Characterizations of *-DMP matrices over rings

Yuefeng GAO, Jianlong CHEN*
School of Mathematics, Southeast University, Nanjing, P.R. China

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Abstract: Let $R$ be a ring with involution $*$. $R^{m \times n}$ denotes the set of all $m \times n$ matrices over $R$. In this paper, we give a characterization of the pseudo core inverse of $A \in R^{m \times n}$ in the form of $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, where $N_l(A) = \{x \in R^{1 \times n} | AX = 0\}$ and $N_r(A) = \{x \in R^{m \times 1} | AX = 0\}$. Then we obtain necessary and sufficient conditions for $A \in R^{m \times n}$, in the form of $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP. If $R$ is a principal ideal domain (resp. semisimple Artinian ring), then matrices expressed as that form include all $n \times n$ matrices over $R$.

Key words: *-DMP matrix, pseudo core inverse, core-EP inverse, Drazin inverse, Moore–Penrose inverse, factorization

1. Introduction

Let $R$ be a ring with involution $*$. $R^{m \times n}$ denotes the set of all $m \times n$ matrices over $R$. Suppose $A = (a_{ij}) \in R^{m \times n}$. Put $A^* = (a_{ji}^*)$. We consider the following equations:

\begin{align}
(1) \quad AXA &= A, \\
(1^k) \quad A^kXA &= A^k \text{ for some positive integer } k, \\
(2) \quad XAX &= X, \\
(3) \quad (AX)^* &= AX, \\
(4) \quad (XA)^* &= XA, \\
(5) \quad AX &=XA.
\end{align}

The Moore–Penrose inverse of $A$, denoted by $A^\dagger$, is the unique matrix $X$ satisfying the above (1), (2), (3), and (4); the $\{1,3\}$-inverse (resp. $\{1,4\}$-inverse) of $A$, denoted by $A^{\{1,3\}}$ (resp. $A^{\{1,4\}}$), is the matrix $X$ satisfying the above (1) and (3) (resp. (1) and (4)). Let $A \in R^{m \times n}$, the group inverse of $A$, denoted by $A^#$, be the unique matrix $X$ satisfying the above (1), (2), and (5); the Drazin inverse of $A$, denoted by $A^D$, is the unique matrix $X$ satisfying the above $(1^k)$, (2), and (5); the smallest positive integer $k$ satisfying the above $(1^k)$, (2), and (5) is called the Drazin index of $A$, denoted by $i(A)$.

*Correspondence: jlchen@seu.edu.cn

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The core inverse of $A \in R^{n \times n}$, denoted by $A^\oplus$, is the unique solution to equations
\[
XA^2 = A, \quad AX^2 = X, \quad \text{and} \quad (AX)^* = AX \quad (\text{see } [23]).
\]
We refer readers to [1] and [18] for a deep study of core inverses.

In [9], the authors introduced the notion of the pseudo core inverse, which extends the notion of core inverse to matrices of an arbitrary index. The pseudo core inverse of $A \in R^{m \times n}$, denoted by $A^\otimes$, is the unique solution to equations
\[
XA^{m+1} = A^m \text{ for some positive integer } m, \quad AX^2 = X \text{ and } (AX)^* = AX.
\]
The smallest positive integer $m$ satisfying the above equations is called the pseudo core index of $A$. If $A$ is pseudo core invertible, then it must be Drazin invertible, and the pseudo core index coincides with the Drazin index (see [9]). Thus, here and subsequently, we denote the pseudo core index of $A$ by $i(A)$. It is clear that if $i(A) = 1$, then the pseudo core inverse of $A$ is the core inverse of $A$. Also, the pseudo core inverse extended the core-EP inverse [13] from complex matrices to matrices over rings in terms of equations (see [9]). For consistency and convenience, we use the terminology ‘pseudo core inverse’ throughout this paper.

Dually, the dual pseudo core inverse of $A \in R^{m \times n}$, denoted by $A_\otimes$, is the unique solution to equations
\[
A^{m+1}X = A^m \text{ for some positive integer } m, \quad X^2A = X \text{ and } (XA)^* =XA \quad [9].
\]
The smallest positive integer $m$ satisfying the above equations is called the dual pseudo core index of $A$, denoted by $i(A)$ as well.

Let $A \in C^{n \times n}$, $A$ be EP if and only if $N(A) = N(A^*)$ if and only if $A^1$ and $A^\#$ exist with $A^1 = A^\#$ (see [2, 20]), where $N(A)$ denotes the null space of $A$ and $A^*$ denotes the conjugate transpose of $A$.

Meanwhile, suppose that $A \in R^{m \times n}$, $N(A) = N(A^*)$ may not imply that $A^1$ and $A^\#$ exist with $A^1 = A^\#$. Hartwig [10] defined that an element $a$ in a *-regular ring (a regular ring with involution such that $a^*a = 0$ implies $a = 0$) is EP if and only if $aR = a^*R$, and he also proved its equivalence with the existence of $a^\#$ together with $a^\# = a^1$. Patricio and Puystjens [15] introduced the notions of *-EP and *-gMP in rings with involution. They said that $a$ is *-EP if $aR = a^*R; a$ is *-gMP if $a^1$ and $a^\#$ exist with $a^1 = a^\#$. As a matter of convenience, we denote a *-gMP element (resp. matrix) as an EP element (resp. matrix) in this paper. A is *-DMP if there exists a positive integer $m$ such that $A^m$ is EP [15]; A is *-DMP with index $m$ if $m$ is the smallest positive integer such that $A^m$ is EP [15]. We refer readers to [10, 12, 14, 16, 18, 19] for a deep study of EP. In [8], the authors gave several characterizations of the *-DMP elements, utilizing the pseudo core inverse and dual pseudo core inverse, in semigroups with involution. Those results are also true for matrices over rings.

Letting $A \in R^{m \times n}$, we will use the following notations:
\[
N_l(A) = \{x \in R^{1 \times m} \mid xA = 0\} \quad \text{and} \quad N_r(A) = \{x \in R^{n \times 1} \mid Ax = 0\}.
\]
From [17] and [11], we find:

(1) if $R$ is a principal ideal domain, then for arbitrary $A \in R^{m \times n}$, it follows that $A = GH, N_r(G) = 0, N_l(H) = 0$ for some matrices $G \in R^{m \times r}, H \in R^{r \times n}$;

(2) if $R$ is a semisimple Artinian ring, then for arbitrary $A \in R^{m \times n}$, it follows that $A = GDH, N_r(G) = 0, N_l(H) = 0, D^2 = D = D^*$ for some matrices $G \in R^{m \times r}, D \in R^{r \times r}, H \in R^{r \times n}$.

In [2, 21, 22], the authors pointed out respectively that a factorization of a matrix $A$ leads to explicit formulae for its Moore–Penrose inverse, Drazin inverse, and generalized inverse $A^{(2)}_{T,S}$. In [3, 4, 7, 16], the
authors gave some characterizations of real or complex EP matrices, EP Hilbert operators, EP Banach algebra elements, and weighted EP Banach space operators through factorizations respectively.

Chen [5] gave the existence criteria and formulae for the \{1,3\}-inverse and Moore–Penrose inverse of \( A \) in the form of \( A = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \). Chen [6] gave the equivalent conditions for \( A \) in the form of \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \) to have a Drazin inverse.

Motivated by the above papers, in Section 2 we give the necessary and sufficient conditions for \( A \) in the form of \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \) to have pseudo core inverses. As applications, in Section 3, we give several characterizations of *-DMP matrices in the form of \( A = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \).

2. Characterizations of pseudo core invertible matrices

In this section, we characterize pseudo core invertibility of \( A \) in the form of \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \). We begin with some useful lemmas.

**Lemma 2.1** [5] Let \( A, G, D, H \) be matrices over \( R \) such that \( A = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \). Then we have the following facts:

1. If \( A[1,3] \neq 0 \), then \( DG^*GD + I - D \) is invertible;
2. If \( A[1,4] \neq 0 \), then \( DHH^*D + I - D \) is invertible;
3. \( A^k \) exists if and only if both \( DG^*GD + I - D \) and \( DHH^*D + I - D \) are invertible.

In this case, \( A^k = (DH)\gamma(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* \).

**Lemma 2.2** [6] Let \( A, G, D, H \) be matrices over \( R \) such that \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D \) for some positive integer \( k \). Then the following are equivalent:

1. \( A^k \) exists with \( i(A) \leq k \);
2. \( DHAGD + I - D \) is invertible.

In this case, \( A^k = GD(DHAGD + I - D)^{-1}DH \).

**Lemma 2.3** [9] Let \( A \in R^{n\times n} \). Then we have the following facts:

1. \( A^\oplus \) exists if and only if \( A^D \) and \( (A^k)^{(1,3)} \) exist, where \( k \geq i(A) \).

In this case, \( A^\oplus = A^D A^k (A^k)^{(1,3)} \).

2. \( A_\oplus \) exists if and only if \( A^D \) and \( (A^k)^{(1,4)} \) exist, where \( k \geq i(A) \).

In this case, \( A_\oplus = (A^k)^{(1,4)} A^k A^D \).

3. \( A^\oplus \) and \( A_\oplus \) exist if and only if \( A^D \) and \( (A^k)^{\dagger} \) exist, where \( k \geq i(A) \).

In this case, \( A^\oplus = A^D A^k (A^k)^{\dagger} \) and \( A_\oplus = (A^k)^{\dagger} A^k A^D \).

Applying Lemmas 2.1–2.3, we derive the following result, which is a characterization of the pseudo core inverse of \( A \), in the form of \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \).

**Theorem 2.4** Let \( A, G, D, H \) be matrices over \( R \) such that \( A^k = GDH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \), \( D^2 = D = D^* \) for some positive integer \( k \). Then the following are equivalent:
(1) \( A \circledast \) exists with \( i(A) \leq k \) if and only if both \( DG^*GD + I - D \) and \( DHAGD + I - D \) are invertible. In this case, \( A \circledast = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^* \).

(2) \( A \circledast \) exists with \( i(A) \leq k \) if and only if both \( DHH^*D + I - D \) and \( DHAGD + I - D \) are invertible. In this case, \( A \circledast = (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH \).

**Proof** (1). From Lemma 2.3, \( A \circledast \) exists if and only if \( A^D \) and \( (A^k)^{(1,3)} \) exist, where \( k \geq i(A) \). Moreover, \( A \circledast = A^DA^k(A^k)^{(1,3)} \). Thus, the necessity of (1) is clear by Lemmas 2.1 and 2.2.

Conversely, since \( DHAGD + I - D \) is invertible, according to Lemma 2.2, \( A^D \) exists with \( i(A) \leq k \) and

\[
(A^D)^k = [GD(DHAGD + I - D)^{-1}DH]^k = GD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DH.
\]

Then \( GDH(GD(DHGD + I - D)^{-1}(DHAGD + I - D)^{-k}DH)GDH = A^k(A^D)^kA^k = A^k = GDH. \)

Since \( N_r(G) = 0 \) and \( N_l(H) = 0 \), we have \( DHGD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DHGD = D \).

From \( (DHAGD + I - D)(DHGD + I - D) = DHAGDGD + I - D = DHGDHAGD + I - D = (DHGD + I - D)(DHAGD + I - D), \)

it follows that \( (DHAGD + I - D)^{-1}(DHGD + I - D) = (DHGD + I - D)(DHAGD + I - D)^{-1}. \) Thus,

\[
D = DHGD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DHGD
= D(DHGD + I - D)(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}(DHGD + I - D)D
= D(DHGD + I - D)^{k+1}(DHAGD + I - D)^{-k}D
= (DHGD + I - D)^{k+1}D(DHAGD + I - D)^{-k}.
\]

Then \( (DHGD)^k = D(DHAGD + I - D)^k = (DHGD + I - D)^{k+1}D = (DHGD)^{k+1}. \)

Therefore, \( (DHAGD + I - D)^k = (DHAGD)^k + I - D = (DHGD)^{k+1} + I - D = (DHGD + I - D)^{k+1}. \)

Since \( DHAGD + I - D \) is invertible, we conclude that \( DHGD + I - D \) is invertible.

Observe that \( GD(DHGD + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* \) is a \( (1,3) \)-inverse of \( A^k \),

and then \( A^k(A^k)^{(1,3)} = GD(DG^*GD + I - D)^{-1}(GD)^* \).

Hence, \( A \circledast = GD(DHGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^* \).

(2). It is analogous. \( \square \)

Let \( D \) be the identity matrix in Theorem 2.4; then we have the following result.

**Corollary 2.5** Let \( A, G, H \) be matrices over \( R \) such that \( A^k = GH \), \( N_r(G) = 0 \), \( N_l(H) = 0 \) for some positive integer \( k \). Then the following are equivalent:

(1) \( A \circledast \) exists with \( i(A) \leq k \) if and only if both \( G^*G \) and \( HAG \) are invertible.

In this case, \( A \circledast = G(HAG)^{-1}HG(G^*G)^{-1}G^* \).

(2) \( A \circledast \) exists with \( i(A) \leq k \) if and only if both \( HH^* \) and \( HAG \) are invertible.

In this case, \( A \circledast = H^*(HH^*)^{-1}HG(HAG)^{-1}H \).

Letting \( D \) be the identity matrix and \( k = 1 \) in Theorem 2.4, then we get the following result, which characterizes the core invertibility of \( A \).
Corollary 2.6 Let $A, G, H$ be matrices over $R$ such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then the following are equivalent:

1. $A^\oplus$ exists if and only if both $G^*G$ and $HG$ are invertible. In this case, $A^\oplus = (G(HG)^{-1}(G^*G)^{-1}G^*)^{-1}$.

2. $A_\ominus$ exists if and only if both $HH^*$ and $HG$ are invertible. In this case, $A_\ominus = H^*(HH^*)^{-1}(HG)^{-1}H$.

3. Characterizations of *-DMP matrices

Pearl [16] pointed out that if $A$, $G$, $H$ are complex matrices with $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$, then $A$ is EP if and only if $G(G^*G)^{-1}G^* = H^*(HH^*)^{-1}H$. Drivaliaris et al. [7] obtained several characterizations of EP operators in Hilbert spaces through operator factorizations. Boasso [3] gave necessary and sufficient conditions for an operator $T$ to be EP in Banach spaces under the assumptions that $T^\dagger$ exists and $T$ is of a operator factorization.

Recall that $A$ is symmetric if $A = A^*$. In what follows, we show several equivalent conditions for $A \in R^{n \times n}$, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP. We begin with an auxiliary lemma.

Lemma 3.1 [8, 12, 15] Let $A \in R^{n \times n}$. Then the following conditions are equivalent:

1. $A$ is *-DMP with index $k$;
2. $A^D$ exists with $i(A) = k$ and $AA^D$ is symmetric;
3. $k$ is the smallest positive integer such that $(A^k)^\dagger$ exists with $A^k(A^k)^\dagger = (A^k)^\dagger A^k$;
4. $A^\oplus$ exists with $i(A) = k$ and $A^\oplus = A^D$;
5. $A_\ominus$ exists with $i(A) = k$ and $A_\ominus = A^D$;
6. $A^D$ exists with $i(A) = k$, $(A^k)^\dagger$ exist and $(A^D)^k = (A^k)^\dagger$;
7. $A^\oplus$ and $A_\ominus$ exist with $i(A) = k$ and $A^\oplus = A_\ominus$.

Applying Lemma 3.1, we obtain several characterizations of *-DMP matrices sequentially. First, we have the following result.

Theorem 3.2 Let $A, G, D, H$ be matrices over $R$ such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D$ for some positive integer $k$. Then $A$ is *-DMP with index $\leq k$ if and only if $DHAGD + I - D$ is invertible and one of the following equivalent conditions holds:

1. $GD(DHGD + I - D)^{-1}DH$ is symmetric;
2. $DH = DH[G(DHGD + I - D)^{-1}DH]^*$;
3. $GD = [GD(DHGD + I - D)^{-1}DH]^*GD$.

Proof From Lemma 3.1, $A$ is *-DMP with index $\leq k$ if and only if $A^D$ exists with $i(A) \leq k$ and $AA^D$ is symmetric. $A^D$ exists with $i(A) \leq k$ if and only if $DHAGD + I - D$ is invertible by Lemma 2.2. According to the proof of Theorem 2.4, we have

$$(A^D)^k = [GD(DHAGD + I - D)^{-1}DH]^k = GD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DH,$$

and $DHGD + I - D$ is invertible with $(DHAGD + I - D)^{-k} = (DHGD + I - D)^{-(k+1)}$.
Thus,

\[(A^D)^k = GD(DHG + I - D)^{-2}DH.\]

Then \(AA^D = A^k(A^D)^k = GDHG(DHG + I - D)^{-2}DH = GD(DHG + I - D)^{-1}DH.\) Applying Lemma 3.1, \(A\) is *-DMP if and only if (1) holds.

(1) \(\Rightarrow\) (2). Since \([GD(DHG + I - D)^{-1}DH]^* = GD(DHG + I - D)^{-1}DH\), then

\[DH = (DHG + I - D)(DHG + I - D)^{-1}DH = DHG(DHG + I - D)^{-1}DH \]

\[= DH[GD(DHG + I - D)^{-1}DH]^*.\]

(2) \(\Rightarrow\) (1). Equality \(DH = DH[GD(DHG + I - D)^{-1}DH]^*\) yields that \(GD(DHG + I - D)^{-1}DH = GD(DHG + I - D)^{-1}DH[GD(DHG + I - D)^{-1}DH]^*\). Hence \(GD(DHG + I - D)^{-1}DH\) is symmetric.

(1) \(\Leftrightarrow\) (3). It is analogous. \(\square\)

Let \(D\) be the identity matrix in Theorem 3.2, and then we have the following result.

**Corollary 3.3** Let \(A, G, H\) be matrices over \(R\) such that \(A^k = GH\), \(N_r(G) = 0\), \(N_l(H) = 0\) for some positive integer \(k\). Then \(A\) is *-DMP with index \(\leq k\) if and only if \(HAG\) is invertible and one of the following equivalent conditions holds:

1. \(G(HG)^{-1}H\) is symmetric;
2. \(H = H[G(HG)^{-1}H]^*\);
3. \(G = [G(HG)^{-1}]^*G\).

Letting \(D\) be the identity matrix and \(k = 1\) in Theorem 3.2, then we get the following result, which gives a characterization for \(A\) to be EP.

**Corollary 3.4** Let \(A, G, H\) be matrices over \(R\) such that \(A = GH\), \(N_r(G) = 0\), \(N_l(H) = 0\). Then \(A\) is EP if and only if \(HG\) is invertible and one of the following equivalent conditions holds:

1. \(G(HG)^{-1}H\) is symmetric;
2. \(H = H[G(HG)^{-1}H]^*\);
3. \(G = [G(HG)^{-1}]^*G\).

The following result gives the second characterization for \(A\), in the form of \(A^k = GDH\), \(N_r(G) = 0\), \(N_l(H) = 0\), \(D^2 = D = D^*\), to be *-DMP.

**Theorem 3.5** Let \(A, G, D, H\) be matrices over \(R\) such that \(A^k = GDH\), \(N_r(G) = 0\), \(N_l(H) = 0\), \(D^2 = D = D^*\) for some positive integer \(k\). Then \(A\) is *-DMP with index \(\leq k\) if and only if \(DG*GD + I - D\) and \(DHH*D + I - D\) are invertible with

\[GD(DG*GD + I - D)^{-1}(GD)^* = (DH)^*(DHH*D + I - D)^{-1}DH.\]

**Proof** According to Lemma 3.1, \(A\) is *-DMP with index \(\leq k\) if and only if there exists a positive integer \(k\) such that \((A^k)^\dagger\) exists with \(A^k(A^k)^\dagger = (A^k)^\dagger A^k\), by Lemma 2.1, which is equivalent to \(DG*GD + I - D\)
and $DHH^*D + I - D$ being invertible with $GDH(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* = (DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*$.

Note that

$$GDH(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* = GD(DG^*GD + I - D)^{-1}(GD)^*$$

and

$$(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*GDH = (DH)^*(DHH^*D + I - D)^{-1}DH,$$

and then we complete the proof. \hfill \Box

Let $D$ be the identity matrix in Theorem 3.5, and then we get the following result.

**Corollary 3.6** Let $A, G, H$ be matrices over $R$ such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer $k$. Then $A$ is *-DMP with index $\leq k$ if and only if $G^*G$ and $HH^*$ are invertible with $G(G^*G)^{-1}G^* = H^*(HH^*)^{-1}H$.

Let $D$ be the identity matrix and $k = 1$ in Theorem 3.5, and then we get the following result, which gives the second characterization for $A$ to be EP.

**Corollary 3.7** Let $A, G, H$ be matrices over $R$ such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then $A$ is EP if and only if $G^*G$ and $HH^*$ are invertible with $G(G^*G)^{-1}G^* = H^*(HH^*)^{-1}H$.

The following result gives the third characterization for $A$, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP.

**Theorem 3.8** Let $A, G, D, H$ be matrices over $R$ such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer $k$. Then $A$ is *-DMP with index $\leq k$ if and only if $DG^*GD + I - D$ and $DHAGD + I - D$ are invertible with

$$(DG^*GD + I - D)^{-1}(GD)^* = (DHGD + I - D)^{-1}DH.$$  

**Proof** According to Theorem 2.4, $DG^*GD + I - D$ and $DHAGD + I - D$ are invertible if and only if $A^\oplus$ exists with $i(A) \leq k$, in which case,

$$A^\oplus = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^*.$$  

Notice that $A^\oplus$ existing implies that $A^D$ exists by Lemma 2.3 and $A^D = GD(DHAGD + I - D)^{-1}DH$ by Lemma 2.2. Applying Lemma 3.1, $A$ is *-DMP with $i(A) \leq k$ if and only if

$$G(DHAGD + I - D)^{-1}(DHGD + I - D)(DG^*GD + I - D)^{-1}(GD)^* = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^* = A^\oplus = A^D$$  

(3.1)

$$= GD(DHAGD + I - D)^{-1}DH = G(DHAGD + I - D)^{-1}DH.$$  

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Since $N_r(G) = 0$, by calculation, equality (3.1) is equivalent to
\[
DG^*GD + I - D)^{-1}DG^* = (DHGD + I - D)^{-1}DH.
\]

Let $D$ be the identity matrix in Theorem 3.8, and then we get the following result.

**Corollary 3.9** Let $A, G, H$ be matrices over $R$ such that $A^k = GH$, $N_r(G) = 0$, $N_i(H) = 0$ for some positive integer $k$. Then $A$ is *-DMP with index $\leq k$ if and only if $G^*G$ and $HAG$ are invertible with $(G^*G)^{-1}G^* = (HG)^{-1}H$.

Let $D$ be the identity matrix and $k = 1$ in Theorem 3.8, and then we get the following result, which gives the third characterization for $A$ to be EP.

**Corollary 3.10** Let $A, G, H$ be matrices over $R$ such that $A = GH$, $N_r(G) = 0$, $N_i(H) = 0$. Then $A$ is EP if and only if $G^*G$ and $HG$ are invertible with $(G^*G)^{-1}G^* = (HG)^{-1}H$.

The following result gives the fourth characterization for $A$, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_i(H) = 0$, $D^2 = D = D^*$, to be *-DMP.

**Theorem 3.11** Let $A, G, D, H$ be matrices over $R$ such that $A^k = GDH$, $N_r(G) = 0$, $N_i(H) = 0$, $D^2 = D = D^*$ for some positive integer $k$. Then $A$ is *-DMP with index $\leq k$ if and only if $DHH^*D + I - D$ and $DGAGD + I - D$ are invertible with
\[
(DH)^*(DHH^*D + I - D)^{-1} = GD(DHGD + I - D)^{-1}.
\]

**Proof** According to Theorem 2.4, $DHH^*D + I - D$ and $DGAGD + I - D$ are invertible if and only if $A_{\oplus}$ exists with $i(A) \leq k$, in which case,
\[
A_{\oplus} = (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH.
\]

Notice that $A_{\oplus}$ existing implies that $A^D$ exists by Lemma 2.3 and $A^D = GD(DHAGD + I - D)^{-1}DH$ by Lemma 2.2. Applying Lemma 3.1, $A$ is *-DMP if and only if
\[
(DH)^*(DHH^*D + I - D)^{-1}(DHGD + I - D)(DHAGD + I - D)^{-1}H
= (DH)^*(DHH^*D + I - D)^{-1}(DHGD + I - D)D(DHAGD + I - D)^{-1}H
= (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH = A_{\oplus} = A^D
= GD(DHAGD + I - D)^{-1}DH = GD(DHAGD + I - D)^{-1}H.
\]

Since $N_i(H) = 0$, by calculation, equality (3.2) is equivalent to
\[
(DH)^*(DHH^*D + I - D)^{-1} = GD(DHGD + I - D)^{-1}.
\]

Let $D$ be the identity matrix in Theorem 3.11, and then we get the following result.
Corollary 3.12 Let $A, G, H$ be matrices over $R$ such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer $k$. Then $A$ is $^*$-DMP with index $\leq k$ if and only if $G^*G$ and $HAG$ are invertible with $H^*(HH^*)^{-1} = G(HG)^{-1}$.

Let $D$ be the identity matrix and $k = 1$ in Theorem 3.11, and then we get the following result, which gives the fourth characterization for $A$ to be EP.

Corollary 3.13 Let $A, G, H$ be matrices over $R$ such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then $A$ is EP if and only if $G^*G$ and $HG$ are invertible with $H^*(HH^*)^{-1} = G(HG)^{-1}$.

The following result gives the fifth characterization for $A$, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be $^*$-DMP.

Theorem 3.14 Let $A, G, D, H$ be matrices over $R$ such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer $k$. Then $A$ is $^*$-DMP with index $\leq k$ if and only if $DG^*GD + I - D$, $DHH^*D + I - D$, and $DHAGD + I - D$ are invertible and one of the following equivalent conditions holds:

1. $(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* = GD(DHGD + I - D)^{-2}DH$;
2. $G(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}G^* = H^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}H$.

Proof $DG^*GD + I - D$, $DHH^*D + I - D$, and $DHAGD + I - D$ are invertible if and only if $(A^k)^\dagger$ and $A^D$ exist with $i(A) \leq k$ by Lemmas 2.1 and 2.2, which is equivalent to $A^{\Box}$ and $A_{\Box}$ existing with $i(A) \leq k$ by Lemma 2.3. Observe that $(A^D)^k = GD(DHGD + I - D)^{-2}DH$ by the proof of Theorem 3.2; $(A^k)^\dagger = (DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*$ by Lemma 2.1; and, by Theorem 2.4,

\[
A^{\Box} = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^*,
\]

\[
A_{\Box} = (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH.
\]

From Lemma 3.1, $A$ is $^*$-DMP with index $\leq k$ if and only if

\[
(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* = (A^k)^\dagger = (A^D)^k = GD(DHGD + I - D)^{-2}DH,
\]

if and only if

\[
G(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}G^* = A^{\Box} = A_{\Box} = H^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}H.
\]

Let $D$ be the identity matrix in Theorem 3.14, and then we get the following result.

Corollary 3.15 Let $A, G, H$ be matrices over $R$ such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer $k$. Then $A$ is $^*$-DMP with index $\leq k$ if and only if $G^*G$, $HH^*$ and $HAG$ are invertible and one of the following equivalent conditions holds:
Let $D$ be the identity matrix and $k = 1$ in Theorem 3.14, and then we get the following result, which gives the fifth characterization for $A$ to be EP.

**Corollary 3.16** Let $A, G, H$ be matrices over $R$ such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then $A$ is EP if and only if $G^*G$, $HH^*$, and $HG$ are invertible and one of the following equivalent conditions holds:

1. $H^*(HH^*)^{-1}(G^*G)^{-1}G^* = G(HG)^{-2}H$;
2. $G(HAG)^{-1}HG(G^*G)^{-1}G^* = H^*(HH^*)^{-1}HG(HAG)^{-1}H$.

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**References**


