A singularly perturbed differential equation with piecewise constant argument of generalized type

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Abstract: The paper considers the extension of Tikhonov Theorem for singularly perturbed differential equation with piecewise constant argument of generalized type. An approximate solution of the problem has been obtained. A new phenomenon of humping has been observed in the boundary layer area. An illustrative example with simulations is provided.

Key words: Piecewise constant argument of generalized type, small parameter, singular perturbation, approximate solution, humping in the boundary layer area

1. Introduction

Singularly perturbed equations are often used as mathematical models describing processes in physics, chemical kinetics, and mathematical biology. This type of equation often arises during investigation of applied problems of technology and engineering [8, 14, 15, 18, 22]. Prandtl [23] was a pioneer to emphasize the significance of singular problems and the necessity of their appearance as mathematical models. He pointed out the importance of the subject while he was developing the theory of the boundary layer in hydrodynamics in 1904. Several scientists such as Friedrichs, Levinson, and Wazow [12, 16, 30] were then interested in singularly perturbed equations. Systematic investigation of singularly perturbed equations by many mathematicians began only after Tikhonov's proof of fundamental and well-known limit theorems for nonlinear systems of ordinary differential equations [26]. We refer to the books [20, 27, 29] for more information on the recent results on singularly perturbed equations. The initial and boundary value problems considered in the studies [9]-[11] are equivalent to the Cauchy problem with the initial jump for differential and integrodifferential equations in the stable case.

Systematic studies of theoretical and practical problems involving piecewise constant arguments were initiated in the early 1980s. Since then, differential equations with piecewise constant arguments have attracted great attention from researchers in mathematics, biology, engineering, and other fields. A mathematical model including a piecewise constant argument was first considered by Busenberg and Cooke [7] in 1982. They constructed a first-order linear equation to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of differential equations with piecewise constant arguments. A system of differential equation with piecewise constant argument of

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generalized type was introduced in [1]–[6].

The contribution of our work relates to a new Tikhonov theorem for singularly perturbed differential equations with piecewise constant arguments.

Tikhonov-type theorems express the limiting behavior of solutions of the singularly perturbed system. It is a powerful instrument for analysis of singular perturbation problems. Due to this fact, it has been studied for many types of differential equations: partial differential equations [17], singularly perturbed differential inclusions [28], and discontinuous differential equations [25]. Although there are some papers that consider singularly perturbed differential equations with piecewise constant arguments [19, 21, 24], there are no articles that discuss the approximation problem, and there is no investigation involving generalized piecewise constant arguments. In the present study, a new interesting phenomenon of humping was observed in the boundary layer area. The hump may be considered a spike and can be applied in neural networks theory [13].

2. Main result
Consider the following system

\[ \varepsilon \dot{x} = F(x, y(\beta(t))), \]
\[ \varepsilon \dot{y} = Q(x, y), \]

with the initial conditions

\[ x(0, \varepsilon) = x_0, \]
\[ y(0, \varepsilon) = \psi, \]

where \( \varepsilon \) is a small positive parameter. System (1) is defined in the domain \( G = \{(x, y, t) ||x|| \leq a, \|y\| \leq a, 0 \leq t \leq T\} \), where \( a \) and \( T \) are fixed positive numbers. The functions \( F \) and \( Q \) are continuously differentiable in the interior of the domain. The piecewise constant argument is determined by the function \( \beta(t) = \theta_i; \) if \( t \in [\theta_i, \theta_{i+1}), i = 1, 2, \ldots, p, 0 < \theta_1 < \theta_2 < \ldots \theta_p < T \).

The following assumptions are required throughout the paper. There exists a point \( (\varphi, \psi) \) in the domain such that

(C1) The function \( F \) satisfies the conditions

\[ F(\varphi, \psi) = 0, \]
\[ F_x(\varphi, \psi) < 0, \]
\[ F_y(\varphi, \psi) < 0. \]

(C2) The function \( Q \) satisfies the conditions

\[ Q(\varphi, \psi) = 0, \]
\[ Q_x(\varphi, \psi) < 0, \]
\[ Q_y(\varphi, \psi) < 0. \]
Theorem 2.1 Assume that the conditions (C1) and (C2) hold. If the initial point \((x_0, \psi)\) is in the domain of attraction of the fixed point \((\varphi, \psi)\), then for sufficiently small \(\varepsilon\), the problem (1)–(2) has a unique solution \(x(t, \varepsilon) = x(t, 0, x_0, \psi, \varepsilon), \ y(t, \varepsilon) = y(t, 0, x_0, \psi, \varepsilon)\) such that the following limiting equalities hold:

\[
\lim_{\varepsilon \to 0} x(t, \varepsilon) = \varphi, \quad \text{for} \quad 0 < t \leq T, \tag{9}
\]

\[
\lim_{\varepsilon \to 0} y(t, \varepsilon) = \psi, \quad \text{for} \quad 0 < t \leq T. \tag{10}
\]

**Proof.** Without loss of generality, we assume that \(x_0 < \varphi, \ y_0 = \psi\). For \(t \in [0, \theta_1]\) the first equation of system (1) takes the form

\[
\varepsilon \frac{dx}{dt} = F(x, \psi). \tag{11}
\]

From the assumptions made above, it implies that \(F(x, \psi) > 0\). Therefore, the last equation is equivalent to the equation

\[
\frac{dt}{dx} = \frac{-\varepsilon}{F(x, \psi)}. \tag{12}
\]

and the initial condition for equation (12) is

\[
t(x_0) = 0. \tag{13}
\]

Setting \(\varepsilon = 0\) in (12), we obtain a regular problem with the unperturbed equation

\[
\frac{dt}{dx} = 0. \tag{14}
\]

If \(\varepsilon\) is sufficiently small, then the problem (12)–(13) has a unique solution in the segment \([x_0, \varphi]\). The solution \(t(x, x_0, \psi, \varepsilon)\) can be made arbitrarily close to the horizontal \(t = 0\), i.e.

\[
\lim_{\varepsilon \to 0} t(x, x_0, y_0, \varepsilon) = 0, \quad x_0 \leq x \leq \varphi.
\]

If we consider \(x(t)\) as a function of the independent variable \(t\), then the integral curve starting at the point \((0, x_0)\) is included in the neighborhood of an arbitrary point on the line \(x = \varphi\). Next, by using the condition (C1), we apply the stability analysis to obtain that the first coordinate of the solution of (1)–(2) approaches the line \(x = \varphi\).

Next we take into account the initial value problem

\[
\varepsilon \frac{dy}{dt} = Q(x(t, \varepsilon), y) \quad y(0) = \psi, \tag{15}
\]

where \(x(t, \varepsilon) < \varphi\) and \(x(t, \varepsilon) \to \varphi\) as \(\varepsilon \to 0\). We will prove that the solution of equation (15) satisfies \(y(t, \varepsilon) > \psi\) and \(y(t, \varepsilon) \to \psi\) as \(\varepsilon \to 0\) for a fixed \(t \in (0, \theta_1]\).

By means of (C1), we have

\[
Q(x(t, \varepsilon), y(t, \varepsilon)) > 0 \tag{16}
\]

for all \(t\) near the initial moment. However, this cannot be true for all \(t > 0\). The equality

\[
Q(x, y) = Q_x(x_1, y_1)(x - \varphi) + Q_y(x_2, y_2)(y - \psi), \tag{17}
\]
is valid, where \( x_i, y_i, i = 1, 2 \), are numbers in a neighborhood of \((\varphi, \psi)\). This is why the condition \( x(t) \to \varphi \) implies that \( y(t) \) increases only to some moment of time \( t = \xi(\varepsilon) \). The moment tends to the zero as \( \varepsilon \to 0 \), and the function \( Q \) is negative for \( t > \xi \). Thus, there is a hump near the moment \( t = \xi \). A similar discussion can be conducted for the coordinate \( x \).

For \( t \in [\theta_1, \theta_2) \) the problem (1)–(2) takes the form

\[
\varepsilon \dot{x} = F(x, y(\theta_1)), \\
\varepsilon \dot{y} = Q(x, y),
\]

with the initial conditions

\[
x(\theta_1, \varepsilon) = x(\theta_1), \quad y(\theta_1, \varepsilon) = y(\theta_1).
\]

Since the values \( x(\theta_1), y(\theta_1) \) approach the point \((\varphi, \psi)\) as \( \varepsilon \to 0 \), the proof made above can be repeated for the interval \([\theta_1, \theta_2)\), with uniform convergence.

Proceeding in this way one can prove the limiting equalities of the Theorem in the intervals \( t \in [\theta_i, \theta_{i+1}), \ i = 2, 3, \ldots, p \).

The theorem is proved.

3. An example

Consider the following system with piecewise constant argument

\[
\varepsilon \dot{x} = -x - y(\beta(t)) + 5, \\
\varepsilon \dot{y} = -xy + 2x + y - 1,
\]

where \( \theta_i = i, i = 1, 2, 3, 4, 5, T = 6, \varphi = 2, \psi = 3 \). Take \( x_0 = 1 \).

![Figure 1](image_url)  

**Figure 1.** Coordinates \( x(t), x(0) = 1 \), and \( y(t) \) of the solution are in blue \((\varepsilon = 0.3)\), magenta \((\varepsilon = 0.2)\), red \((\varepsilon = 0.1)\), and green \((\varepsilon = 0.05)\).
One can easily see that the conditions of Theorem 2.1 are fulfilled. Indeed, system (20) is of the form (1) with \( F(x, y) = -x - y + 5, Q(x, y) = -xy + 2x + y - 1 \), and \( F(2, 3) = 0, Q(2, 3) = 0, F_x(2, 3) = -1 < 0, F_y(2, 3) = -1 < 0, Q_x(2, 3) = -1 < 0, Q_y(2, 3) = -1 < 0 \).

We provide results of simulations with two different values of \( x_0 \). The first one with \( x_0 = 1 \) is considered in Figure 1. One can observe the hump in the boundary layer. Another choice with \( x_0 = 3 \) is simulated in Figure 2, and the hump is observed in the simulations.

Thus, according to the main assertion, the solution of the initial value problem approaches the equilibrium, as the parameter decreases to zero. In both simulations the solutions approach the equilibrium as the parameter decreases.

References


