Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1726 – 1737
© TÜBİTAK
doi:10.3906/mat-1610-104

On the global L^p boundedness of a general class of h-Fourier integral operators

Chafika Amel AITEMRAR[1], **Abderrahmane SENOUSSAOUI**[2,*]

[1]Faculty of Mathematics and Informatics, Mohamed Boudiaf University of Sciences and Technologies, El M'naouar, Oran, Algeria

[2]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), Ahmed Ben Bella University of Oran 1, El M'naouar, Oran, Algeria

Received: 26.10.2016 • **Accepted/Published Online:** 28.03.2016 • **Final Version:** 24.07.2018

Abstract: In this paper, we study the L^p-boundedness of a class of semiclassical Fourier integral operators.

Key words: h-Admissible Fourier integral operators, symbol and phase, L^p-boundedness

1. Introduction

It is known that h-pseudodifferential operators appear in Cauchy problems associated with semiclassical hyperbolic partial differential equations [12, 14, 16]. However, the theory of pseudodifferential operators has not been limited to this type of problem. Various extensions and generalizations of the class of pseudodifferential operators are considered with respect to the studied problems. Our investigation concerns the class of h-Fourier integral operators [8, 9] defined by

$$(I_h(a,\phi)f)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi,\ \ f\in\mathcal{S}(\mathbb{R}^n),$$

where $a(x,\xi)$ is the amplitude, $\phi(x,\xi)$ is the phase function and $h\in\,]0,h_0]$.

However, the problem related to the introduction of these classes of operators is their boundedness and in which functional spaces. This boundedness depends on the choice of the amplitude $a(x,\xi)$ and the phase function $\phi(x,\xi)$. The general case was to consider the class of amplitudes introduced by Hörmander [13], denoted $S^m_{\rho,\delta}$ the space of $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$,

$$\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)\right| < +\infty,\ \ m\in\mathbb{R}, 0<\rho,\delta\leq 1,$$

and the class of phase functions $\phi(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\})$; those are positively homogeneous of degree 1 in the frequency variable ξ. A first result of boundedness of Fourier integral operators in $L^2(\mathbb{R}^n)$ was given in Asada and Fujiwara [2, 3], assuming some conditions on the amplitude and the phase function. Adding the assumption that $a(x,\xi)\in S^m_{\rho,1-\rho}$ is of compact support with respect to x and a relation between the parameters ρ, m , n , p, Seeger et al. obtained in [20] an interesting result of local L^p-boundedness of these operators. For

*Correspondence: senoussaoui_abdou@yahoo.fr

2010 *AMS Mathematics Subject Classification:* 35S30, 35S05, 47G30 the global L^p-boundedness, Cordero et al. have considered Fourier integral operators with amplitudes in $S^m_{1,0}$; see [4]. Another result of global L^p-boundedness of Fourier integral operators, due to Coriasco and Ruzhanky [5, 6], was to consider their amplitudes in some subspaces of $S^0_{1,0}$. In the case $p=2$ (i.e. L^2-boundedness), the result is extended to a general class of amplitude in $S^m_{\rho,\delta}$, satisfying some conditions; see in this sense the papers by Messirdi and Senoussaoui [15] and Ruzhansky and Sugimoto [18]. We note that a class of unbounded Fourier integral operators with an amplitude in the Hörmander's class $S^0_{1,1}$ and in $\bigcap\limits_{0<\rho<1} S^0_{\rho,1}$ was given in [11] and [22].

Similar results of L^2-boundedness and L^2-compactness are obtained for h-Fourier integral operators, assuming some conditions on the amplitude and the phase function; see [1] and [10], [21].

As we have said, in the previous results one has considered the classes of amplitudes introduced by Hörmander [13], $S^m_{\rho,\delta}$; those consist of smooth functions $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, whose derivatives can be estimated by the weight function $1+|\xi|$. In a recent work [17, Example 2.4], see also Dos Santos Ferreira and Staubach [7], the authors have studied the L^p-boundedness of a class of Fourier integral operators associated with amplitudes not necessary C^∞ functions, but their derivatives have a behavior of type L^p. They have obtained results of boundedness of these operators from L^q into L^r, for the numbers q, r satisfying some relations with the others parameters $p,\ n,\ m,\ \rho,\ \delta$.

The aim of our work is to give an analogous study established by Rodriguez-Lopez and Staubach in [17, Example 2.4], for h-Fourier integral operators introduced as in [1] and [10].

2. A general class of amplitudes and nondegenerate phase functions

We recall in this section the classical definitions and notations of the theory of pseudodifferential operators; then we introduce some definitions relating to our study of the boundedness of pseudodifferential operators.

For convenience we use the notation $\langle\xi\rangle$ for $\left(1+|\xi|^2\right)^{\frac{1}{2}}$.

Definition 1 *Let $1 \le p \le \infty, m \in \mathbb{R}$, and $0 \le \rho \le 1$. We denote by $L^p S^m_\rho(\mathbb{R}^n)$, the space of functions $a(x,\xi), x, \xi \in \mathbb{R}^n$ such that $a(x,\xi)$ is measurable in $x \in \mathbb{R}^n$, $a(x,\xi) \in C^\infty(\mathbb{R}^n)$ a.e. $x \in \mathbb{R}^n$, and for each multi-index α, there exists a constant C_α such that*

$$\left\|\partial^\alpha_\xi a(.,\xi)\right\|_{L^p(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}.$$

Example 1 *For the existence of such functions see [17, Example 2.4].*

Definition 2 *We denote by $\Phi^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, the space of real valued functions $\phi(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\})$, such that $\phi(x,\xi)$ is positively homogeneous of degree 1 in the frequency variable ξ, and satisfies the following condition:*

Notation 1 $\forall \alpha, \beta \in \mathbb{N}^n$, *with* $|\alpha|+|\beta| \ge k$, $\exists C_{\alpha,\beta} > 0$:

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\}} |\xi|^{-1+|\alpha|} \left|\partial^\alpha_\xi \partial^\beta_x \phi(x,\xi)\right| \le C_{\alpha,\beta}.$$

Definition 3 *A real valued phase* $\phi(x,\xi) \in C^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ *satisfies the strong nondegeneracy condition (or the SND condition for short), if there exists a constant* $c > 0$ *such that*

$$\left|\det \frac{\partial^2 \phi(x,\xi)}{\partial x_j \partial \xi_k}\right| \geq c \text{ for all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

Example 2 *The phase function*

$$\phi(x,\xi) = |\xi| + \langle x, \xi \rangle$$

satisfies the SND condition and belongs to the class Φ^2.

Definition 4 *We call* h*-Fourier integral operator every* C^∞ *application* I_h *of* $]0,h_0]$ *in* $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), L^p(\mathbb{R}^n))$ *of the form*

$$(I_h(a,\phi) f)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)} a(x,\xi) \widehat{f}(\xi)\, d\xi,$$

where $a \in L^p S^m_\rho(\mathbb{R}^n)$ *and* $\phi \in \Phi^k(\mathbb{R}^n)$. *Here*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle} f(x)\, dx.$$

3. Assumption and preliminaries

We consider the following Littlewood–Paley partition of unity (see [20])

$$\Psi_0(\xi) + \sum_{j=1}^{\infty} \Psi_j(\xi) = 1,$$

where

$$supp\Psi_0 \subset \{\xi \in \mathbb{R}^n; |\xi| \leq 2\},$$

and $\Psi_j(\xi) = \Psi\left(2^{-j}\xi\right)$, with

$$supp\Psi \subset \left\{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\right\}.$$

However, for our study we need another decomposition in the following way.

For each j we fix a collection of unit vectors $\left\{\xi_j^\nu\right\}_\nu$ that satisfy

(1) $\left|\xi_j^\nu - \xi_j^{\nu'}\right| \geq 2^{-\frac{j}{2}}$, if $\nu \neq \nu'$.

(2) If $\xi \in \mathbb{S}^{n-1}$, then there exists a ξ_j^ν so that $\left|\xi - \xi_j^\nu\right| < 2^{-\frac{j}{2}}$.

Let Γ_j^ν denote the cone

$$\Gamma_j^\nu = \left\{\xi \in \mathbb{R}^n; \left|\frac{\xi}{|\xi|} - \xi_j^\nu\right| \leq 2 \cdot 2^{-\frac{j}{2}}\right\}.$$

We can also construct an associated partition of unity given by functions χ_j^ν, homogeneous of degree 0 in ξ with support in Γ_j^ν, i.e.

$$\sum_\nu \chi_j^\nu(\xi) = 1, \text{ for all } \xi \neq 0 \text{ and all } j,$$

and such that

$$\left|\partial_\xi^\alpha \chi_j^\nu(\xi)\right| \leq C_\alpha 2^{\frac{|\alpha| j}{2}} |\xi|^{-|\alpha|},$$

with the improvement

$$\left|\partial_{\xi_1}^N \chi_j^\nu(\xi)\right| \leq C_N |\xi|^{-N}, \text{ for } N \geq 1,$$

If one chooses the axis in ξ space such that ξ_1 is in the direction of ξ_j^ν and $\xi' = (\xi_2, \ldots, \xi_n)$ is perpendicular to ξ_j^ν.

Thus we can construct a Littelewood–Paley partition of unity by using the functions Ψ_j, χ_j^ν

$$\Psi_0(\xi) + \sum_{j=1}^{\infty} \sum_\nu \chi_j^\nu(\xi) \Psi_j(\xi) = 1.$$

Now let us consider an h-Fourier integral operator

$$(I_h(a,\phi) f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)} a(x,\xi) \widehat{f}(\xi) d\xi, \tag{3.1}$$

where $a \in L^p S_\rho^m(\mathbb{R}^n)$ and $\phi \in \Phi^k(\mathbb{R}^n)$. Using the Littlwood–Paley decomposition above, we decompose this operator as

$$I_h^0 f(x) + \sum_{j=1}^{\infty} \sum_\nu (I_h)_j^\nu f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)} a(x,\xi) \Psi_0(\xi) \widehat{f}(\xi) d\xi$$
$$+ (2\pi)^{-n} \sum_{j=1}^{\infty} \sum_\nu \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)} a(x,\xi) \chi_j^\nu(\xi) \Psi_j(\xi) \widehat{f}(\xi) d\xi. \tag{3.2}$$

Remark 1 *I_h^0 is called the low frequency part and $(I_h)_j^\nu$ the high frequency part of the h-Fourier integral operator I_h.*

Let us introduce the phase function $\Phi(x,\xi)$ and the amplitude $A_j^\nu(x,\xi)$ by

$$\Phi(x,\xi) = \phi(x,\xi) - \left\langle (\nabla_\xi \phi)\left(x, \xi_j^\nu\right), \xi \right\rangle,$$

$$A_j^\nu(x,\xi) = e^{\frac{i}{h}\Phi(x,\xi)} a(x,\xi) \chi_j^\nu(\xi) \Psi_j(\xi). \tag{3.3}$$

One can verify (see [23, p. 407]) that $\Phi(x,\xi)$ satisfies for $N \geq 2$ with the support of $A_j^\nu(x,\xi)$,

(1) $\left|\left(\frac{\partial}{\partial \xi_1}\right)^N \Phi(x,\xi)\right| \leq C_N 2^{-Nj}$,

(2) $\left|(\nabla_{\xi'})^N \Phi(x,\xi)\right| \leq C_N 2^{\frac{-Nj}{2}}$,

Thus we can rewrite $(I_h)_j^\nu$ as

$$(I_h)_j^\nu f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} A_j^\nu(x,\xi)\, e^{\frac{i}{h}\left\langle (\nabla_\xi \phi)\left(x,\xi_j^\nu\right),\xi\right\rangle} \widehat{f}(\xi)\, d\xi.$$

The following lemma is similar to lemma 3.1 of [17]. For the proof see [7, lemma 1.2.3].

Lemma 1 *Any h-Fourier integral operator I_h with amplitude $a(x,\xi) \in L^p S_\rho^m(\mathbb{R}^n)$ and phase function $\varphi(x,\xi) \in \Phi^2(\mathbb{R}^n)$ can be written as a finite sum of h-Fourier integral operators*

$$(2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{\frac{i}{h}\psi(x,\xi) + \frac{i}{h}\langle \nabla_\xi \phi(x,\zeta),\xi\rangle} \widehat{f}(\xi)\, d\xi,$$

where ζ is a point on the unit sphere $\mathbb{S}^{n-1}, \psi(x,\xi) \in \Phi^1(\mathbb{R}^n)$ and $a(x,\xi) \in L^p S_\rho^m(\mathbb{R}^n)$ is localized in the ξ variable around the point ζ.

We will also need the following lemmas, which are in [17].

Lemma 2 *Let $K \subset \mathbb{R}^n$ be a compact set, $U \supset K$ an open set, and k a nonnegative integer. Let φ be a real valued function in $C^\infty(U)$ such that*

(1) $|\nabla\varphi| > 0$,

(2) $\exists C_1 > 0$, $|\partial^\alpha \varphi| \le C_1 |\nabla\varphi|$ *for all multi-indices α with $|\alpha| \ge 1$,*

(3) $\exists C_2 > 0$, $\left|\partial^\alpha \left(|\nabla\varphi|^2\right)\right| \le C_2 |\nabla\varphi|^2$, *for all multi-indices α with $|\alpha| \ge 1$.*

Then, for $f \in C_0^\infty(K)$, any integer $k \ge 0, h \in]0,h_0]$, and $\lambda > 0$,

$$\lambda^k \left| \int_{\mathbb{R}^n} f(\xi)\, e^{\frac{i}{h}\lambda\varphi(\xi)} d\xi \right| \le C_{k,n,K} \sum_{|\alpha|\le k} \int_K |\partial^\alpha f(\xi)|\, |\nabla\varphi(\xi)|^{-k} d\xi.$$

Lemma 3 *If $a \in L^p S_\rho^{m_1}(\mathbb{R}^n)$ and $b \in L^q S_\rho^{m_2}(\mathbb{R}^n)$ then $a \cdot b \in L^r S_\rho^{m_1+m_2}(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, 1 \le p,q \le \infty$. Moreover, if $\eta(\xi) \in C_0^\infty$ and $a_\varepsilon(x,\xi) = a(x,\xi)\,\eta(\varepsilon\xi)$ and $\varepsilon \in [0,1)$, then one has*

$$\sup_{0<\varepsilon\le 1} \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-m+\rho|\alpha|} \left\| \partial_\xi^\alpha a_\varepsilon(.,\xi) \right\|_{L^p(\mathbb{R}^n)} \le c_{m,\rho,p,\alpha,\eta}.$$

4. Global boundedness of h-Fourier integral operators

The main result of this work is to prove boundedness results for h-Fourier integral operators with smooth strongly nondegenerate phase function.

According to the decomposition in (3.2), we will estimate each term in this sum. First consider the low frequency portion of the h-Fourier integral operator given by

$$\left(I_h^0(a,\phi) f\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)} a(x,\xi)\, \Psi_0(\xi)\, \widehat{f}(\xi)\, d\xi. \tag{4.1}$$

Since $\Psi_0 \in C_0^\infty$ is supported near the origin, we can see I_h^0 as an h-Fourier integral operator I_a whose amplitude $a(x,\xi)$ is compactly supported in the frequency variable ξ. To begin, we need the following lemma.

Lemma 4 *Let $\eta(\xi)$ be a C_0^∞ function and set*

$$K(x,z) = \int_{\mathbb{R}^n} \eta(\xi)\, e^{\frac{i}{h}(\psi(x,\xi)+\langle z,\xi\rangle)} d\xi,$$

where $\psi(x,\xi) \in \Phi^1(\mathbb{R}^n)$, $h \in]0,h_0]$. Then for any $\alpha \in (0,1)$, there exists a positive constant C such that

$$|K(x,z)| \leq C(1+|z|)^{-n-\alpha}.$$

Proof The proof is similar to Lemma 4.1 of [17]. □

Theorem 1 *Suppose that $0 < r \leq \infty$, $1 \leq p$, $q \leq \infty$ satisfy the relation $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. Assume that $\phi \in \Phi^2(\mathbb{R}^n)$ satisfies the SND condition and let $a \in L^p S_\rho^m$ with $m \leq 0, 0 \leq \rho \leq 1$, such that $supp_\xi a(x,\xi)$ is compact. Then the h-Fourier integral I_h defined in (3.1) is bounded from L^q to L^r.*

Proof Consider a closed cube Q of side-length L such that $supp_\xi a(x,\xi) \subset \overset{\circ}{Q}$. Denote by $\widetilde{a}(x,\xi) \in C^\infty(\mathbb{R}^n)$, the extension of $a(x,.)_{|Q}$ periodically of period L with respect to ξ. Let $\eta \in C_0^\infty$ with support in Q such that $\eta = 1$ on $supp_\xi a(x,\xi)$, and so $a(x,\xi) = \widetilde{a}(x,\xi)\eta(\xi)$. By the expansion of $\widetilde{a}(x,\xi)$ in a Fourier series, we have

$$I_h f(x) = \sum_{k\in\mathbb{Z}^n} a_k(x)\, I_\eta(f_k)(x), \tag{4.2}$$

where

$$\begin{aligned} a_k(x) &= \frac{1}{L^n}\int_{\mathbb{R}^n} a(x,\xi)\, e^{-\frac{i2\pi}{hL}\langle k,\xi\rangle} d\xi, \\ f_k(x) &= f\left(x - \frac{2\pi k}{L}\right) \\ I_\eta(f_k)(x) &= (2\pi)^{-n}\int \eta(\xi)\, e^{\frac{i}{h}\phi(x,\xi)} \widehat{f}_k(\xi)\, d\xi \end{aligned}$$

Take $l = 1,\cdots,n$ such that $k_l \neq 0$; then by integration by parts we obtain

$$a_k(x) = \frac{C_{n,N}}{|k_l|^N}\int_{\mathbb{R}^n} \partial_{\xi_l}^N a(x,\xi)\, e^{\frac{i2\pi}{hL}\langle k,\xi\rangle} d\xi.$$

However, from the hypothesis on the amplitude a and Lemma 3, we have

$$\max_{s=0,\ldots,N}\int_{\mathbb{R}^n} \left\|\partial_{\xi_l}^s a(.,\xi)\right\|_{L^p} d\xi \leq C_{n,N,\rho}\, |a|_{p,m,N},$$

for $N = [\max(n,n/r)] + 1$, then

$$\|a_k\|_{L^p} \leq C_{n,N,\rho}\, |a|_{p,m,N}(1+|k|)^{-N}. \tag{4.3}$$

Let us suppose for a moment that I_η is bounded on L^q and first consider the case $r \geq 1$. Minkowski's and Hölder's inequalities yield

$$\|I_h f\|_{L^r} \leq \sum_{k\in\mathbb{Z}^n} \|a_k\|_{L^p}\, \|I_\eta(f_k)\|_{L^q}.$$

Since the translations are isometries on L^q, we get that $\|I_\eta(f_k)\|_{L^q} \leq C_{\eta,\phi}\|f\|_{L^q}$. Therefore (4.3) yields for some constant $C > 0$

$$\|I_h f\|_{L^r} \leq C\,|a|_{p,m,N} \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \|f\|_{L^q} \approx \|f\|_{L^q}.$$

Assume now that $0 < r < 1$. Using (4.2) and Hölder's inequality we have

$$\int |I_h f(x)|^r\, dx \leq \sum_{k\in\mathbb{Z}^n} \int |I_\eta(f_k)(x)|^r\, |a_k(x)|^r\, dx \leq \sum_{k\in\mathbb{Z}^n} \|a_k\|_{L^p}^r \|I_\eta(f_k)\|_{L^q}^r,$$

which gives with (4.3) for some constant $C > 0$

$$\int |I_h f(x)|^r\, dx \leq C\,|a|_{p,m,N}^r \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-Nr} \|f\|_{L^q}^r \approx \|f\|_{L^q}^r.$$

Thus the operator I_h is bounded from L^q to L^r assuming that the introduced operator I_η is bounded on L^q. Then to finish the proof we shall prove this assumption. By Lemma 1 we can assume without loss of generality that

$$\phi(x,\xi) = \psi(x,\xi) + \langle t(x), \xi\rangle,$$

with a smooth map $t\colon \mathbb{R}^n \to \mathbb{R}^n$. For $f \in C_0^\infty(\mathbb{R}^n)$ one has

$$I_\eta(f)(x) = \frac{1}{(2\pi)^n} \int \eta(\xi)\, e^{\frac{i}{h}\langle \xi, t(x)\rangle} e^{\frac{i}{h}\psi(x,\xi)} \widehat{f}(\xi)\, d\xi = \int K(x, t(x)-y)\, f(y)\, dy,$$

with

$$K(x,z) = \frac{1}{(2\pi)^n} \int \eta(\xi)\, e^{\frac{i}{h}\langle \xi, z\rangle} e^{\frac{i}{h}\psi(x,\xi)} d\xi.$$

From Lemma 4 we have for any $\alpha \in (0,1)$, there exists a constant C such that

$$|K(x,z)| \leq C(1+|z|)^{-n-\alpha}, \tag{4.4}$$

and therefore $\sup_x \int |K(x,t(x)-y)|\, dy < \infty$. This shows the boundedness of the operator I_η on L^∞. Using the change of variables $z = t(x)$, the SND condition yields that $|\det Dt(x)| \geq C > 0$, and so the Schwartz global inverse function theorem (see [19, Theorem 1.22]) implies that t is a global diffeomorphism on $\mathbb{R}^n$ and $|\det J(z)| \leq \frac{1}{C}$, where $J(z)$ denotes the Jacobian of the change of variables. Thus using also (4.4) we obtain

$$\begin{aligned} \sup_y \int |K(x,t(x)-y)|\, dx &= \sup_y \int \left|K\left(t^{-1}(z), z-y\right)\right| |\det J(z)|\, dz \\ &\leq \frac{1}{C} \sup_y \int (1+|z-y|)^{-n-\alpha} dz < \infty. \end{aligned}$$

Therefore, by Schur's Lemma, I_h is bounded on L^q for all $1 \leq q \leq \infty$. The proof of the theorem is complete. □

Theorem 2 *Suppose that* $0 < r \leq \infty,\ 1 \leq p,\ q \leq \infty$ *satisfy the relation* $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. *Assume that* $\phi \in \Phi^2(\mathbb{R}^n)$ *satisfies the SND condition and let* $a \in L^p S^m_\rho(\mathbb{R}^n)$ *with* $0 \leq \rho \leq 1$ *and*

$$m < -\frac{(n-1)}{2}\left(\frac{1}{s} + \frac{1}{\min(p, s')}\right) - \frac{n(1-\rho)}{s},$$

where $s = \min(2, p, q)$ *and* $\frac{1}{s} + \frac{1}{s'} = 1$. *Then the* h*-Fourier integral operator* I_h *defined in* (3.1) *is bounded from* L^q *to* L^r.

Proof From Theorem 1 and the decomposition (3.2), we shall prove the estimate only for the high frequency part of the operator I_h. Consider the case $q < \infty$, and let us prove that

$$\|I_h f\|_{L^r(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)},\ \forall f \in C_0^\infty(\mathbb{R}^n). \tag{4.5}$$

We have

$$\begin{aligned}(I_h)_j^\nu f(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} A_j^\nu(x,\xi)\, e^{\frac{i}{h}\left\langle(\nabla_\xi\phi)\left(x,\xi_j^\nu\right),\xi\right\rangle}\widehat{f}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n} K_j^\nu\left(x, (\nabla_\xi\phi)\left(x,\xi_j^\nu\right) - y\right) f(y)\,dy,\end{aligned} \tag{4.6}$$

where

$$K_j^\nu(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} A_j^\nu(x,\xi)\, e^{\frac{i}{h}\langle z,\xi\rangle}d\xi.$$

Let L be the differential operator given by

$$L = I - 2^{2j}\frac{\partial^2}{\partial\xi_1^2} - 2^{2j}\triangle_{\xi'}.$$

Using the definition of $A_j^\nu(x,\xi)$ in (3.3) and the hypothesis on χ_j^ν and Φ, we can show that

$$\left\|L^N A_j^\nu(.,\xi)\right\|_{L^p} \leq C_N 2^{j(m+2N(1-\rho))},\ \forall\nu, \forall\xi \in supp_\xi A_j^\nu.$$

Let $t_j^\nu(x) = (\nabla_\xi\phi)\left(x,\xi_j^\nu\right)$ and $\alpha \in (0,\infty)$. As before, the SND condition on the phase function yields that $\left|\det Dt_j^\nu(x)\right| \geq c > 0$. Setting

$$g(y) = \left(2^{2j}y_1^2 + 2^{2j}|y'|^2\right)^{\frac{\alpha}{2}},$$

we can split

$$\begin{aligned}I_1 + I_2 &= \sum_\nu\left(\int_{g(y)\leq 2^{-j\rho}} + \int_{g(y)>2^{-j\rho}}\right)\left|K_j^\nu(x,y) f\left(t_j^\nu(x) - y\right)\right|dy \\ &= \sum_\nu\int\left|K_j^\nu(x,y) f\left(t_j^\nu(x) - y\right)\right|dy.\end{aligned} \tag{4.7}$$

Hölder's inequality in ν and y simultaneously and thereafter, since $1 \leq s \leq 2$, the Hausdorff–Young inequality in the y variable of the second integral yield for some constant $C > 0$

$$\begin{aligned} I_1 &\leq \left[\sum_\nu \int_{g(y)\leq 2^{-j\rho}} \left|f\left(t_j^\nu(x)-y\right)\right|^s dy\right]^{\frac{1}{s}} \left[\sum_\nu \int \left|K_j^\nu(x,y)\right|^{s'} dy\right]^{\frac{1}{s'}} \\ &\leq C\left[\sum_\nu \int_{g(y)\leq 2^{-j\rho}} \left|f\left(t_j^\nu(x)-y\right)\right|^s dy\right]^{\frac{1}{s}} \left[\sum_\nu \int \left(\left|A_j^\nu(x,y)\right|^s d\xi\right)^{\frac{s'}{s}}\right]^{\frac{1}{s'}}. \end{aligned}$$

Set $F_j^\nu(x,y) = f\left(t_j^\nu(x)-y\right)$, and raise both sides to the r-th power; by Hölder's inequality we get that

$$\|I_1\|_{L^r} \leq \tag{4.8}$$

$$C\left[\int\left(\sum_\nu \int_{g(y)\leq 2^{-j\rho}} \left|F_j^\nu(x,y)\right|^s dy\right)^{\frac{q}{s}} dx\right]^{\frac{1}{q}} \left[\left(\sum_\nu \left(\int \left|A_j^\nu(x,y)\right|^s d\xi\right)^{\frac{s'}{s}}\right)^{\frac{p}{s'}} dx\right]^{\frac{1}{p}}.$$

Using Minkowski's inequality, we obtain that the first term is bounded by

$$\left[\sum_\nu \int_{g(y)\leq 2^{-j\rho}} \left(\int \left|F_j^\nu(x,y)\right|^q dx\right)^{\frac{s}{q}} dy\right]^{\frac{1}{s}}.$$

Set $t_j^\nu(x) = t$; since $\left|\det Dt_j^\nu(x)\right| \geq c > 0$, we have

$$\left(\int \left|F_j^\nu(x,y)\right|^q dx\right)^{\frac{1}{q}} = \left(\int |f(t-y)|^q \left|\det Dt_j^\nu(x)\right|^{-1} dt\right)^{\frac{1}{q}} \leq C^{-\frac{1}{q}} \|f\|_{L^q}. \tag{4.9}$$

Thus, the first term on the right-hand side of (4.8) is bounded by a constant multiple of

$$\begin{aligned} \left[\sum_\nu \int_{g(y)\leq 2^{-j\rho}} dy\right]^{\frac{1}{s}} \|f\|_{L^q} &\leq 2^{j\frac{n-1}{2s}} 2^{-j\frac{n+1}{2s}} \left[\int_{|y|\leq 2^{-j\frac{\rho}{\alpha}}} dy\right]^{\frac{1}{s}} \|f\|_{L^q} \\ &\leq 2^{j\frac{n-1}{2s}} 2^{-j\frac{n+1}{2s}} 2^{-j\frac{n\rho}{\alpha s}} \|f\|_{L^q}. \end{aligned}$$

For the estimate of the second term we distinguish two cases. If $p \geq s'$, Minkowski's inequality yields that this term is bounded by

$$\left\{\sum_\nu \left[\int \left(\int \left|A_j^\nu(x,y)\right|^s d\xi\right)^{\frac{p}{s}} dx\right]^{\frac{s'}{p}}\right\}^{\frac{1}{s'}}$$

$$\leq \left\{\sum_\nu \left[\int \left(\int \left|A_j^\nu(x,y)\right|^p dx\right)^{\frac{s}{p}} d\xi\right]^{\frac{s'}{s}}\right\}^{\frac{1}{s'}}$$

$$\leq C 2^{jm} \left(\sum_\nu \left|\sup_\xi p A_j^\nu\right|^{\frac{s'}{s}}\right)^{\frac{1}{s'}} \leq C' 2^{jm} 2^{j\frac{n+1}{2s}} 2^{j\frac{n-1}{2s'}}$$

where we have used the fact that the measure of the ξ-support of A_j^ν is $O\left(2^{j\frac{n+1}{2}}\right)$.

If $p < s'$, this second term is bounded by

$$\begin{aligned}\left\{\sum_\nu \int \left(\int \left|A_j^\nu (x,y)\right|^s d\xi\right)^{\frac{p}{s}} dx\right\}^{\frac{1}{p}} &\leq \left\{\sum_\nu \left[\int \left(\int \left|A_j^\nu (x,y)\right|^p dx\right)^{\frac{s}{p}} d\xi\right]^{\frac{p}{s}}\right\}^{\frac{1}{p}} \\ &\leq C2^{jm}\left(\sum_\nu \left|\sup_\xi p A_j^\nu\right|^{\frac{p}{s}}\right)^{\frac{1}{p}} \\ &\leq C'2^{jm}2^{j\frac{n+1}{2s}}2^{j\frac{n-1}{2p}}.\end{aligned}$$

Therefore, (4.8) and the previous estimates give

$$\left\|I_1\right\|_{L^r} \leq C'2^{j\left(m-\rho\frac{n}{\alpha s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min\left(p,s'\right)}\right)\right)}\left\|f\right\|_{L^q}. \tag{4.10}$$

where is not depending on α.

Let us now estimate the L^r-norm of I_2. Define $h(y) = 1 + 2^{2j}y_1^2 + 2^j\left|y'\right|^2$ and let $M > \frac{n}{2s}$. By Hölder's inequality, we have

$$\begin{aligned}\left\|I_2\right\|_{L^r} \leq& \left[\int \left(\sum_\nu \int_{g(y)>2^{-j\rho}} \left|F_j^\nu (x,y)\right|^s h(y)^{-sM} dy\right)^{\frac{q}{s}} dx\right]^{\frac{1}{q}} \\ &\times \left[\int \left(\sum_\nu \int \left|K_j^\nu (x,y)\, h(y)^M\right|^{s'} dy\right)^{\frac{p}{s'}} dx\right]^{\frac{1}{p}}.\end{aligned} \tag{4.11}$$

Using Minkowski's inequality and (4.9), the first term of the right-hand side is bounded by a constant times

$$\left\|f\right\|_{L^q}\left[\sum_\nu \int_{g(y)>2^{-j\rho}} h(y)^{-sM} dy\right]^{\frac{1}{s}} \leq C\left\|f\right\|_{L^q} 2^{j\frac{n-1}{2s}}2^{\frac{-j(n+1)}{2s}}2^{j\frac{\rho}{\alpha}\left(2M-\frac{n}{s}\right)}.$$

To estimate the second term, suppose M is an integer. By the same argument as in the estimate of I_1, we obtain from Hausdorff–Young's inequality and Minkowski's inequality, the following bound of the second term:

$$\begin{aligned}&\left\{\int \left[\sum_\nu \int \left|K_j^\nu (x,y)\, h(y)^M\right|^{s'} dy\right]^{\frac{p}{s'}} dx\right\}^{\frac{1}{p}} \\ \leq& \left\{\int \left[\sum_\nu \left(\int \left|L^M A_j^\nu (x,y)\right|^s d\xi\right)^{\frac{s'}{s}}\right]^{\frac{p}{s'}} dx\right\}^{\frac{1}{p}} \\ \leq& \; C2^{j(m+2M(1-\rho))}2^{j\frac{n+1}{2s}}2^{j\frac{n-1}{2\min\left(s',p\right)}}.\end{aligned}$$

We can obtain the same result for nonintegal values of M, writing $M = [M] + \{M\}$, where $[M]$ is the integer part of M, and using Hölder's inequality with conjugate exponents $\frac{1}{\{M\}}$, $\frac{1}{1-\{M\}}$ and the result of the previous case.

Hence, (4.11) and the previous estimates yield

$$\|I_2\|_{L^r} \leq C2^{j(m+2M(1-\rho))}2^{j\frac{\rho}{\alpha}\left(2M-\frac{n}{s}\right)}2^{j\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)}\|f\|_{L^q}, \tag{4.12}$$

with a constant independent of α.

Therefore, the estimates (4.10) and (4.12) of I_1 and I_2, and the formula (4.7) give that for any $\alpha > 0$

$$\begin{aligned}\left\|(I_h)_j^{\nu} f\right\|_{L^r} \leq C \Bigg(& 2^{j\left(m+2M(1-\rho)+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(\rho,s')}\right)\right)}2^{j\frac{\rho}{\alpha}\left(2M-\frac{n}{s}\right)} \\ & +2^{j\left(m-\rho\frac{n}{\alpha s}+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)\right)}\|f\|_{L^q}\Bigg),\end{aligned}$$

where C does not depend on α. Letting α tend to ∞, we get

$$\left\|(I_h)_j^{\nu} f\right\|_{L^r} \leq C \leq 2^{j\left(m+2M(1-\rho)+\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(\rho,s')}\right)\right)}\|f\|_{L^q}.$$

Now if we let $R = \min(r, 1)$, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}(I_h)_j^{\nu} f\right\|_{L^r}^R & \leq \sum_{j=1}^{\infty}\left\|(I_h)_j^{\nu} f\right\|_{L^r}^R \\ & \leq C\sum_{j=1}^{\infty}2^{jR\left[\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)+m+2M(1-\rho)\right]}\|f\|_{L^q}^R \\ & \leq C'\|f\|_{L^q}^R.\end{aligned}$$

provided $m < -\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)+2M(\rho-1)$. Since $m < -\frac{n-1}{2}\left(\frac{1}{s}+\frac{1}{\min(p,s')}\right)+\frac{n}{s}(\rho-1)$, we can find $M > \frac{n}{2s}$, which satisfies the above condition. $\square$

In this case of $q = 2 \leq p$, Theorem 2 can be improved in the same manner as in [17, theorem 4.6].

Theorem 3 *Let $2 \leq p \leq \infty$ and define $r = \frac{2p}{p+2}$. Assume that $\varphi \in \Phi^2(\mathbb{R}^n)$ satisfies the SND condition and let $a \in L^pS_\rho^m(\mathbb{R}^n)$ with $0 \leq \rho \leq 1$ and*

$$m < \frac{n(\rho-1)}{2}.$$

Then the h-Fourier integral operator I_h defined as in (3.1) is bounded from L^2 to L^r.

References

[1] Aitemrar CA, Senoussaoui A. h-Admissible Fourier integral opertaors. Turk J Math 2016; 40: 553-568.

[2] Asada K, Fujiwara D. On some oscillatory transformations in $L^2(\mathbb{R}^n)$. Japanese J Math 1978; 4: 299-361.

[3] Calderón AP, Vaillancourt R. On the boundedness of pseudodifferential operators. J Math Soc Japan 1971; 23: 374-378.

[4] Cordero E, Nicola F, Rodino L. On the global boundedness of Fourier integral opertaors. Ann Global Anal Geom 2010; 38: 373-398.

[5] Coriasco S, Ruzhansky M. On the boundedness of Fourier integral opertaors on $L^p(\mathbb{R}^n)$. Comptes Rendus Math matique, Acad mie des Sciences Paris 2010; 348: 847-851.

[6] Coriasco S, Ruzhansky M. Global L^p-continuity of Fourier integral operators. T Am Math Soc 2014; 366: 2575-2596.

[7] Dos Santos Ferreira D, Staubach W. Global and local regularity of Fourier integral operators on weighted and unweighted spaces. Memoirs of the Amer Math Soc 2014; 229, Memo1074.

[8] Duistermaat JJ. Fourier integral operators. New York, NY, USA: Courant Institute Lecture Notes, 1973.

[9] Egorov Yu V. Microlocal analysis. In Partial Differential Equations IV. Berlin, Germany: Springer-Verlag, 1993, pp. 1-147.

[10] Harrat C, Senoussaoui A. On a class of h-Fourier integral operators. Demonstratio Mathematica 2014; XLVII: 596-607.

[11] Hasanov M. A class of unbounded Fourier integral operators. J Math Anal Appl 1998; 225: 641-651.

[12] Helffer B. Théorie spectrale pour des opérateurs globalement elliptiques. Société Mathématiques de France Astérisque 112,1984.

[13] Hörmander L. Fourier integral operators I. Acta Math 1971; 127: 79-183.

[14] Hörmander L. The Weyl calculus of pseudodifferential operators. Comm Pure Appl Math 1979; 32: 359-443.

[15] Messirdi B, Senoussaoui A. On the L^2 boundedness and L^2 compactness of a class of Fourier integral operators. Elec J Diff Equ 2006; vol 2006: 1-12.

[16] Robert D. Autour de l'Approximation Semi-Classique. Boston, MA, USA: Birkäuser, 1987.

[17] Rodriguez-López S, Staubach W. Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators. J Functional Analysis 2013; 264: 2356-2385.

[18] Ruzhansky M, Sugimoto M. Global L^2 boundedness theorems for a class of Fourier integral operators. Comm Partial Diff Eq 2006; 31: 547-569.

[19] Schwartz JT. Nonlinear functional analysis. Gordon and Breach Science Publishers, New York, 1969.

[20] Seeger A, Sogge CD, Stein EM. Regularity properties of Fourier integral operators. Ann Math 1991; 134: 231-251.

[21] Senoussaoui A. Opérateurs h-admissibles matriciels à symbole opérateur. African Diaspora J Math 2007; 4: 7-26.

[22] Senoussaoui A. On the unboundedness of a class of Fourier integral operators on $L^2(\mathbb{R}^n)$. J Math Anal Appl 2013; 405: 700-705.

[23] Stein EM. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton, NJ, USA: Princeton Mathematical Series, 1993, Vol 43.