive integers $(k_n)_n$ such that $y_n \to y$ and $\lim_n ST^{k_n}S^{-1}y_n = z$. Since $S^{-1}y_n \to S^{-1}y$, we get $S^{-1}z \in J(T, S^{-1}y)$ and so $z \in S(J(T, S^{-1}y))$. Thus,

$$J(STS^{-1}, y) \subseteq S(J(T, S^{-1}y)).$$

Conversely, if $z \in S(J(T, S^{-1}y))$, then there exist $x \in X$, $(u_n)_n \subseteq X$, and a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$z = Sx, u_n \to S^{-1}y \text{ and } \lim_n T^{k_n}u_n = x.$$

Putting $v_n = Su_n$, then $v_n \to y$ and we have

$$z = Sx = \lim_n ST^{k_n}S^{-1}v_n.$$

Hence, $z \in J(STS^{-1}, y)$. Now if $W \in B(Y)$ leaves invariant the sets $J(STS^{-1}, y)$, then for all $y \in Y$, we have

$$WS(J(T, S^{-1}y)) \subseteq S(J(T, S^{-1}y)).$$

Therefore, we get

$$S^{-1}WS(J(T, S^{-1}y)) \subseteq J(T, S^{-1}y).$$

By the $J$-reflexivity of $T$, there exists a sequence $(n_k)_k$ of positive integers such that

$$S^{-1}WSx = \lim_k T^{n_k}x$$

for all $x \in X$. Putting $u = Sx$, we have

$$Wu = \lim_k (STS^{-1})^{n_k}u$$

for all $u \in Y$ and the proof is complete. \hfill \Box

In the following theorem, the sufficient conditions for the $J$-reflexivity of an operator are given.

**Theorem 3.6** Let $T \in B(X)$. If $T$ satisfies one of the following statements, then $T$ is $J$-reflexive: 

a) $T$ is a power bounded operator that has no periodic point and $x \in J(T, x)$ for all $x \in X$.

b) $T$ is power bounded and invertible, and there exists $m \in \mathbb{N}$ such that $T^m x \in J(T, x)$ for all $x \in X$.

c) $T$ is invertible and there exists positive integer $m$ such that $T^m = I$.

**Proof** Let $T$ satisfy in (a) and suppose on the contrary that $T$ is not $J$-reflexive. Then there exists an operator $S \in B(X) \setminus orb(T)$ such that

$$S(J(T, x)) \not\subseteq J(T, x).$$
for all \( x \in X \). Therefore, there exist \( x_0 \in X \) and \( \delta > 0 \) such that \( B(Sx_0; \delta) \cap \text{orb}(T, x_0) \) is a finite set. Since \( x_0 \in J(T, x_0),\) \( Sx_0 \in J(T, x_0) \), and this means that there exists a strictly increasing sequence of positive integers \((k_n)_n\) such that
\[
Sx_0 = \lim_n T^{k_n}x_0.
\]
This is possible only if \( x_0 \) is a periodic point for \( T \), which is a contradiction. Now suppose that (b) holds. Let \( W \in B(X) \) be such that \( W(J(T, x)) \subseteq J(T, x) \) for all \( x \). The surjectivity of \( T \) yields the surjectivity of \( T^m \) and therefore for every \( z \in X \) there exists \( x \in X \) such that \( z = T^m x \). Since for all \( x \), \( T^m x \in J(T, x) \), we have \( W(T^m x) \in J(T, x) \) and therefore there exists a strictly increasing sequence of positive integers \((k_n)_n\) such that
\[
Wz = W(T^m x) = \lim_n T^{k_n} x = \lim_n T^{k_n - m} z.
\]
Thus, \( W \in \text{orb}(T)^{\text{SOT}} \) and \( T \) is \( J \)-reflexive. Finally, assume that (c) holds. Since \( T^m = I \), \( \text{orb}(T) \) is a finite set and therefore \( T \) is power bounded. On the other hand, for all \( x \), \( Tx \in J(T^m, Tx) \), and then by Lemma 2.1 (c), \( Tx \in J(T, Tx) \). Eventually, Lemma 2.1 (a) (equality holds) yields that \( Tx \in J(T, x) \). Now by using (b), we get the desired result. \( \square \)

Recall that the space of convergent sequences is usually denoted by \( c \). This is a Banach space over \( \mathbb{C} \) or \( \mathbb{R} \) under the supremum norm.

**Example 3.7** Define \( T : c \to c \) by
\[
T(x_1, x_2, x_3, \cdots) = (x_2, x_1, x_3, \cdots).
\]
Clearly \( T \) is invertible, and \( ||T|| = 1 \). Since for any \( x \in X \) and all \( n \in \mathbb{N} \) we have \( T^{2n+1} x = Tx \) and \( (T^2)^n(Tx) = Tx \) and so \( Tx \in J(T^2, x) \). Now \( Tx \in J(T, x) \) by Lemma 2.1 (c). Indeed, \( J(T, x) = \{x, Tx\} \) for all \( x \). Hence, \( T \) is \( J \)-reflexive by Theorem 3.6 (b).

**Example 3.8** Let \( p(A) \) be a permutation \( p \) on a finite set \( A \). Define \( S_{ij} \) on \( c \) by
\[
S_{ij}(x_1, x_2, \cdots) = (x_1, \cdots, x_{i-1}, p\{x_i, \cdots, x_j\}, x_{j+1}, \cdots).
\]
If \( m \) is the smallest positive integer satisfying \( p^{m+1} = p \), then \( S_{ij}^{m+1} x = S_{ij} x \). Thus, \( S_{ij} x \in J(S_{ij}^m, x) \) and by Lemma 2.1 (c), \( S_{ij} x \in J(S_{ij}, x) \). Indeed,
\[
J(S_{ij}, x) = \{x, S_{ij} x, \cdots, S_{ij}^{m-1} x\}.
\]
Now by Theorem 3.6 (b), it follows that \( S_{ij} \) is \( J \)-reflexive.

**Example 3.9** Let \( H \) be a real Hilbert space and \( \Lambda : B(H) \to B(H) \) be defined by \( \Lambda(T) = T^* \). Then \( \Lambda^2 = I \) and \( \Lambda \) is \( J \)-reflexive by Theorem 3.6 (c).

Next we present some examples of \( J \)-reflexive and non-\( J \)-reflexive operators on finite dimensional spaces.

**Example 3.10** Let \( T \) be defined on the space of complex \( n \times n \) matrices by \( T(A) = A^t \) where \( A^t \) is the transpose of \( A \). Since \( T^2 = I \), \( T \) is \( J \)-reflexive by Theorem 3.6 (c).
Example 3.11 Suppose that $\dim X = 1$ and $T$ is defined on $X$ by $Tx = (1/2)x$. Then $\|T\| < 1$, and by Lemma 3.4, it follows that $T$ is not $J$-reflexive.

The concept of $J$-reflexivity can be expressed in terms of subsets as follows.

Definition 3.12 Let $T$ be a bounded linear operator on a Banach space $X$ and $M$ be a subset of $X$. We call $T$ $M$-$J$-reflexive if every $W \in B(X)$ that leaves invariant $J(T, x)$ for all $x$ in $M$ is contained in $\overline{\text{orb}(T)}_{SOT}$. $M$-$J^{mix}$-reflexivity can also be defined in a similar way.

It is clear that $M$-$J$-reflexivity implies $J$-reflexivity. The converse is true for power bounded operators and dense subsets.

Theorem 3.13 If $T \in B(X)$ is $J$-reflexive and power bounded and $M$ is a dense subset of $X$, then $T$ is $M$-$J$-reflexive.

Proof Assume that $W \in B(X)$ leaves invariant $J(T, z)$ for all $z \in M$. It is enough to show that $W$ leaves invariant $J(T, x)$ for all $x \in X$. Suppose that $x \in X$ and $y \in J(T, x)$. Then there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that $y = \lim_n T^{k_n}x$. By density of $M$, we can find sequences $(x_n)_n$ and $(y_n)_n$ in $M$ such that $x_n \to x$ and $y_n \to y$. Thus, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|x_n - x\| < \epsilon/3L, \quad \|y_n - y\| < \epsilon/3, \quad \|y - T^{k_n}x\| < \epsilon/3$$

, where $L$ is a power bound of $T$. Now for $m, n \geq N$, we get

$$\|y_m - T^{k_n}x_m\| \leq \|y_m - y\| + \|y - T^{k_n}x\| + \|T^{k_n}\|\|x - x_m\| < \epsilon.$$ 

Thus, $y_m \in J(T, x_m)$ and therefore $Wy_m \in J(T, x_m)$ for $m \geq N$. Since $Wy_m$ tends to $Wy$, by applying the same method that has been used in the proof of Lemma 2.5 in [5], we obtain that $Wy \in J(T, x)$. This completes the proof.

References


