Diagonal lift in the semi-cotangent bundle and its applications

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Abstract: The present paper is devoted to some results concerning the diagonal lift of tensor fields of type (1,1) from manifold M to its semi-cotangent bundle t*M. In this context, cross-sections in the semi-cotangent (pull-back) bundle t*M of cotangent bundle T*M by using projection (submersion) of the tangent bundle TM can be also defined.

Key words: Vector field, complete lift, diagonal lift, pull-back bundle, cross-section, semi-cotangent bundle

1. Introduction
Let \( M_n \) be an \( n \)-dimensional differentiable manifold of class \( C^\infty \), and let \((T(M_n), \pi_1, M_n)\) be a tangent bundle over \( M_n \). We use the notation \((x^i) = (x^\alpha, x^\beta)\), where the indices \( i, j, \ldots \) run from 1 to \( 2n \), the indices \( \alpha, \beta, \ldots \) from 1 to \( n \), and the indices \( \alpha, \beta, \ldots \) from \( n + 1 \) to \( 2n \), while \( x^\alpha \) are coordinates in \( M_n \) and \( x^\alpha = y^\alpha \) are fiber coordinates of the tangent bundle \( T(M_n) \) (for definition of the pull-back bundle, see, for example, [1],[3],[4],[5],[6]).

Now let \((T^*(M_n), \bar{\pi}, M_n)\) be a cotangent bundle with base space \( M_n \) and let \( T(M_n) \) be a tangent bundle determined by a natural projection (submersion) \( \pi_1 : T(M_n) \to M_n \). The semi-cotangent \((\bar{8},\bar{9})\) bundle (induced or pull-back bundle) of the cotangent bundle \((T^*(M_n), \bar{\pi}, M_n)\) is the bundle \((t^*(M_n), \pi_2, T(M_n))\) over tangent bundle \( T(M_n) \) with a total space

\[
t^*(M_n) = \left\{ (x^\alpha, x^\beta) \in T(M_n) \times T^*_2(M_n) : \pi_1(x^\alpha, x^\beta) = \bar{\pi}(x^\alpha, x^\beta) = (x^\alpha) \right\}
\]

and with the projection map \( \pi_2 : t^*(M_n) \to T(M_n) \) defined by \( \pi_2(x^\alpha, x^\beta) = (x^\alpha, x^\beta) \), where \( T^*_2(M_n) (x = \pi_1(x), \bar{x} = (x^\alpha, x^\beta) \in T(M_n)) \) is the cotangent space at a point \( x \) of \( M_n \), where \( x^\beta = p_\beta (\bar{\alpha}, \bar{\beta}, \ldots = 2n + 1, \ldots, 3n) \) are fiber coordinates of the cotangent bundle \( T^*(M_n) \). If \((x^\gamma) = (x^\alpha, x^\beta, x^\gamma)\) is another system of local adapted coordinates in the semi-cotangent bundle \( t^*(M_n) \), then we have

\[
\begin{align*}
x^\gamma &= \frac{\partial x^\gamma}{\partial x^\beta} y^\beta, \\
x^\alpha' &= x^\alpha + \frac{\partial x^\gamma}{\partial x^\alpha} (y^\beta) , \\
x^\beta' &= \frac{\partial x^\gamma}{\partial x^\beta} p_\beta .
\end{align*}
\]

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The Jacobian of (1.1) has components

\[ \overline{A} = (A_j') = \begin{pmatrix} A_\alpha' & A_\beta' y^\sigma & 0 \\ 0 & A_\beta' & 0 \\ 0 & p\sigma A_\beta' A_\sigma' & A_\alpha' \end{pmatrix}, \tag{1.2} \]

where

\[ A_\alpha' = \frac{\partial^2 x^\alpha}{\partial x^\beta \partial x^\sigma}, \quad A_\beta' = \frac{\partial^2 x^\alpha}{\partial x^\beta \partial x^\alpha}. \]

We denote by \( \mathcal{I}_{p,q}(T(M_n)) \) and \( \mathcal{I}_{p,q}(M_n) \) the modules over \( F(T(M_n)) \) and \( F(M_n) \) of all tensor fields of type \((p,q)\) on \( T(M_n) \) and \( M_n \), respectively, where \( F(T(M_n)) \) and \( F(M_n) \) denote the rings of real-valued \( C^\infty \)-functions on \( T(M_n) \) and \( M_n \), respectively.

Let \( \theta \) be a covector field on \( T(M_n) \). Then the transformation \( p \to \theta_p, \) \( \theta_p \) being the value of \( \theta \) at \( p \in T(M_n) \), determines a cross-section \( \beta_\theta \) of a semi-cotangent bundle. Thus, if \( \sigma : M_n \to T^*(M_n) \) is a cross-section of \( (T^*(M_n), \pi, M_n) \), such that \( \pi \circ \sigma = I(M_n) \), an associated cross-section \( \beta_\theta : T(M_n) \to t^*(M_n) \) of semi-cotangent (pull-back) bundle \( (t^*(M_n), \pi_2, T(M_n)) \) of cotangent bundle by using projection (submersion) of the tangent bundle \( T(M_n) \) defined by [[2], p. 217–218], [[7], p. 301]:

\[ \beta_\theta(x^\sigma, x^\alpha) = (x^\sigma, x^\alpha, \sigma \circ \pi_1(x^\sigma, x^\alpha)) = (x^\sigma, x^\alpha, \sigma(x^\alpha)) = (x^\sigma, x^\alpha, \theta_\alpha(x^\beta)). \]

If the covector field \( \theta \) has the local components \( \theta_\alpha(x^\beta) \), the cross-section \( \beta_\theta(T(M_n)) \) of \( t^*(M_n) \) is locally expressed by

\[ x^\sigma = y^\alpha = V^\alpha(x^\beta), \quad x^\alpha = x^\alpha, \quad x^\sigma = p_\alpha = \theta_\alpha(x^\beta) \tag{1.3} \]

with respect to the coordinates \( x^A = (x^\sigma, x^\alpha, x^\bar{\sigma}) \) in \( t^*(M_n) \). \( x^\sigma = y^\alpha \) are considered as parameters. Taking the derivative of (1.3) with respect to \( x^\sigma = y^\alpha \), we have vector fields \( B_{(\bar{\sigma})} (\bar{\beta} = 1, \ldots, n) \) with components

\[ B_{(\bar{\sigma})} = \frac{\partial x^A}{\partial x^\bar{\sigma}} = \frac{\partial V^\alpha}{\partial x^\bar{\sigma}} = \begin{pmatrix} \frac{\partial V^\alpha}{\partial x^\bar{\sigma}} \\ \frac{\partial x^\alpha}{\partial x^\bar{\sigma}} \\ \frac{\partial \theta_\alpha}{\partial x^\bar{\sigma}} \end{pmatrix}, \]

which are tangent to the cross-section \( \beta_\theta(T(M_n)) \).

Thus, \( B_{(\bar{\sigma})} \) have components

\[ B_{(\bar{\sigma})} : \begin{pmatrix} B_{(\bar{\sigma})} \end{pmatrix} = \begin{pmatrix} \frac{\partial^\alpha}{\partial x^\bar{\sigma}} \\ 0 \\ 0 \end{pmatrix} \]

with respect to the coordinates \( (x^\sigma, x^\alpha, x^\bar{\sigma}) \) in \( t^*(M_n) \), where

\[ \frac{\partial^\alpha}{\partial x^\bar{\sigma}} = A^\alpha_{\bar{\beta}} = \frac{\partial x^\alpha}{\partial x^\bar{\sigma}}. \]
Let \( X \in \Im_1^1(T(M_n)) \), i.e. \( X = X^\alpha \partial_\alpha \). We denote by \( BX \) the vector field with local components

\[
BX : \left( B^A_{(\beta)} \right) \bar{X} = \begin{pmatrix}
\delta^\alpha_\beta X^\gamma \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
A^2_\beta X^\gamma \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
X^\alpha \\
0 \\
0
\end{pmatrix}
\tag{1.4}
\]

with respect to the coordinates \((x^\pi, x^\alpha, x^\overline{\pi})\) in \( t^*(M_n) \), which is defined globally along \( \beta_\theta(T(M_n)) \). Then a mapping

\[
B : \Im_0^1(T(M_n)) \rightarrow \Im_0^1(\beta_\theta(T(M_n)))
\]

is defined by (1.4). The mapping \( B \) is the differential of \( \beta_\theta : T(M_n) \rightarrow t^*(M_n) \) and so an isomorphism of \( \Im_0^1(T(M_n)) \) onto \( \Im_0^1(\beta_\theta(T(M_n))) \).

Since a cross-section is locally expressed by \( x^\pi = y^\alpha = \text{const.}, \ x^\overline{\pi} = p_\alpha = \text{const.}, \ x^\alpha = x^\alpha, \ x^\overline{\alpha} \) being considered as parameters. Taking the derivative of (1.3) with respect to \( x^\alpha \), we have vector fields \( C_\beta \) \((\beta = n + 1, ..., 2n) \) with components

\[
C_\beta = \frac{\partial x^A}{\partial x^\beta} = \partial_\beta x^A = \begin{pmatrix}
\partial_\beta V^\alpha \\
\delta^\alpha_\beta \\
\delta^\alpha_\beta
\end{pmatrix}
\]

which are tangent to the cross-section \( \beta_\theta(T(M_n)) \).

Thus, \( C_\beta \) have components

\[
C_\beta : \left( C^A_\beta \right) = \begin{pmatrix}
\partial_\beta V^\alpha \\
\delta^\alpha_\beta \\
\delta^\alpha_\beta
\end{pmatrix}
\]

with respect to the coordinates \((x^\pi, x^\alpha, x^\overline{\pi})\) in \( t^*(M_n) \), where

\[
\delta^\alpha_\beta = A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x^\beta}.
\]

Let \( X \in \Im_0^1(T(M_n)) \). Then we denote by \( CX \) the vector field with local components

\[
CX : \left( C^A_\beta \right) \bar{X} = \begin{pmatrix}
X^\beta \partial_\beta V^\alpha \\
X^\alpha \\
X^\beta \partial_\beta \theta_\alpha
\end{pmatrix}
\tag{1.5}
\]

with respect to the coordinates \((x^\pi, x^\alpha, x^\overline{\pi})\) in \( t^*(M_n) \), which is defined globally along \( \beta_\theta(T(M_n)) \). Then a mapping

\[
C : \Im_0^1(T(M_n)) \rightarrow \Im_0^1(\beta_\theta(T(M_n)))
\]

is defined by (1.5). The mapping \( C \) is the differential of \( \beta_\theta : T(M_n) \rightarrow t^*(M_n) \) and so an isomorphism of \( \Im_0^1(T(M_n)) \) onto \( \Im_0^1(\beta_\theta(T(M_n))) \).

Now, considering \( \omega \in \Im_0^1(M_n) \) and vector field \( X \in \Im_0^1(T(M_n)) \), then \( \bar{\bar{\omega}} \) (vertical lift), \( X \) (complete lift), and \( \bar{H}X \) (horizontal lift) have, respectively, components on the semi-cotangent bundle \( t^*(M_n) \) [8]:

\[
\bar{\bar{\omega}} = \begin{pmatrix}
0 \\
0 \\
\omega_\alpha
\end{pmatrix}, \quad X = \begin{pmatrix}
y^{\pi}_\alpha \partial_\alpha X^\alpha \\
X^\alpha \\
-\partial_\alpha (\partial_\alpha X^\pi)
\end{pmatrix}, \quad \bar{H}X = \begin{pmatrix}
-\Gamma^\beta_\alpha X^\beta \\
X^\alpha \\
X^\beta \Gamma_\alpha^\beta
\end{pmatrix}
\tag{1.6}
\]
with respect to the coordinates \((x^\pi, x^\alpha, x^\overline{\beta})\), where

\[
\Gamma^\alpha_\beta = V^\epsilon\Gamma^\alpha_\epsilon_\beta, \quad \Gamma^\alpha_\beta = \theta^\epsilon\Gamma^\alpha_\beta_\epsilon.
\]

On the other hand, the fiber is locally represented by

\[
x^\pi = y^\alpha = \text{const,} \quad x^\alpha = \text{const,} \quad x^\overline{\beta} = p^\alpha = \text{const},
\]

\(p^\alpha\) being considered as parameters. Thus, on differentiating with respect to \(p^\alpha\), we easily see that the vector fields \(E^v_{(\beta)} = \frac{\partial}{\partial x^\beta} (dx^\beta) (\beta = 2n + 1, \ldots, 3n)\) with components

\[
E^v_{(\beta)} : \left( E^A_{(\beta)} \right) = \partial_{(\beta)} x^A = \begin{pmatrix}
\frac{\partial x^\alpha}{\partial x^\beta} \\
\frac{\partial x^\beta}{\partial x^\alpha} \\
\frac{\partial p^\alpha}{\partial x^\beta}
\end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
\delta^\beta_\alpha
\end{pmatrix}
\]

are tangent to the fiber, where

\[
\delta^\beta_\alpha = A^\beta_\alpha = \frac{\partial x^\beta}{\partial x^\alpha}.
\]

Let \(\omega\) be a 1-form with local components \(\omega^\alpha\) on \(M_n\), so that \(\omega\) is a 1-form with local expression \(\omega = \omega^\alpha dx^\alpha\). We denote by \(E\omega\) the vector field with local components

\[
E\omega : \left( E^A_{(\beta)} \omega^\beta \right) = \begin{pmatrix} 0 \\
0 \\
\omega^\alpha
\end{pmatrix}, \tag{1.7}
\]

which is tangent to the fiber. Then a mapping

\[
E : \mathfrak{S}^0_1(M_n) \to \mathfrak{S}^1_0(t^*(M_n))
\]

is defined by (1.7) and so an isomorphism of \(\mathfrak{S}^0_1(M_n)\) in to \(\mathfrak{S}^1_0(t^*(M_n))\).

From (1.4), (1.5), and (1.7), we obtain:

**Theorem 1** Let \(X\) and \(Y\) be vector fields on \(T(M_n)\). For the Lie product, we have

(i) \([CX, CY] = C[X, Y]\),

(ii) \([BX, BY] = 0\),

(iii) \([E\psi, E\omega] = 0\),

for any \(\psi, \omega \in \mathfrak{S}^0_1(M_n)\).
Proof

(i) If $X$ and $Y$ are vector fields on $T(M_n)$ and $egin{pmatrix} [CX, CY] & [CX, CY]^\beta 
 [CX, CY]^\alpha 
 \end{pmatrix}$ are components of $[CX, CY]$ with respect to the coordinates $(x^\alpha, x^\beta, x^\gamma)$ in $t^*(M_n)$, then we have

$$[CX, CY]^J = (CX)^I \partial_I(CY)^J - (CY)^I \partial_I(CX)^J.$$ 

First, if $J = \overline{J}$, we have

$$[CX, CY]^\overline{J} = (CX)^I \partial_I(CY)^\overline{J} - (CY)^I \partial_I(CX)^\overline{J}$$

$$= (CX)^\overline{I} \partial_{\overline{I}}(CY)^\overline{J} + (CX)^\alpha \partial_{\alpha}(CY)^\overline{J} + (CX)^\overline{\beta} \partial_{\overline{\beta}}(CY)^\overline{J}$$

$$- (CY)^\overline{I} \partial_{\overline{I}}(CX)^\overline{J} - (CY)^\alpha \partial_{\alpha}(CX)^\overline{J} - (CY)^\overline{\beta} \partial_{\overline{\beta}}(CX)^\overline{J}$$

$$= X^\beta \partial_{\beta} Y^\gamma \partial_{\gamma} V^\beta + X^\alpha \partial_{\alpha} Y^\gamma \partial_{\gamma} V^\beta$$

$$- Y^\beta \partial_{\beta} X^\gamma \partial_{\gamma} V^\beta - Y^\alpha \partial_{\alpha} X^\gamma \partial_{\gamma} V^\beta$$

$$= (X^\alpha \partial_{\alpha} Y^\gamma - Y^\alpha \partial_{\alpha} X^\gamma) \partial_{\gamma} V^\beta$$

$$= [X, Y]^\gamma \partial_{\gamma} V^\beta$$

by virtue of (1.5). Second, if $J = J$, we have

$$[CX, CY]^J = (CX)^I \partial_I(CY)^J - (CY)^I \partial_I(CX)^J$$

$$= (CX)^J \partial_J(CY)^J + (CX)^\alpha \partial_{\alpha}(CY)^J + (CX)^\beta \partial_{\beta}(CY)^J$$

$$- (CY)^J \partial_J(CX)^J - (CY)^\alpha \partial_{\alpha}(CX)^J - (CY)^\beta \partial_{\beta}(CX)^J$$

$$= X^\gamma \partial_{\gamma} Y^\beta + X^\alpha \partial_{\alpha} Y^\beta + X^\beta \partial_{\beta} \theta_\alpha \partial_{\alpha} Y^\beta$$

$$- Y^\gamma \partial_{\gamma} X^\beta - Y^\alpha \partial_{\alpha} X^\beta - Y^\beta \partial_{\beta} \theta_\alpha \partial_{\alpha} X^\beta$$

$$= X^\alpha \partial_{\alpha} Y^\beta - Y^\alpha \partial_{\alpha} X^\beta$$

$$= [X, Y]^\beta$$

by virtue of (1.5). Third, if $J = \overline{J}$, then we have

$$[CX, CY]^\overline{J} = (CX)^I \partial_I(CY)^\overline{J} - (CY)^I \partial_I(CX)^\overline{J}$$

$$= (CX)^\overline{I} \partial_{\overline{I}}(CY)^\overline{J} + (CX)^\alpha \partial_{\alpha}(CY)^\overline{J} + (CX)^\overline{\beta} \partial_{\overline{\beta}}(CY)^\overline{J}$$

$$- (CY)^\overline{I} \partial_{\overline{I}}(CX)^\overline{J} - (CY)^\alpha \partial_{\alpha}(CX)^\overline{J} - (CY)^\overline{\beta} \partial_{\overline{\beta}}(CX)^\overline{J}$$

$$= X^\gamma \partial_{\gamma} Y^\beta \theta_\beta + X^\alpha \partial_{\alpha} Y^\gamma \partial_{\gamma} \theta_\beta + X^\beta \partial_{\beta} \theta_\alpha \partial_{\alpha} Y^\gamma \partial_{\gamma} \theta_\beta$$

$$- Y^\gamma \partial_{\gamma} X^\beta \partial_{\beta} \theta_\alpha \partial_{\alpha} X^\gamma \partial_{\gamma} \theta_\beta$$
\[ X^\alpha \partial_\alpha Y^\gamma \partial_\gamma \theta_\beta - Y^\alpha \partial_\alpha X^\gamma \partial_\gamma \theta_\beta = (X^\alpha \partial_\alpha Y^\gamma - Y^\alpha \partial_\alpha X^\gamma) \partial_\gamma \theta_\beta = [X, Y]^\gamma \partial_\gamma \theta_\beta \]

by virtue of (1.5). On the other hand, we know that \( C [X, Y] \) have components

\[
C [X, Y] = \begin{pmatrix}
[X, Y]^\gamma \partial_\gamma \varphi \\
[X, Y]^\beta \\
[X, Y]^\gamma \partial_\gamma \theta_\beta
\end{pmatrix}
\]

with respect to the coordinates in \( t^* (M_n) \). Thus, we have \( [CX, CY] = C [X, Y] \).

(ii) \( X, Y \in \mathcal{X}_1^0 (T(M_n)) \) and \( [BX, BY]^\beta \) are components of \( [BX, BY] \) with respect to the coordinates \( (x^\beta, x^\gamma, x^\rho) \) in \( t^* (M_n) \), and then we have

\[
[BX, BY]^J = (BX)^I \partial_I (BY)^J - (BY)^I \partial_I (BX)^J.
\]

First, if \( J = \overline{\beta} \), we have

\[
[BX, BY]^\overline{\beta} = (BX)^I \partial_I (BY)^\overline{\beta} - (BY)^I \partial_I (BX)^\overline{\beta}
\]

\[
= (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta} + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta} + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta}
\]

\[
-(BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta} - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta} - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta}
\]

\[
= X^\alpha \partial_\alpha Y^\beta - Y^\alpha \partial_\alpha X^\beta
\]

\[
= 0
\]

by virtue of (1.4). Second, if \( J = \beta \), we have

\[
[BX, BY]^\beta = (BX)^I \partial_I (BY)^\beta - (BY)^I \partial_I (BX)^\beta
\]

\[
= (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\beta + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\beta + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\beta
\]

\[
-(BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\beta - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\beta - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\beta
\]

\[
= 0
\]

by virtue of (1.4). Third, if \( J = \overline{\beta} \), then we have

\[
[BX, BY]^\overline{\beta} = (BX)^I \partial_I (BY)^\overline{\beta} - (BY)^I \partial_I (BX)^\overline{\beta}
\]

\[
= (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta} + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta} + (BX)^\overline{\alpha} \partial_\overline{\alpha} (BY)^\overline{\beta}
\]

\[
-(BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta} - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta} - (BY)^\overline{\alpha} \partial_\overline{\alpha} (BX)^\overline{\beta}
\]

\[
= 0
\]

by virtue of (1.4). Thus, we have \( [BX, BY] = 0 \).
(iii) If \( \psi, \omega \in \mathfrak{X}_1^0(M_n) \) and \( \left[ \begin{array} {c} [E \psi, E \omega]^\overline{\beta} \\ [E \psi, E \omega]^\beta \\ [E \psi, E \omega]^\overline{\beta} \end{array} \right] \) are components of \([E \psi, E \omega]\) with respect to the coordinates \((x^\overline{\beta}, x^\beta, x^\overline{\beta})\) in \(t^*(M_n)\), then we have

\[
[E \psi, E \omega]^J = (E \psi)^J \partial_x (E \omega)^J - (E \omega)^J \partial_x (E \psi)^J \\
= (E \psi)^\overline{\beta} \partial_{\overline{\beta}} (E \omega)^J + (E \psi)^\beta \partial_\beta (E \omega)^J + (E \psi)^\overline{\beta} \partial_{\overline{\beta}} (E \omega)^J \\
- (E \omega)^\beta \partial_\beta (E \psi)^J - (E \omega)^\overline{\beta} \partial_{\overline{\beta}} (E \psi)^J - (E \omega)^\overline{\beta} \partial_{\overline{\beta}} (E \psi)^J \\
= \psi_\alpha \partial_{\overline{\beta}} (E \omega)^J - \omega_\alpha \partial_{\overline{\beta}} (E \psi)^J.
\]

First, if \( J = \overline{\beta} \), we have

\[
[E \psi, E \omega]^\overline{\beta} = \psi_\alpha \partial_{\overline{\beta}} (E \omega)^\overline{\beta} - \omega_\alpha \partial_{\overline{\beta}} (E \psi)^\overline{\beta} \\
= 0
\]

by virtue of (1.7). Second, if \( J = \beta \), we have

\[
[E \psi, E \omega]^\beta = \psi_\alpha \partial_\beta (E \omega)^\beta - \omega_\alpha \partial_\beta (E \psi)^\beta \\
= 0
\]

by virtue of (1.7). Third, if \( J = \overline{\beta} \), then we have

\[
[E \psi, E \omega]^\overline{\beta} = \psi_\alpha \partial_{\overline{\beta}} (E \omega)^\overline{\beta} - \omega_\alpha \partial_{\overline{\beta}} (E \psi)^\overline{\beta} \\
= 0
\]

by virtue of (1.7). Thus, we have \([E \psi, E \omega] = 0\). 

We consider in \( \pi^{-1}(U) \) \( 3n \) local vector fields \( B_{(\overline{\beta})} \), \( C_{(\beta)} \), and \( E_{(\overline{\beta})} \) along \( \beta_\theta(T(M_n)) \), which are respectively represented by

\[
B_{(\overline{\beta})} = B \frac{\partial}{\partial x^{\overline{\beta}}}, \quad C_{(\beta)} = C \frac{\partial}{\partial x^{\beta}}, \quad E_{(\overline{\beta})} = Edx^{\overline{\beta}}.
\]

**Theorem 2** Let \( X \) be a vector field on \( T(M_n) \). We have along \( \beta_\theta(T(M_n)) \) the formula

\[
ccc X = CX + B(L_V X) + E(-L_X \theta),
\]

where \( L_V X \) denotes the Lie derivative of \( X \) with respect to \( V \), and \( L_X \theta \) denotes the Lie derivative of \( \theta \) with respect to \( X \).
Proof Using (1.4), (1.5), and (1.7), we have

\[
CX + B(L_V X) + E(-L_X \theta) = \begin{pmatrix} 
X^\beta \partial_\beta V^\alpha \\
X^\alpha \\
X^\beta \partial_\beta \theta_\alpha
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
-\beta_\beta \partial_\alpha X^\beta
\end{pmatrix} = \begin{pmatrix}
V^\beta \partial_\beta X^\alpha - X^\beta \partial_\beta V^\alpha \\
0 \\
0
\end{pmatrix} = c^c X.
\]

Thus, we have Theorem 2.

On the other hand, on putting \(C(\tilde{\pi}) = E(\tilde{\pi})\), we write the adapted frame of \(\beta_\theta (T(M_n))\) as \(\{B(\tilde{\pi}), C(\tilde{\beta}), C(\tilde{\alpha})\}\). The adapted frame \(\{B(\tilde{\pi}), C(\tilde{\beta}), C(\tilde{\alpha})\}\) of \(\beta_\theta (T(M_n))\) is given by the matrix

\[
\tilde{A} = (\tilde{A}_B^A) = \begin{pmatrix}
\delta_\beta^\alpha \\
0 \\
\partial_\beta \theta_\alpha
\end{pmatrix}.
\]

(1.8)

Since the matrix \(\tilde{A}\) in (1.8) is nonsingular, it has the inverse. Denoting this inverse by \((\tilde{A})^{-1}\), we have

\[
(\tilde{A})^{-1} = (\tilde{A}_B^A)^{-1} = \begin{pmatrix}
\delta_\beta^\alpha \\
0 \\
\partial_\beta \theta_\alpha
\end{pmatrix} = 0
\]

(1.9)

where \(\tilde{A}(\tilde{A})^{-1} = (\tilde{A}_B^A) (\tilde{A}_C^B)^{-1} = \delta_\alpha^\beta = I\), where \(A = (\pi, \alpha, \tilde{\pi})\), \(B = (\tilde{\pi}, \beta, \tilde{\alpha})\), \(C = (\tilde{\pi}, \theta, \tilde{\alpha})\).

Proof From (1.8) and (1.9), we easily see that

\[
\tilde{A}(\tilde{A})^{-1} = (\tilde{A}_B^A) (\tilde{A}_C^B)^{-1} = \begin{pmatrix}
\delta_\beta^\alpha \\
0 \\
\partial_\beta \theta_\alpha
\end{pmatrix} = \begin{pmatrix}
\delta_\beta^\alpha \\
\delta_\beta^\alpha \\
\partial_\beta \theta_\alpha
\end{pmatrix} = \begin{pmatrix}
\delta_\alpha^\beta \\
\delta_\alpha^\beta \\
\partial_\alpha \theta_\alpha
\end{pmatrix} = \begin{pmatrix}
\delta_\alpha^\beta \\
\delta_\alpha^\beta \\
\partial_\alpha \theta_\alpha
\end{pmatrix} = \delta_\alpha^\beta = I.
\]

\[\square\]
Then we see from Theorem 2 that the complete lift $ccX$ of a vector field $X \in \mathfrak{X}(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form

$$ccX : \begin{pmatrix} L_V X^\alpha \\ X^\alpha \\ -L_X \theta_\alpha \end{pmatrix}$$

with respect to the adapted frame $\{B(\overline{\beta}), C(\beta), C(\overline{\beta})\}$.

$BX$, $CX$, and $E\omega$ also have components

$$BX = \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix}, \quad CX = \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, \quad E\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix},$$

respectively, with respect to the adapted frame $\{B(\overline{\beta}), C(\beta), C(\overline{\beta})\}$ of the cross-section $\beta_\theta(T(M_n))$ determined by a 1-form $\theta$ on $T(M_n)$.

2. Complete lift of tensor fields of type (1,1) on a cross-section in a semi-cotangent bundle

Suppose now that $F \in \mathfrak{X}^1(T(M_n))$ and $F$ has local components $F_\beta^\alpha$ in a neighborhood $U$ of $M_n$, $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then the semi-cotangent (pull-back) bundle $t^*(M_n)$ of cotangent bundle $T^*(M_n)$ by using projection of the tangent bundle $T(M_n)$ admits the complete lift $ccF$ of $F$ with components [8]

$$ccF = (ccF^A) = \begin{pmatrix} F_\beta^\alpha & y^\gamma \partial_\gamma F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ p_\sigma (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix},$$

with respect to the coordinates $(x^\alpha, x^\beta, x^\overline{\beta})$ on $t^*(M_n)$. Then $ccF$ has components $F_B^A$ given by

$$ccF = (ccF_B^A) = \begin{pmatrix} F_\beta^\alpha & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \phi_F \theta & F_\alpha^\beta \end{pmatrix},$$

with respect to the adapted frame $\{B(\overline{\beta}), C(\beta), C(\overline{\beta})\}$ of the cross-section $\beta_\theta(T(M_n))$ determined by a 1-form $\theta$ in $T(M_n)$, where $A = (\alpha, \alpha, \overline{\alpha})$, $B = (\overline{\beta}, \beta, \overline{\beta})$. Also, the component $ccF_\beta^\overline{\alpha}$ of $ccF_B^A$ is defined as the Tachibana operator $\phi_F \theta$ of $F$, i.e.

$$ccF_\beta^\overline{\alpha} = \phi_F \theta = (\partial_\overline{\beta} F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) \partial_\sigma - F_\gamma^\overline{\alpha} \partial_\gamma \theta_\alpha + F_\alpha^\beta \partial_\beta \theta_\gamma,$$

and $L_V F_\beta^\alpha$ denotes the Lie derivative of $F_\beta^\alpha$ with respect to $V$, i.e.

$$L_V F_\beta^\alpha = V^\gamma \partial_\gamma F_\beta^\alpha + F_\sigma^\alpha \partial_\beta V^\gamma - F_\gamma^\overline{\alpha} \partial_\gamma V^\alpha.$$
Proof Let $F \in \mathcal{S}_1(T(M_n))$. Then we have by (1.8), (1.9), and (2.1):

\[
\begin{align*}
\lll F &= \left( \bar{A}_A^B \right)^{-1} \left( \lll F_C^A \right) \left( \bar{A}_D^C \right) \\
&= \begin{pmatrix}
\delta^\alpha_\alpha & -\partial_\alpha V^\beta & 0 \\
0 & \delta^\beta_\beta & 0 \\
0 & -\partial_\alpha \theta_\beta & \delta^\gamma_\delta
\end{pmatrix}
\begin{pmatrix}
F^\alpha_\gamma & V^\varepsilon \partial_\varepsilon F^\alpha_\gamma & 0 \\
0 & F^\alpha_\gamma & 0 \\
\theta_\sigma \left( \partial_\gamma F^\sigma_\alpha - \partial_\alpha F^\sigma_\gamma \right) & F^\gamma_\alpha & 0
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_j & \partial_j V^\gamma & 0 \\
0 & \delta^\gamma_j & 0 \\
0 & \partial_\psi \theta_\gamma & \delta^\psi_
u
\end{pmatrix} \\
&= \begin{pmatrix}
F^\beta_\gamma & V^\varepsilon \partial_\varepsilon F^\beta_\gamma & 0 \\
0 & F^\beta_\gamma & 0 \\
0 & -\partial_\alpha \theta_\beta + \theta_\sigma \partial_\sigma F^\sigma_\beta - \theta_\beta \partial_\gamma F^\gamma_\beta & 0
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_j & \partial_j V^\gamma & 0 \\
0 & \delta^\gamma_j & 0 \\
0 & \partial_\psi \theta_\gamma & \delta^\psi_
u
\end{pmatrix} \\
&= \begin{pmatrix}
F^\beta_\gamma & F^\beta_\gamma \partial_\psi V^\gamma + V^\varepsilon \partial_\varepsilon F^\beta_\gamma & 0 \\
0 & F^\beta_\gamma & 0 \\
0 & -\partial_\alpha \theta_\beta + \theta_\sigma \partial_\sigma F^\sigma_\beta \partial_\gamma - \theta_\beta \partial_\gamma F^\gamma_\beta + F^\gamma_\beta \partial_\psi \theta_\gamma & F^\psi_\beta
\end{pmatrix} \\
&= \begin{pmatrix}
F^\beta_\gamma \partial_\psi V^\gamma & 0 \\
0 & F^\beta_\gamma & 0 \\
\varphi F \theta & F^\psi_\beta
\end{pmatrix}
= \left( \lll F_D^B \right),
\end{align*}
\]

where $A = (\pi, \alpha, \bar{\pi})$, $B = (\tilde{\beta}, \beta, \bar{\beta})$, $C = (\bar{\gamma}, \gamma, \bar{\gamma})$, $D = (\bar{\psi}, \psi, \bar{\psi})$, $\square$

Using (2.2), we have along $\beta_\theta (T(M_n))$: 

**Theorem 3** If $F$ and $X$ are affinor and vector fields on $T(M_n)$, and $\omega \in \mathcal{S}_1^0(M_n)$, then:

(i) $\lll F (BX + CX) = B (FX) + C (FX) + B ((L_\psi F) X) + E (P_X),$ 

(ii) $\lll F (E \omega) = E (\omega \circ F),$

where $P \in \mathcal{S}_1^0(M_n)$ with local components

\[
P_{\beta \alpha} = \phi_F \theta = \left( \partial_\beta F^\sigma_\alpha - \partial_\alpha F^\sigma_\beta \right) \theta_\sigma - F^\gamma_\beta \partial_\gamma \theta_\alpha + F^\beta_\gamma \partial_\psi \theta_\gamma,
\]

$\theta_\beta$ being local components of $\theta$, and $P_X \in \mathcal{S}_1^0(M_n)$ defined by $P_X (Y) = P (X, Y)$, for $Y \in \mathcal{S}_1^0(T(M_n))$. 

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Proof (i) If $F$ and $X$ are affinor and vector fields on $T(M_n)$, then by (1.10) and (2.2), we have

$$c^cF(BX + CX) = \left( \begin{array}{ccc} F_\alpha^\beta & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ \phi_F \theta & F_\alpha^\beta & 0 \end{array} \right) \left( \begin{array}{c} X^\beta \\ X^\beta \\ \phi_F \theta X^\beta \end{array} \right)$$

$$= \left( \begin{array}{c} F_\beta^\alpha X^\beta + L_V F_\beta^\alpha X^\beta \\ F_\alpha^\beta X^\beta \\ X^\beta \partial_\beta F_\alpha^\beta \theta_\sigma - X^\beta \partial_\alpha F_\beta^\sigma \theta_\sigma - F_\beta^\alpha X^\beta \partial_\gamma \theta_\alpha + F_\alpha^\beta X^\beta \partial_\beta \theta_\gamma \end{array} \right)$$

$$= \left( \begin{array}{c} (FX)^\alpha + V^\gamma \partial_\gamma F_\beta^\alpha X^\beta + F_\beta^\gamma \partial_\gamma V^\gamma X^\beta - F_\beta^\gamma \partial_\gamma V^\alpha X^\beta \\ (FX)^\alpha \\ X^\beta \partial_\beta F_\alpha^\beta \theta_\sigma - X^\beta \partial_\alpha F_\beta^\sigma \theta_\sigma - F_\beta^\alpha X^\beta \partial_\gamma \theta_\alpha + F_\alpha^\beta X^\beta \partial_\beta \theta_\gamma \end{array} \right)$$

$$= \left( \begin{array}{c} (FX)^\alpha \\ 0 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ (FX)^\alpha \\ 0 \end{array} \right) + \left( \begin{array}{c} (L_V F) X \\ 0 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \\ P_X \end{array} \right)$$


Thus, we have $c^cF(BX + CX) = B(FX) + C(FX) + B((L_V F) X) + E(P_X).$

(ii) If $\omega \in S^1_1(M_n)$, $F$ is an affinor field on $T(M_n)$, and then by (1.10) and (2.2), we have

$$c^cF(E\omega) = \left( \begin{array}{ccc} F_\alpha^\beta & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ \phi_F \theta & F_\alpha^\beta & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ \omega_\beta \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ (\omega \circ F)_\alpha \end{array} \right) = E(\omega \circ F),$$

which gives equation (ii) of Theorem 3.

When $c^cF(BX + CX)$ is always tangent to $\beta_\theta(T(M_n))$ for any vector field $X \in S^1_1(T(M_n))$, $c^cF$ is said to leave the cross-section $\beta_\theta(T(M_n))$ invariant.

Thus, we have:

**Theorem 4** The complete lift $c^cF$ of an element of $F \in S^1_1(T(M_n))$ leaves the cross-section $\beta_\theta(T(M_n))$ invariant if and only if:

(i) $(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma = 0$ (i.e., $\phi_F \theta = 0$),

(ii) $V^\gamma \partial_\gamma F_\alpha^\sigma + F_\beta^\gamma \partial_\gamma V^\gamma - F_\beta^\gamma \partial_\gamma V^\alpha = 0$ (i.e., $L_V F = 0$),

where $F_\beta^\alpha$, $\theta_\beta$, and $V^\alpha$ are local components of $F$, $\theta$, and $V$, respectively.

3. Adapted frames and diagonal lifts of affinor fields

Let $\nabla$ be a symmetric affine connection in $M_n$. In each coordinate neighborhood $\{U, x^\alpha\}$ of $M_n$, we put

$$X(\alpha) = \frac{\partial}{\partial x^\alpha}, \quad \theta^{(\alpha)} = dx^\alpha.$$
Then \(3n\) local vector fields \(Y(\alpha), HX(\alpha), \text{ and } \theta(\alpha)\) have respectively components of the form

\[
Y(\alpha) : \begin{pmatrix} \delta^\alpha_\beta \\ 0 \\ 0 \end{pmatrix}, \quad HX(\alpha) : \begin{pmatrix} -\Gamma^\alpha_{\beta\gamma} \\ \delta^\alpha_\beta \\ \Gamma^\alpha_{\beta\gamma} \end{pmatrix}, \quad v\theta(\alpha) : \begin{pmatrix} 0 \\ 0 \\ \delta^\alpha_\beta \end{pmatrix}
\]

with respect to the induced coordinates \((x^\pi, x^\alpha, x^\bar{\beta})\) in \(\pi^{-1}(U)\), where we have used \((1.6)\). We call the set \(\{Y(\alpha), HX(\alpha), v\theta(\alpha)\}\) the frame adapted to the symmetric affine connection \(\nabla\) in \(\pi^{-1}(U)\). On putting

\[
\bar{e}(\pi) = Y(\alpha), \quad \bar{e}(\alpha) = HX(\alpha), \quad \bar{e}(:\pi) = v\theta(\alpha)
\]

we write the adapted frame as

\[
\{\bar{e}(\alpha)\} = \{\bar{e}(\pi), \bar{e}(\alpha), \bar{e}(\bar{\pi})\}.
\]

The adapted frame \(\{\bar{e}(\alpha)\} = \{\bar{e}(\pi), \bar{e}(\alpha), \bar{e}(\bar{\pi})\}\) is given by the matrix

\[
\hat{A} = \begin{pmatrix} A_B^A \\ \delta^A_C \end{pmatrix} = \begin{pmatrix} \delta^\alpha_\beta \\ -\Gamma^\alpha_{\beta\gamma} \\ 0 \\ 0 \\ \delta^\alpha_\beta \\ 0 \\ 0 \\ \Gamma^\alpha_{\beta\gamma} \\ \delta^\alpha_\beta \end{pmatrix}.
\]

(3.4)

Since the matrix \(\hat{A}\) in \(3.4\) is nonsingular, it has the inverse. Denoting this inverse by \(\hat{A}^{-1}\), we have

\[
\hat{A}^{-1} = (A_B^A)\begin{pmatrix} \delta^A_C \end{pmatrix}^{-1} = \begin{pmatrix} \delta^\alpha_\beta \\ 0 \\ -\Gamma^\alpha_{\beta\gamma} \\ 0 \\ \delta^\alpha_\beta \\ 0 \\ -\Gamma^\alpha_{\beta\gamma} \\ \delta^\alpha_\beta \end{pmatrix},
\]

(3.5)

where \(\hat{A}^{-1} = (A_B^A)\begin{pmatrix} \delta^A_C \end{pmatrix}^{-1} = \delta^A_C = \bar{I}\), where \(A = (\pi, \alpha, \bar{\pi}), \ B = (\bar{\beta}, \beta, \bar{\beta}),\ C = (\bar{\theta}, \theta, \bar{\theta})\).

Proof From \(3.4\) and \(3.5\), we easily see that

\[
\hat{A}^{-1} = (A_B^A)\begin{pmatrix} \delta^A_C \end{pmatrix}^{-1} = \begin{pmatrix} \delta^\alpha_\beta \\ 0 \\ -\Gamma^\alpha_{\beta\gamma} \\ 0 \\ \delta^\alpha_\beta \\ 0 \\ 0 \\ \Gamma^\alpha_{\beta\gamma} \\ \delta^\alpha_\beta \end{pmatrix} \begin{pmatrix} \delta^\alpha_\beta \\ 0 \\ 0 \\ 0 \\ \delta^\alpha_\beta \\ 0 \\ 0 \\ 0 \\ \delta^\alpha_\beta \end{pmatrix} = \begin{pmatrix} \delta^\alpha_\beta \\ 0 \\ -\Gamma^\alpha_{\beta\gamma} \\ 0 \\ \delta^\alpha_\beta \\ 0 \\ 0 \\ \Gamma^\alpha_{\beta\gamma} \\ \delta^\alpha_\beta \end{pmatrix} = \delta^A_C = \bar{I}.
\]

(3.6)

If we take account of \(3.3\), we see that the diagonal lift \(\bar{D}F\) of \(F \in \mathfrak{H}_1(T(M_n))\) has components

\[
\bar{D}F = \begin{pmatrix} \bar{D}F^I \\ \bar{D}F^J \end{pmatrix} = \begin{pmatrix} -F^\alpha_\beta -\Gamma^\alpha_{\beta\gamma} F^\gamma_\epsilon - \Gamma^\alpha_{\beta\gamma} F^\gamma_\epsilon \\ 0 \\ \delta^\alpha_\beta F^\alpha_\beta + \Gamma^\alpha_{\beta\gamma} F^\gamma_\epsilon - F^\alpha_\beta \end{pmatrix},
\]

(3.6)
with respect to the coordinates \((x^\alpha, x^\beta, x^\gamma)\) on \(t^*(M_n)\), where
\[
\Gamma^\alpha_\epsilon = y^\gamma \Gamma^\alpha_\gamma \epsilon, \quad \Gamma_{\alpha\sigma} = p_\gamma \Gamma^\alpha_\gamma \epsilon.
\]

**Proof** Let \(F \in \mathfrak{X}(T(M_n))\). Then we have by (3.4), (3.5), and (3.6):

\[
\begin{align*}
^{DD}F &= \left( \hat{A} \right) \left( ^{DD}F \right) \left( \hat{A} \right)^{-1} \\
&= \begin{pmatrix}
\delta_\alpha^\beta & \Gamma^\beta_\alpha & 0 \\
0 & \delta_\beta^\alpha & 0 \\
0 & \Gamma_{\alpha\beta} & \delta_\beta^\alpha
\end{pmatrix}
\begin{pmatrix}
-F^\alpha_\gamma & -F^\alpha_\epsilon F^\gamma_\epsilon - F^\alpha_\gamma F^\beta_\epsilon & 0 \\
0 & -F^\alpha_\gamma F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\sigma\epsilon} F^\beta_\sigma + \Gamma_{\alpha\sigma} F^\sigma_\epsilon - F^\beta_\alpha
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_\psi & \Gamma^\gamma_\psi & 0 \\
0 & \delta^\gamma_\psi & 0 \\
0 & 0 & -\Gamma_{\epsilon\psi} \delta^\epsilon_\gamma
\end{pmatrix}
\end{align*}
\]

\[
= \begin{pmatrix}
-F_\beta^\gamma & -\Gamma^\beta_\gamma F^\gamma_\epsilon - \Gamma^\beta_\epsilon F^\gamma_\epsilon - \Gamma^\beta_\gamma F^\beta_\epsilon & 0 \\
0 & -F_\beta^\gamma F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\alpha\beta} F^\gamma_\sigma + \Gamma_{\beta\sigma} F^\sigma_\gamma - F^\gamma_\beta
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_\psi & \Gamma^\gamma_\psi & 0 \\
0 & \delta^\gamma_\psi & 0 \\
0 & 0 & -\Gamma_{\epsilon\psi} \delta^\epsilon_\gamma
\end{pmatrix}
\begin{pmatrix}
-F^\beta_\psi & -\Gamma^\beta_\psi F^\gamma_\epsilon - \Gamma^\beta_\epsilon F^\gamma_\epsilon & 0 \\
0 & -F^\beta_\psi F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\mu\beta} F^\mu_\gamma + \Gamma_{\beta\mu} F^\mu_\psi - F^\mu_\beta
\end{pmatrix}
\end{align*}
\]

which proves (3.6).

We now see, from (3.3), that the diagonal lift \(^{DD}F\) of \(F \in \mathfrak{X}(T(M_n))\) has components of the form

\[
^{DD}F = \left( ^{AA}F \right) = \begin{pmatrix}
-F^\beta_\gamma & 0 & 0 \\
0 & F^\beta_\gamma & 0 \\
0 & 0 & -F^\beta_\alpha
\end{pmatrix}
\]

with respect to the adapted frame \(\left\{ \hat{e}_{(B)} \right\} \) in \(t^*(M_n)\).

**Proof** Let \(F \in \mathfrak{X}(T(M_n))\). Then we have by (3.4), (3.5), and (3.6):

\[
^{DD}F = \left( \hat{A} \right)^{-1} \left( ^{DD}F \right) \left( \hat{A} \right)
\]

\[
= \begin{pmatrix}
\delta_\alpha^\beta & \Gamma^\beta_\alpha & 0 \\
0 & \delta_\beta^\alpha & 0 \\
0 & \Gamma_{\alpha\beta} & \delta_\beta^\alpha
\end{pmatrix}
\begin{pmatrix}
-F^\alpha_\gamma & -F^\alpha_\epsilon F^\gamma_\epsilon - F^\alpha_\gamma F^\beta_\epsilon & 0 \\
0 & -F^\alpha_\gamma F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\sigma\epsilon} F^\beta_\sigma + \Gamma_{\alpha\sigma} F^\sigma_\epsilon - F^\beta_\alpha
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_\psi & \Gamma^\gamma_\psi & 0 \\
0 & \delta^\gamma_\psi & 0 \\
0 & 0 & -\Gamma_{\epsilon\psi} \delta^\epsilon_\gamma
\end{pmatrix}
\end{align*}
\]

\[
= \begin{pmatrix}
-F_\beta^\gamma & -\Gamma^\beta_\gamma F^\gamma_\epsilon - \Gamma^\beta_\epsilon F^\gamma_\epsilon + \Gamma^\beta_\gamma F^\beta_\epsilon & 0 \\
0 & -F_\beta^\gamma F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\alpha\beta} F^\gamma_\sigma + \Gamma_{\beta\sigma} F^\sigma_\gamma - F^\gamma_\beta
\end{pmatrix}
\begin{pmatrix}
\delta^\gamma_\psi & \Gamma^\gamma_\psi & 0 \\
0 & \delta^\gamma_\psi & 0 \\
0 & 0 & -\Gamma_{\epsilon\psi} \delta^\epsilon_\gamma
\end{pmatrix}
\begin{pmatrix}
-F^\beta_\psi & -\Gamma^\beta_\psi F^\gamma_\epsilon - \Gamma^\beta_\epsilon F^\gamma_\epsilon + \Gamma^\beta_\psi F^\beta_\epsilon & 0 \\
0 & -F^\beta_\psi F^\gamma_\epsilon & 0 \\
0 & 0 & \Gamma_{\mu\beta} F^\mu_\gamma + \Gamma_{\beta\mu} F^\mu_\psi - F^\mu_\beta
\end{pmatrix}
\end{align*}
\]
This completes the proof.

We now obtain from (3.6) that the diagonal lift $^{DD}F$ of an affinor field $F \in \mathfrak{I}^1(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form

$$^{DD}F : \begin{pmatrix} -F^\alpha_\beta & -\nabla_\xi V^\alpha F^\xi_\beta - (\nabla_\beta V^\xi) F^\xi_\beta & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & 0 & -F^\alpha_\beta \end{pmatrix},$$

(3.7)

with respect to the adapted frame $\{B_{(\xi)}, C_{(\beta)}, C_{(\beta)}\}$.

Proof Let $F \in \mathfrak{I}^1(T(M_n))$. Then we have by (1.8), (1.9), and (3.7):

$$^{DD}F = \begin{pmatrix} \delta^\beta_\alpha - \partial_\xi V^\beta & 0 & 0 \\ 0 & \delta^\beta_\alpha & 0 \\ 0 & 0 & \delta^\beta_\alpha \end{pmatrix} \begin{pmatrix} -F^\alpha_\gamma & -\Gamma_\gamma^\alpha F_\gamma^\xi - \Gamma_\xi^\alpha F_\xi^\gamma & 0 \\ 0 & F^\alpha_\gamma & 0 \\ 0 & \Gamma_\gamma^\sigma F^\sigma_\alpha + \Gamma_\alpha^\sigma F^\sigma_\gamma & -F^\gamma_\alpha \end{pmatrix} \begin{pmatrix} \delta_\psi^\gamma & \partial_\psi V^\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi \theta_\gamma & \delta_\psi^\gamma \end{pmatrix}$$

$$= \begin{pmatrix} -F^\alpha_\gamma & -\Gamma_\gamma^\alpha F_\gamma^\xi - \Gamma_\xi^\alpha F_\xi^\gamma & 0 \\ 0 & F^\alpha_\gamma & 0 \\ 0 & \Gamma_\gamma^\sigma F^\sigma_\alpha + \Gamma_\alpha^\sigma F^\sigma_\gamma & -F^\gamma_\alpha \end{pmatrix} \begin{pmatrix} \delta_\psi^\gamma & \partial_\psi V^\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi \theta_\gamma & \delta_\psi^\gamma \end{pmatrix}$$

$$= \begin{pmatrix} -F^\beta_\psi & -\partial_\psi V^\gamma F^\gamma_\beta - \Gamma_\beta^\gamma F_\psi^\gamma & 0 \\ 0 & F^\beta_\psi & 0 \\ 0 & \partial_\psi \theta_\beta & \partial_\psi \theta_\gamma \end{pmatrix}$$

Thus, the proof is complete.

Then we see from (1.6) that the horizontal lift $^{HH}X$ of a vector field $X \in \mathfrak{I}^1_0(T(M_n))$ has along $\beta_\theta(T(M_n))$ components of the form

$$^{HH}X : \begin{pmatrix} -X^\beta (\nabla_\beta V^\alpha) \\ X^\alpha \\ -\nabla_\beta \theta_\alpha X^\beta \end{pmatrix},$$

(3.8)

with respect to the adapted frame $\{B_{(\xi)}, C_{(\beta)}, C_{(\beta)}\}$.
Proof Let \( X \in \mathcal{X}_1(T(M_n)) \). Then we have by (1.6) and (1.9):

\[
\begin{aligned}
HH X &= (\hat{A})^{-1} (HH X) = \\
&= \begin{pmatrix} \delta^\alpha_\beta & -\delta^\beta_\alpha & 0 \\
0 & \delta^\beta_\alpha & 0 \\
0 & 0 & \delta^\beta_\alpha \\
\end{pmatrix} \begin{pmatrix} -V^\gamma \Gamma^\beta_{\gamma \alpha} X^\alpha \\
X^\beta \\
X^\alpha \theta^\gamma_{\alpha \beta} \\
\end{pmatrix} \\
&= \begin{pmatrix} -V^\gamma \Gamma^\beta_{\gamma \theta} X^\theta - \delta^\beta_\alpha X^\alpha \\
X^\alpha \\
-\delta^\beta_\alpha X^\beta + X^\theta \theta^\gamma_{\alpha \beta} \\
\end{pmatrix} = \begin{pmatrix} -X^\beta (\nabla_\beta V^\alpha) \\
X^\alpha \\
- (\nabla_\alpha X^\beta) \\
\end{pmatrix},
\end{aligned}
\]

which gives (3.8).

Using (1.6), (3.7), and (3.8), we have along \( \beta_0(T(M_n)) \):

**Theorem 5** If \( F \) and \( X \) are affinor and vector fields on \( T(M_n) \), and \( \omega \in \mathcal{X}_1(M_n) \), then with respect to a symmetric affine connection \( \nabla \) in \( M_n \), we have

(i) \( D^D F(HH X) = H^H (FX) \),

(ii) \( D^D F(\nu) = -\nu (\omega \circ F) \).

**Proof**

(i) If \( F \in \mathcal{X}_1(T(M_n)) \) and \( X \in \mathcal{X}_0(T(M_n)) \), then by (3.7) and (3.8), we have

\[
\begin{aligned}
D^D F(HH X) &= \begin{pmatrix} -F^\alpha_\beta \\
0 \\
0 \\
\end{pmatrix} \\
&= \begin{pmatrix} -F^\alpha_\beta \\
0 \\
0 \\
\end{pmatrix} = H^H (FX).
\end{aligned}
\]

Thus, we have \( D^D F(HH X) = H^H (FX) \).

(ii) If \( \omega \in \mathcal{X}_1(M_n) \) and \( F \in \mathcal{X}_1(T(M_n)) \), then by (1.6), (1.10), and (3.7), we have

\[
\begin{aligned}
D^D F(\nu) &= \begin{pmatrix} -F^\alpha_\beta \\
0 \\
0 \\
\end{pmatrix} \\
&= \begin{pmatrix} -F^\alpha_\beta \\
0 \\
0 \\
\end{pmatrix} = -\nu (\omega \circ F).
\end{aligned}
\]

Thus, we have (ii) of Theorem 5. \( \square \)
References