On small covers over a product of simplices

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Abstract: In this paper, we give a formula for the number of $\mathbb{Z}_n^2$-equivariant homeomorphism classes of small covers over a product of simplices. We also give an upper bound for the number of small covers over a product of simplices up to homeomorphism.

Key words: Small cover, equivariant homeomorphism, polytope, acyclic digraph

1. Introduction

A small cover is a smooth closed manifold $M^n$ that admits a locally standard $\mathbb{Z}_n^2$-action whose orbit space is a simple convex polytope. The notion of a small cover was introduced by Davis and Januszkiewicz [5] as a generalization of real toric manifolds. In [5], it was shown that every small cover over a simple convex polytope $P^n$ can be obtained from a characteristic function on the set of facets of $P^n$. There is a free action of the general linear group $GL(n, \mathbb{Z}_2)$ on the set of characteristic functions and the orbit space of this action is in one-to-one correspondence with the Davis–Januszkiewicz equivalence classes of small covers. Recently, several studies have been done to calculate the number of Davis–Januszkiewicz equivalence classes of small covers over a specific polytope (see [1, 3, 6]). In [6], Garrison and Scott used a computer program to find the number of small covers over a dodecahedron up to Davis–Januszkiewicz equivalence. In [3], Choi constructed a bijection between the set of Davis–Januszkiewicz equivalence classes of small covers over an $n$-cube and the set of acyclic digraphs with $n$-labeled nodes. He also gave a formula for the number of small covers over a product of simplices up to Davis–Januszkiewicz equivalence in terms of acyclic digraphs with labeled nodes.

There is a standard action of the automorphism group of the face poset of $P^n$ on the set of characteristic functions on $P^n$. Lü and Masuda [7] showed that there is a bijection between the set of orbits of this action and the set of $\mathbb{Z}_n^2$-equivariant homeomorphism classes of small covers over $P^n$. By Burnside’s lemma, the number of orbits of an action is the average number of the points fixed by an element of the group. Therefore, one can find the number of $\mathbb{Z}_n^2$-equivariant homeomorphism classes of small covers over $P^n$ by enumerating the number of fixed points of elements of the automorphism group. Using the Burnside lemma, Choi [3] gave a formula for the number of $\mathbb{Z}_n^2$-equivariant homeomorphism classes of small covers over a cube, which is the product of 1-simplices. When $P^n$ is a product of simplices of dimension greater than 1, the action of the automorphism group of the face poset is free. Therefore, the number of equivariant small covers over a product of simplices of dimension greater than 1 is the quotient of the number of the small covers and the order of the automorphism group.

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group of the face poset. In [2], Chen and Wang directly counted the number of equivariant homeomorphism classes of small covers over $\Delta^1 \times \Delta^{n_1} \times \Delta^{n_2}$ and $\Delta_1 \times \Delta^{n_3}$, where $\Delta^{n_i}$ is an $n_i$-simplex with $n_i \geq 1$ for $1 \leq i \leq 3$. In this paper, we use Choi’s argument to generalize these formulas to an arbitrary product of simplices.

The paper is organized as follows. In Section 2 we recall the basic theory about the small covers over a simple polytope and vector matrices. In Section 3 we obtain a formula for the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes over a product of equidimensional simplices. In Section 4 we give an upper bound for the number of small covers over a product of equidimensional simplices up to homeomorphism.

2. Preliminaries

An $n$-dimensional convex polytope $P$ is said to be simple if every vertex of $P$ is the intersection of precisely $n$ facets. A small cover over $P$ is a smooth closed $n$-manifold $M^n$ that admits a $\mathbb{Z}_2^n$-action that is locally isomorphic to a standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and the orbit space of the action is $P$.

Given a simple convex polytope $P$ of dimension $n$, let $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$ be the set of facets of $P$. A function $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ is called a characteristic function if it satisfies the nonsingularity condition that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_k}$ is nonempty, the set \{\(\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\)\} forms a basis for $\mathbb{Z}_2^n$. For a given point $p \in P$, let $\mathbb{Z}_2^n(p)$ be the subgroup of $\mathbb{Z}_2^n$ generated by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})$ where the intersection $\bigcap_{j=1}^k F_{i_j}$ is the minimal face containing $p$ in its relative interior. Then the manifold $M(\lambda) = (P \times \mathbb{Z}_2^n) / \sim$ where

\[(p, g) \sim (q, h) \text{ if } p = q \text{ and } g^{-1}h \in \mathbb{Z}_2^n(p)\]

is a small cover over $P$.

**Theorem 2.1 ([5])** For every small cover $M$ over $P$, there is a characteristic function $\lambda$ with $\mathbb{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on $P$.

Two small covers $M_1$ and $M_2$ over $P$ are said to be DJ-equivalent (Davis-Januszkiewicz equivalent) if there is a weakly $\mathbb{Z}_2^n$-homeomorphism $f : M_1 \to M_2$ covering the identity on $P$. Following [7], let $\Lambda(P)$ be the set of all characteristic functions on $P$. There is a free action of $GL(n, \mathbb{Z}_2)$ on $\Lambda(P)$ defined by $g \cdot \lambda = g \circ \lambda$. By the above theorem, DJ-equivalence classes of small covers over $P$ bijectively correspond to the coset $GL(n, \mathbb{Z}_2) \backslash \Lambda(P)$. In particular, $|\Lambda(P)|$ is equal to the product of $|GL(n, \mathbb{Z}_2) \backslash \Lambda(P)|$ and $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1})$.

On the other hand, the equivariant classes of small covers over $P$ are characterized by the action of the automorphism group of the face poset of $P$. More precisely, let $\text{Aut}(\mathcal{F}(P))$ be the group of bijections from the set of faces of $P$ to itself, which preserves the poset structure. Then $\text{Aut}(\mathcal{F}(P))$ acts on $\Lambda(P)$ on the right by $\lambda \cdot h = \lambda \circ h$. In [7], Lu and Masuda proved the following theorem.

**Theorem 2.2** The set of $\mathbb{Z}_2^n$-homeomorphism classes of small covers over $P$ corresponds bijectively to the coset $\Lambda(P) / \text{Aut}(\mathcal{F}(P))$.

By the above theorem, to find the number of equivariant classes of small covers over $P$, we need to find the number of orbits of $\Lambda(P)$ under the action of $\text{Aut}(\mathcal{F}(P))$. The Burnside lemma reduces this problem to
the enumeration of fixed points

\[ \Lambda(P)_h = \{ \lambda \in \Lambda(P) \mid \lambda(h(F)) = \lambda(F) \text{ for all } F \in \mathcal{F}(P) \} \]

by elements \( h \in \text{Aut}(\mathcal{F}(P)) \).

Lemma 2.3 (Burnside lemma) Let \( G \) be a finite group acting on a set \( X \). Then the number of \( G \)-orbits of \( X \) is equal to \( \frac{1}{|G|} \sum_{g \in G} |X^g| \), where \( X^g = \{ x \in X \mid gx = x \} \).

Therefore, one can find the number of \( \mathbb{Z}_2^n \)-equivariant homeomorphism classes of small covers over \( P^n \) by enumerating \( \Lambda(P)_h \) for all \( h \in \text{Aut}(\mathcal{F}(P)) \).

As a combination of the above theorems, we have the following result.

Theorem 2.4 The number of weakly \( \mathbb{Z}_2^n \)-homeomorphism classes of small covers over \( P \) is the size of the double coset \( \text{GL}(n, \mathbb{Z}_2) \backslash \Lambda(P)/\text{Aut}(\mathcal{F}(P)) \).

3. The number of \( \mathbb{Z}_2^n \)-equivariant homeomorphism classes

Let \( P = \Delta^{n_1} \times \cdots \times \Delta^{n_m} \), where \( \Delta^{n_i} \) is the standard \( n_i \)-simplex. Let \( G_m \) be the set of acyclic digraphs with \( m \) labeled nodes with labeled vertex set \( V(G) = \{ v_1, \ldots, v_m \} \). Here, a digraph is a graph with at most one edge directed from vertex \( v_i \) to \( v_j \). A directed graph is said to be acyclic if there is no directed cycle. The outdegree \( \text{outdeg}(v) \) (the indegree \( \text{indeg}(v) \)) of a vertex \( v \) is the number of edges directed from (to) \( v \). In [3], Choi gave the following formula for the number of small covers over \( P \).

Theorem 3.1 (Theorem 2.8, [3]) The number of DJ-equivalence classes of small covers over \( P = \Delta^{n_1} \times \cdots \times \Delta^{n_m} \) with \( \sum_{i=1}^m n_i = n \) is

\[ |\text{GL}(n, \mathbb{Z}_2) \backslash \Lambda(P)| = \sum_{G \in G_m} \prod_{v_i \in V(G)} (2^{n_i} - 1)^{\text{outdeg}(v_i)}. \]

It is well known that the automorphism group of the face poset of \( \Delta^n \) is the group of permutations on the set of facets, i.e. \( \text{Aut}(\mathcal{F}(\Delta^n)) \cong S_{n+1} \), where \( S_{n+1} \) is the symmetric group of degree \( n+1 \). To understand the automorphism group of \( \mathcal{F}(P) \), we need to take the number of \( \Delta^n \) occurring in \( P \) into account. For this reason, we write

\[ P = \prod_{i=1}^l P_i, \text{ where } P_i = \Delta_1^{n_{i_1}} \times \cdots \times \Delta_{m_i}^{n_{i_k}}, \]

with \( 1 \leq n_1 < n_2 < \cdots < n_l \) and \( \sum_{i=1}^l n_i m_i = n \). Then the set of facets of \( P^i \) is

\[ \{ f_{j,k}^i = \Delta_1^{n_{i_1}} \times \cdots \times \Delta_{j-1}^{n_{i_{j-1}}} \times \Delta_{j+1}^{n_{i_{j+1}}} \times \cdots \times \Delta_{m_i}^{n_{i_{m_i}}} \mid 0 \leq k \leq n_i, \ 1 \leq j \leq m_i \} \]

where \( \{ f_{j,0}, \ldots, f_{j,n_i} \} \) is the set of facets of the simplex \( \Delta_j^{n_i} \). Therefore, we have

\[ \mathcal{F}(P) = \{ F_{j,k}^i \mid 0 \leq k \leq n_i, \ 1 \leq j \leq m_i, \ 1 \leq i \leq l \} \]
where $F^i_{j,k} = P_i \times \cdots \times P_{i-1} \times f^i_{j,k} \times P_{i+1} \times \cdots \times P_l$. Note that there are $(n + m)$-facets, where $m = \sum_{i=1}^l m_i$.

Since $\text{Aut}(F(D^n)) \cong S_{n+1}$, $\text{Aut}(F(P))$ is the wreath product of $S_{n+1}$ with $S_{m_i}$, where $\mu \in S_{m_i}$ sends $f^i_{j,k}$ to $f^\mu_{j,k}$. More precisely, $\text{Aut}(F(P)) = S_{n+1} \wr S_{m_i}$ is equal to $S_{n+1} \times \cdots \times S_{n+1} \times S_{m_i}$, as a set where the group multiplication is defined by

$$\left(\sigma_1, \ldots, \sigma_{m_i}, \mu\right)\left(\sigma'_1, \ldots, \sigma'_{m_i}, \mu'\right) = \left(\sigma_1\sigma'_{\mu^{-1}(1)}, \ldots, \sigma_{m_i}\sigma'_{\mu^{-1}(m_i)}, \mu\mu'\right)$$

for any $\sigma_i, \sigma'_i \in S_{n+1}$ and $\mu, \mu' \in S_{m_i}$. Since $n_1 < n_2 < \cdots < n_l$, we have the following.

**Lemma 3.2** $\text{Aut}(F(P)) \cong \prod_{i=1}^l \left(S_{n+1} \wr S_{m_i}\right)$.

By the nonsingularity condition, a characteristic function must send any set obtained by taking $\{F^i_{j,k} \mid 0 \leq k \leq n_i\}$ for each $1 \leq j \leq m_i$ and $1 \leq i \leq l$ to a basis of $\mathbb{Z}^n_2$. When $1 < n_1$, more than one element is arbitrarily chosen from each set. However, for every nontrivial element $g$ of $\text{Aut}(F(P))$, there exist $1 \leq j \leq m_i$ and $1 \leq i \leq l$ for which at least two elements from the set $\{F^i_{j,k} \mid 0 \leq k \leq n_i\}$ are not fixed by $g$. Therefore, $g$ cannot fix any characteristic function. This means that the action of $\text{Aut}(F(P))$ on $F(P)$ is free and hence the number of equivariant homeomorphism classes of small covers over $P$ with $n_1 > 1$ is

$$\frac{|\Lambda(P)|}{|\text{Aut}(F(P))|} = \frac{|\Lambda(P)|}{\prod_{i=1}^l [(n_i + 1)!]^{m_i}(m_i)!}.$$ 

Since $|\text{GL}(n, \mathbb{Z}_2)| = \prod_{k=1}^n (2^n - 2^{k-1})$, by the above theorem we have:

**Corollary 3.3** Let $P = \prod_{i=1}^l \Delta^{n_i}_i \times \cdots \times \Delta^{n_l}_l$ with $\sum_{i=1}^l m_i = m$ and $\sum_{i=1}^l n_i m_i = n$. Define a function $n : \{1, \ldots, m\} \to \{n_1, \ldots, n_l\}$ by $n(s) = n_i$ whenever $k_1 + \cdots + k_{i-1} + 1 \leq s \leq k_1 + \cdots + m_i$.

Then the number of equivariant homeomorphism classes of small covers over $P$ with $n_1 > 1$ is

$$\frac{|\Lambda(P)|}{|\text{Aut}(F(P))|} = \frac{\left(\prod_{k=1}^n (2^n - 2^{k-1})\right)\left(\sum_{G \in \mathcal{G}_m} \prod_{v_j \in V(G)} (2^{n(s)} - 1)^{\text{outdeg}(v_j)}\right)}{\prod_{i=1}^l [(n_i + 1)!]^{m_i}(m_i)!}.$$ 

When $n_1 = 1$, the only elements of $\text{Aut}(F(P))$ that have a fixed point are the ones of the form

$$\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}, \quad \epsilon_i \in \mathbb{Z}_2$$

where $\chi_1, \cdots, \chi_{m_1}$ are the reflections in $\text{Aut}(F(P))$. To count the number of elements in $\Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}$, first note that it is a $\text{GL}(n, \mathbb{Z}_2)$-invariant subset of $\Lambda(P)$. Since the action of $\text{GL}(n, \mathbb{Z}_2)$ is free, we have

$$|\Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}| = |\text{GL}(n, \mathbb{Z}_2)| \times |\text{GL}(n, \mathbb{Z}_2)\backslash \Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}|.$$
To find $|GL(n,\mathbb{Z}_2)\setminus \Lambda(P)_{\chi_1^{s_1}\cdots \chi_{m_1}^{s_{m_1}}}|$ we use the correspondence given by Choi [3]. By the nonsingularity condition, for any $\lambda \in \Lambda(P)$, the vectors

$$\lambda(F^1_{1,1}), \ldots, \lambda(F^1_{m_1,1}), \lambda(F^2_{1,1}), \lambda(F^2_{1,2}), \ldots, \lambda(F^2_{1,n_2}), \ldots, \lambda(F^l_{m_1,1}), \ldots, \lambda(F^l_{m_1,n_l})$$

form a basis for $\mathbb{Z}_2^n$. For each coset in $GL(n,\mathbb{Z}_2)\setminus \Lambda(P)_{\chi_1^{s_1}\cdots \chi_{m_1}^{s_{m_1}}}$, choose a representative $\lambda$ for which the vectors in (1) correspond to the standard basis elements

$$\{\epsilon_i = (1,0,\ldots,0), \ldots, \epsilon_n = (0,\ldots,0,1)\},$$

respectively. More precisely, we have

$$\lambda(F^i_{j,k}) = \epsilon_{m_1n_1+\cdots+m_{i-1}n_{i-1}+(j-1)n_i+k}$$

for $1 \leq i \leq l$, $1 \leq j \leq m_i$ and $1 \leq k \leq n_i$. Let $A(\epsilon_1, \ldots, \epsilon_{m_1})$ be the set of such representatives. For the remaining facets, we write $F^i_{j,0} = F^i_{m_1+\cdots+m_{i-1}+j}$ for $1 \leq i \leq l$ and $1 \leq j \leq m_i$. Then we have

$$\lambda(F^i_{p}) = \sum_{q=1}^{n} a_{pq}\epsilon_q.$$ 

We can view the corresponding $(n \times m)$-matrix $\Lambda = [a_{pq}]$ as an $(m \times m)$-vector matrix $[v_{pq}]$ whose entries in the $p$th row are vectors in $\mathbb{Z}_2^{n(p)}$ where $n(p)$ is defined as in Corollary 3.3. We refer reader to [4] for details. Let $\Lambda_{s_1\cdots s_m}$ be the $(m \times m)$-submatrix of $\Lambda$ whose $i$th row is the $s_i$th row of $[v_{pq}]$. Then $\lambda$ satisfies the singularity condition if and only if every principal minor of $\Lambda_{s_1\cdots s_m}$ is 1 for any $1 \leq s_1 \leq n(1), \ldots, 1 \leq s_m \leq n(m)$.

**Theorem 3.4** $|\Lambda(P)_{\chi_1^{s_1}\cdots \chi_{m_1}^{s_{m_1}}}| = \left(\prod_{k=1}^{n} (2^n - 2^{k-1})\right) \left(\sum_{G \in \mathcal{G}_m(\epsilon_1, \ldots, \epsilon_{m_1}) \atop v_i \in V(G)} \prod_{v_i \in V(G)} (2^{n_i} - 1)^{\text{outdeg}(v_i)}\right)$

where $\mathcal{G}_m(\epsilon_1, \ldots, \epsilon_{m_1})$ is the set of acyclic digraphs with $m$ labeled nodes $\{v_1, \ldots, v_m\}$ such that $\text{indeg}(v_i) = 0$ whenever $\epsilon_i = 1$ for $1 \leq i \leq m_1$.

**Proof** Without loss of generality, we assume that $\epsilon_i = 1$ for $1 \leq i \leq t \leq m_1$ and $\epsilon_i = 0$ for $t < i \leq m_1$. Let $A = A(1, \ldots, 1, 0, \ldots, 0)$. For $\lambda \in A$, let $\Lambda = [v_{ij}]$ be the $(m \times m)$-vector matrix corresponding to $\lambda$.

Let $B(A) =: [b_{ij}]$ be the $\mathbb{Z}_2$-matrix whose $(i, j)$th entry is 1 if $v_{ij}$ is nonzero and 0 otherwise. By Lemma 5.1 in [4], $\Lambda$ is conjugate to a unipotent upper triangular vector matrix. Therefore, $B(A) - I_m$, where $I_m$ is the $(m \times m)$ identity matrix, is an adjacency matrix of an acyclic digraph. Define $\phi$ from $A$ to $\mathcal{G}_m$ by $\phi(\lambda) = G$ where the adjacency matrix of $G$ is $B(A) - I_m$.

Since $\lambda \in A$, $b_{ij} = 0$ for $i \neq j$ where $1 \leq j \leq t$ and $1 \leq i \leq n$. Therefore, the image of $\phi$ is indeed $\mathcal{G}_m(\epsilon_1, \ldots, \epsilon_{m_1})$. For $G \in \mathcal{G}_m(\epsilon_1, \ldots, \epsilon_{m_1})$, we have

$$|\phi^{-1}(G)| = \prod_{v_i \in V(G)} (2^{n_i} - 1)^{\text{outdeg}(v_i)},$$

as shown in the proof of Theorem 2.8 in [3].

Therefore, by the Burnside lemma, we have the following result.

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Theorem 3.5 The number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P$ with $n_1 = 1$ is

$$
\left( \sum_{(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m} \prod_{i=1}^m \left( \frac{2^{n_1} - 1}{\prod ((n_i + 1)!)} \right)^{\text{outdeg}(v_i)} \right) \prod_{k=1}^n (2^n - 2^{-k-1})
$$

where $G_m(\epsilon_1, \ldots, \epsilon_m)$ is the set of acyclic digraphs with $m$ labeled nodes $\{v_1, \ldots, v_m\}$ such that $\text{indeg}(v_i) = 0$ whenever $\epsilon_i = 1$ for $1 \leq i \leq m$.

Let $A_{mr}$ be the number of acyclic digraphs with $m$ labeled nodes and $r$ edges where the labeled vertex set is $\{v_1, \ldots, v_m\}$. For $\alpha \subseteq \{v_1, \ldots, v_m\}$, let $A^{\alpha}_{mr}$ be the number of acyclic digraphs with $m$ labeled nodes $\{v_1, \ldots, v_m\}$ such that $\text{indeg}(v) = 0$ for all $v \in \alpha$ and $A^{\alpha}_{mr}$ be the number of such acyclic digraphs with $r$ edges.

Corollary 3.6 (Theorem 3.3, [3]) If $P = I^n$ then the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P$ is

$$
\left( \sum_{i=0}^n \binom{n}{i} 2^{i(n-i)} A_i \frac{2^n - 1}{2^m n!} \right) \prod_{k=1}^n (2^n - 2^{-k-1}).
$$

Proof Let $\alpha(\epsilon_1, \cdots, \epsilon_n) = \{v_i | \epsilon_i = 1\}$. Then

$$
|A(P)_{x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}}| = \left( \prod_{k=1}^n (2^n - 2^{-k-1}) \right) A^\alpha_{n}(\epsilon_1, \cdots, \epsilon_n).
$$

By (4) in [8], for any $\alpha \subseteq \{v_1, \cdots, v_n\}$,

$$
A^\alpha_n = \sum_{r \geq 0} \sum_{k=0}^r \binom{|\alpha|}{r-k} A_n - |\alpha|, k.
$$

Therefore, we have

$$
\sum_{(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n} A^\alpha_{n}(\epsilon_1, \cdots, \epsilon_n) = \sum_{\alpha \subseteq \{v_1, \ldots, v_n\}} A^\alpha_n = \sum_{\alpha \subseteq \{v_1, \ldots, v_n\}} \sum_{r \geq 0} \sum_{k=0}^r \binom{|\alpha|}{r-k} A_n - |\alpha|, k
$$

$$
= \sum_{i=0}^n \binom{n}{i} \sum_{r \geq 0} \sum_{k=0}^r \binom{i(n-i)}{r-k} A_i, k
$$

$$
= \sum_{i=0}^n \binom{n}{i} \sum_{k \geq 0} \left( \sum_{r \geq k} \binom{i(n-i)}{r-k} \right) A_i, k
$$

$$
= \sum_{i=0}^n \binom{n}{i} \sum_{k \geq 0} 2^{i(n-i)} A_i, k = \sum_{i=0}^n \binom{n}{i} 2^{i(n-i)} A_i,
$$

as desired. $\Box$
Let $P = I \times \Delta^n$ with $n \geq 2$. There are three acyclic digraphs with 2 labeled nodes $\{v_1, v_2\}$:

\[ G_1 : \bullet \rightarrow \bullet, \quad G_2 : \bullet \rightarrow \bullet, \quad \text{and} \quad G_3 : \bullet \leftarrow \bullet. \]

Since $m_1 = 1$ in the formula of Theorem 3.5, we have $\mathcal{G}(0) = \{G_1, G_2, G_3\}$ and $\mathcal{G}(1) = \{G_1, G_2\}$. Thus, we obtain:

**Corollary 3.7** (Theorem 4.2, [2]) If $P = I \times \Delta^n$ with $n \geq 2$, the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P$ is

\[ \left( \frac{2^n + 3}{2(n + 1)!} \right) \prod_{i=1}^{n+1} (2^{n+1} - 2^{i-1}). \]

In a similar way, by listing the acyclic digraphs of 3 vertices, one can obtain the following result due to Chen and Wang.

**Corollary 3.8** (Theorem 4.1, [2]) If $P = I \times \Delta^n \times \Delta^m$ then the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P$ is

1. \[ \frac{\prod_{i=1}^{n+m+1} (2^{n+m+1} - 2^{i-1})}{2(n + 1)!} \left( 2^{2n+m} + 2^{n+2m} + 2^{2n} + 2^{2m} + 3 \cdot 2^{n+1} + 3 \cdot 2^{m+1} - 2^{n+m} - 7 \right) \text{ if } 1 < n < m, \]

2. \[ \frac{\prod_{i=1}^{n+m+1} (2^{n+m+1} - 2^{i-1})}{4(n + 1)!} \left( 2^{3n+1} + 2^{2n} + 3 \cdot 2^{n+2} - 7 \right) \text{ if } 1 < n = m, \]

3. \[ \frac{\prod_{i=1}^{n+2} (2^{n+2} - 2^{i-1})}{8(m + 1)!} \left( 3 \cdot 2^{2m} + 3 \cdot 2^{m+2} + 8 \right) \text{ if } 1 = n < m. \]

4. The number of weakly equivariant homeomorphism classes

By Theorem 2.4 the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a simple polytope $P$ is equal to the size of the double coset on $\Lambda(P)$ by $\text{GL}(n,\mathbb{Z}_2)$ and $\text{Aut}(\mathcal{F}(P))$. Therefore, the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P = \prod_{i=1}^{l} P_i$, where $P_i = \Delta_{n_i} \times \cdots \times \Delta_{n_i}$ with $1 \leq n_1 < n_2 < \cdots < n_l$, $\sum_{i=1}^{l} m_i = m$ and $\sum_{i=1}^{l} n_i m_i = n$, is

\[ |A(P)/\prod_{i=1}^{l} (S_{n_{i+1}} \wr S_{m_i})| \]

where $A(P) = A(0, \cdots, 0)$.

Consider the subgroup $H = \prod_{i=1}^{l} S_{m_i} \leq \text{Aut}(\mathcal{F}(P))$. Note that an element $\mu \in S_{m_i}$ acts on $A(P)$ by $\lambda \cdot \mu = \lambda_\mu$ where $\lambda_\mu \in A(P)$ corresponds to a class represented by the characteristic function that sends $F_{q,r}^p$ to $\lambda(P_{(q,r)})$ if $p = i$ and to $\lambda(F_{q,r}^p)$ otherwise. Let $\varpi \in S_{n_i}$ be the permutation that sends
\[ m_1^n + \cdots + m_{i-1}^n + (j-1)n_i + k \] to \[ m_1^n + \cdots + m_{i-1}^n + (\mu(j) - 1)n_i + k \] for \( 1 \leq j \leq m_i \), \( 1 \leq k \leq n_i \) and fixes other elements. Then the matrix \( \Lambda_\mu \) corresponding to \( \lambda_\mu \) is

\[ P(\overline{\sigma})^{-1} \Lambda P(\mu) \]

where \( P(\sigma) \) denotes the permutation matrix corresponding to a permutation \( \sigma \). This is the conjugation action of \( H = \prod_{i=1}^l S_{m_i} \leq S_m \) on the set of \((m \times m)\)-vector matrices. It corresponds to an action of \( H \leq S_m \) on the acyclic digraph with \( m \)-labeled nodes \( \{v_1, \ldots, v_m\} \) given by

\[ \mu \cdot v_j = \begin{cases} v_{\mu(j)} & \text{if } m_1 + \cdots + m_{i-1} + 1 \leq j \leq m_1 + \cdots + m_i \\ v_j & \text{otherwise} \end{cases} \]

for any \( \mu \in S_{m_i} \). Therefore, when \( l = 1 \), we have the following generalization of Theorem 4.1 in [3].

**Theorem 4.1** The number of weakly \( \mathbb{Z}_2^n \)-equivariant homeomorphism classes of small covers over \( P = \Delta_1^k \times \cdots \times \Delta_m^k \) with \( mk = n \) is less than or equal to

\[ \sum_{G \in \mathcal{G}_m} (2^k - 1)^{|E(G)|} \]

where \( \mathcal{G}_m \) is the set of acyclic digraphs with \( m \) unlabeled nodes and \( E(G) \) is the set of edges of the graph \( G \).

**Corollary 4.2** The number of homeomorphism classes of small covers over \( P = \Delta_1^k \times \cdots \times \Delta_m^k \) with \( mk = n \) is less than or equal to

\[ \sum_{G \in \mathcal{G}_m} (2^k - 1)^{|E(G)|} \]

where \( \mathcal{G}_m \) is the set of acyclic digraphs with \( m \) unlabeled nodes.

**References**