A study on (strong) order-congruences in ordered semihypergroups

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Abstract: In this paper, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup $S$, and obtain the relationship between strong order-congruences and pseudoorders on $S$. Furthermore, we characterize the (strong) order-congruences by the $\rho$-chains, where $\rho$ is a (strong) congruence on $S$. Moreover, we give a method of constructing order-congruences, and prove that every hyperideal $I$ of an ordered semihypergroup $S$ is congruence class of one order-congruence on $S$ if and only if $I$ is convex. Finally, we define and study the strong order-congruence generated by a strong congruence. As an application of the results of this paper, we solve an open problem on ordered semihypergroups given by Davvaz et al.

Key words: Ordered semihypergroup, pseudoorder, (strong) order-congruence, $\rho$-chain

1. Introduction

In mathematics, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation “$\leq$” such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. As we know, congruences on ordered semigroups play an important role in studying the structures of ordered semigroups, for example, see [13-15, 22-24]. For any congruence $\rho$ on an ordered semigroup $S$, in general, we do not know whether the quotient semigroup $S/\rho$ is also an ordered semigroup. Even if $S/\rho$ is an ordered semigroup, the order on $S/\rho$ is not necessarily relative to the order on the original ordered semigroup $S$. As to the above-mentioned questions, Kehayopulu and Tsingelis [13, 14] introduced the concept of pseudoorder on an ordered semigroup $S$ and proved that if $\sigma$ is a pseudoorder on $S$, then there exists a congruence $\sigma$ on $S$ such that $S/\sigma$ is an ordered semigroup. In the same papers a necessary and sufficient condition such that $S$ is a subdirect product of some ordered semigroups was given and two isomorphism theorems of $S$ were established. Since then, Xie [23] introduced the concept of regular congruences on an ordered semigroup $S$, and proved that $\rho$ is a regular congruence on $S$ if and only if there exists a pseudoorder $\sigma$ on $S$ such that $\rho = \sigma \cap \sigma^{-1}$.

On the other hand, algebraic hyperstructures, particularly hypergroups, were introduced by Marty [17] in 1934. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a set. Thus algebraic hyperstructures are a suitable

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generalization of classical algebraic structures. Surveys of hyperstructure theory can be found in the books by Corsini [4], Corsini and Leoreanu [5], Davvaz and Leoreanu-Fotea [8], and Vougiouklis [20]. In the hyperstructure theory, semihypergroups are the simplest algebraic hyperstructures that are a generalization of the concept of semigroups. At present, many researchers have studied different aspects of semihypergroups. For more details, the reader is referred to [1, 3, 6, 9, 10, 12, 16, 18, 25]. Especially, regular and strong regular relations on semihypergroups have been introduced and investigated in [4].

A theory of hyperstructures on ordered semigroups has been recently developed. In [11], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Later on, a lot of papers on ordered semihypergroups have been written; for instance, see [2, 7, 19, 26]. It is worth pointing out that Davvaz et al. [7] introduced the concept of a pseudoorder on an ordered semihypergroup, and extended some results in [13] on ordered semigroups to ordered semihypergroups. In particular, they posed an open problem about ordered semihypergroups: Is there a regular relation $\rho$ on an ordered semihypergroup $(S, *, \leq)$ for which $S/\rho$ is an ordered semihypergroup? As a further study, in this paper we define and study the order-congruences and strong order-congruences on an ordered semihypergroup, and extend some results in ordered semigroups to ordered semihypergroups. The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory. In Section 3, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup $S$, and establish the relationship between strong order-congruences and pseudoorders on $S$. Moreover, we described the least pseudoorder containing a strong order-congruence on an ordered semihypergroup, and give out a homomorphism theorem of ordered semihypergroups by pseudoorders. In Section 4, we characterize the strong order-congruences (resp. order-congruences) by the $\rho$-chains, where $\rho$ is a strong congruence (resp. congruence). Furthermore, we provide a method of constructing order-congruences, and prove that every hyperideal $I$ of an ordered semihypergroup $S$ is congruence class of one order-congruence on $S$ if and only if $I$ is convex. By this constructing method of order-congruences, we answer to the open problem given by Davvaz et al. in [7]. Finally, we define and discuss the strong order-congruence generated by a strong congruence.

2. Preliminaries and some notations

Recall that a hypergroupoid $(S, *)$ is a nonempty set $S$ together with a hyperoperation, that is a map $*: S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all the nonempty subsets of $S$. The image of the pair $(x, y)$ is denoted by $x*y$. If $x \in S$ and $A, B$ are nonempty subsets of $S$, then $A*B$ is defined by $A*B = \bigcup_{a \in A, b \in B} a*b$.

Also $A*x$ is used for $A\{x\}$ and $x*A$ for $\{x\}*A$. Generally, the singleton $\{x\}$ is identified by its element $x$.

We say that a hypergroupoid $(S, *)$ is a semihypergroup if the hyperoperation "*" is associative, that is, $(x*y)*z = x*(y*z)$ for all $x, y, z \in S$ (see [4]).

We now recall the notion of ordered semihypergroups from [11].

**Definition 2.1** An algebraic hyperstructure $(S, *, \leq)$ is called an ordered semihypergroup (also called po-semihypergroup in [11]) if $(S, *)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a*x \leq a*y$ and $x*a \leq y*a$. Here, if $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Clearly, every ordered semigroup is an ordered semihypergroup.
Let \((S, \leq)\) be a partially ordered set (or briefly poset). For \(\emptyset \neq H \subseteq S\), we define

\[\{t \in S \mid t \leq h \text{ for some } h \in H\}\]  

A nonempty subset \(A\) of a poset \(S\) is called convex if \(a \leq b \leq c\) implies \(b \in A\) for all \(a, c \in A, b \in S\). A nonempty subset \(B\) of a poset \(S\) is called strongly convex if \(B = (B)\) (equivalently, \(a \in S, b \in B\) and \(a \leq b\) imply \(a \in B\)). Any strongly convex subset of \(S\) is clearly a convex subset; however, the converse does not hold in general.

By a subsemihypergroup of an ordered semihypergroup \(S\) we mean a nonempty subset \(A\) of \(S\) such that \(A \ast A \subseteq A\). A nonempty subset \(A\) of a semihypergroup \((S, \ast)\) is called a left (resp. right) hyperideal of \(S\) if \(A \ast A \subseteq A\) (resp. \(A \ast S \subseteq A\)). If \(A\) is both a left and a right hyperideal of \(S\), then it is called a hyperideal of \(S\). A nonempty subset \(A\) of an ordered semihypergroup \((S, \ast, \leq)\) is called an ordered hyperideal of \(S\) if \(A\) is a strongly convex hyperideal of \(S\).

Let \(\rho\) be an equivalence relation on a semihypergroup \((S, \ast)\) or an ordered semigroup \((S, \ast, \leq)\). If \(A\) and \(B\) are nonempty subsets of \(S\), then we write \(A\bar{\rho}\) to denote that for every \(a \in A\), there exists \(b \in B\) such that \(a \bar{\rho} b\) and for every \(b \in B\) there exists \(a \in A\) such that \(a \bar{\rho} b\). We write \(A\bar{\rho}B\) if for every \(a \in A\) and for every \(b \in B\) we have \(a \bar{\rho} b\). The equivalence relation \(\rho\) is called congruence (also called regular relation in [4, 7]) if for every \((x, y) \in S \times S\) the implication \(x \rho y \Rightarrow a \ast x \bar{\rho} a \ast y\) and \(x \ast a \bar{\rho} y \ast a\), for all \(a \in S\), is valid. \(\rho\) is called strong congruence (also called strongly regular relation in [4, 7]) if for every \((x, y) \in S \times S\), from \(x \rho y\), it follows that \(a \ast x \bar{\rho} a \ast y\) and \(x \ast a \bar{\rho} y \ast a\) for all \(a \in S\).

**Lemma 2.2** ([4]) Let \((S, \ast)\) be a semihypergroup and \(\rho\) an equivalence relation on \(S\). Then

(i) If \(\rho\) is a congruence, then \((S/\rho, \bar{\ast})\) is a semihypergroup with respect to the following hyperoperation:

\[(a)_{\rho} \bar{\ast} (b)_{\rho} = \bigcup_{c \in a \ast b} (c)_{\rho}, \text{ and it is called a factor semihypergroup.}\]

(ii) If \(\rho\) is a strong congruence, then \((S/\rho, \bar{\ast})\) is a semigroup with respect to the following operation: \((a)_{\rho} \bar{\ast} (b)_{\rho} = (c)_{\rho}\) for all \(c \in a \ast b\), and it is called a factor semigroup.

Let \(I\) be a hyperideal of a semihypergroup \((S, \ast)\). The relation \(\rho_I\) on \(S\) is defined as follows:

\[\rho_I := \{(x, y) \in S \setminus I \times S \setminus I \mid x = y\} \cup (I \times I)\]

Clearly, \(\rho_I\) is an equivalence relation on \(S\). Moreover, we have the following lemma.

**Lemma 2.3** Let \((S, \ast)\) be a semihypergroup and \(I\) a hyperideal of \(S\). Then \(\rho_I\) is a congruence on \(S\) and it is called Rees congruence induced by \(I\).

**Proof** Let \(x, y \in S\) and \(x \rho_I y\). Then \(x = y \in S \setminus I\) or \(x, y \in I\). We consider the following cases:

Case 1. If \(x = y \in S \setminus I\), then, for any \(z \in S\), \(x \ast z = y \ast z\). Hence \(x \ast z \bar{\rho_I} y \ast z\).

Case 2. Let \(x, y \in I\). Since \(I\) is a hyperideal of \(S\), we have \(x \ast z \subseteq I, y \ast z \subseteq I\) for any \(z \in S\). Thus, for any \(a \in x \ast z, b \in y \ast z\), we have \((a, b) \in I \times I \subseteq \rho_I\). Therefore, \(x \ast z \bar{\rho_I} y \ast z\). \(\square\)

Similarly, we can show that \(z \ast x \bar{\rho_I} z \ast y\) for any \(z \in S\). We have thus shown that \(\rho_I\) is a congruence on \(S\).
Remark 2.4  (1)  \( S/\rho_l = \{ \{x\} \mid x \in S \setminus I \} \cup \{ I \} \).

(2) By Lemmas 2.2 and 2.3, \( (S/\rho_l, \otimes_I) \) forms a factor semi-hypergroup, which is called Rees factor semi-hypergroup (also called Rees quotient semi-hypergroup). Here the hyperoperation \( \otimes_I \) on \( S/\rho_l \) is defined by 
\[
(a)_{\rho_l} \otimes_I (b)_{\rho_l} = \bigcup_{c \in a \ast b} (c)_{\rho_l}.
\]

A relation \( \rho \) on an ordered semi-hypergroup \( (S, \ast, \leq) \) is called pseudoorder if it satisfies the following conditions: (1) \( \leq \subseteq \rho \), (2) \( ab \rho \) and \( bpc \) imply \( apc \), i.e. \( \rho \circ \rho \subseteq \rho \) and (3) \( ab \rho \) implies \( a \ast c \triangleright \bar{\rho} b \ast c \) and \( c \ast a \triangleright \bar{\rho} c \ast b \), for all \( c \in S \) (see [7]).

Lemma 2.5 Let \( (S, \ast, \leq) \) be an ordered semi-hypergroup and \( \rho \) a pseudoorder on \( S \). Then \( (S/\rho^*, \otimes, \leq^*_\rho) \) is an ordered semigroup, where \( \rho^* \) (\( = \rho \cap \rho^{-1} \)) is a strong congruence on \( S \), and the order relation \( \leq^*_\rho \) is defined as follows:
\[
\leq^*_\rho = \{(x)_{\rho^*}, (y)_{\rho^*} \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.
\]

Let \( (S, \ast, \leq) \) and \( (T, \circ, \leq) \) be two ordered semi-hypergroups, \( f : S \rightarrow T \) a mapping from \( S \) to \( T \). \( f \) is called isotone if \( x \leq y \) implies \( f(x) \leq f(y) \), for all \( x, y \in S \). \( f \) is called reverse isotone if \( x, y \in S \), \( f(x) \leq f(y) \) implies \( x \leq y \). \( f \) is called homomorphism (resp. strong homomorphism) if it is isotone and satisfies \( f(x) \circ f(y) = \bigcup_{z \in x \circ y} f(z) \) (resp. \( f(x) \circ f(y) = f(z) \), \( \forall z \in x \circ y \)), for all \( x, y \in S \). \( f \) is called isomorphism (resp. strong isomorphism) if it is homomorphism (resp. strong homomorphism), onto, and reverse isotone.

The ordered semi-hypergroups \( S \) and \( T \) are called strongly isomorphic, in symbol \( S \cong T \), if there exists a strong isomorphism between them.

Remark 2.6 Let \( S \) and \( T \) be two ordered semi-hypergroups. Then

(1) If \( f \) is a strong homomorphism and reverse isotone mapping from \( S \) to \( T \), then \( S \cong \text{Im}(f) \).

(2) In particular, if \( S \) and \( T \) are both ordered semigroups, then, in this case, the concepts of strong isomorphisms of ordered semi-hypergroups and isomorphisms of ordered semigroups coincide.

The reader is referred to [5, 21] for notation and terminology not defined in this paper.

3. Strong order-congruences and order-congruences on ordered semi-hypergroups

As we know, pseudoorders on ordered semi-hypergroups play an important role in studying the structures of ordered semi-hypergroups (see [7]). To investigate the properties of pseudoorders on ordered semi-hypergroups in detail, in this section we shall introduce the concepts of order-congruences and strong order-congruences on an ordered semi-hypergroup, and study the relationship between strong order-congruences and pseudoorders.

Definition 3.1 Let \( (S, \ast, \leq) \) be an ordered semi-hypergroup. A congruence (resp. strong congruence) \( \rho \) is called an order-congruence (resp. a strong order-congruence) if there exists an order relation “\( \leq \)” on \( (S/\rho, \otimes) \) such that:

(1) \( (S/\rho, \otimes, \leq) \) is an ordered semi-hypergroup (resp. ordered semigroup), where the hyperoperation “\( \otimes \)” is defined as one in Lemma 2.2.
(2) The canonical epimorphism $\varphi : S \to S/\rho, x \mapsto (x)_\rho$ is isotope, that is, $\varphi$ is a homomorphism (resp. strong homomorphism) from $S$ onto $S/\rho$.

It is clear that the equality relation $1_S$ and the universal relation $S \times S$ on $S$ are both order-congruences. In general, an example of order-congruence is given as follows:

**Example 3.2** We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation “*” and the order “$\leq$”:

<table>
<thead>
<tr>
<th>*</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>${b, d}$</td>
<td>${b, d}$</td>
<td>${d}$</td>
<td>${d}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${b, d}$</td>
<td>${b, d}$</td>
<td>${d}$</td>
<td>${d}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${d}$</td>
<td>${d}$</td>
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<td>${d}$</td>
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<td>$d$</td>
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<td>${d}$</td>
<td>${d}$</td>
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</tr>
<tr>
<td>$e$</td>
<td>${d}$</td>
<td>${d}$</td>
<td>${d}$</td>
<td>${d}$</td>
<td>${d}$</td>
</tr>
</tbody>
</table>

$\leq := \{(a,a), (a,b), (b,b), (c,c), (c,e), (d,b), (d,d), (d,e), (d,d), (e,e)\}$.

We give the covering relation “$\sqsubseteq$” and the figure of $S$ as follows:

$\sqsubseteq := \{(a,b), (d,b), (d,c), (c,e)\}$.

Then $(S, *, \leq)$ is an ordered semihypergroup. Let $\rho_1, \rho_2$ be congruences on $S$ defined as follows:

$\rho_1 := \{(a,a), (b,b), (c,c), (d,d), (e,e), (d,e), (e,d)\}$,

$\rho_2 := \{(a,a), (b,b), (c,c), (d,d), (e,e), (c,e), (e,c)\}$.

Then $S/\rho_1 = \{\{a\}, \{b\}, \{c\}, \{d,e\}\}, S/\rho_2 = \{\{a\}, \{b\}, \{c,e\}, \{d\}\}$. Moreover, we have

1. $\rho_1$ is not an order-congruence on $S$. In fact, if $\rho_1$ is an order-congruence on $S$, then there exists an order “$\preceq_1$” on $S/\rho_1$ such that $(S/\rho_1, \preceq_1, \leq_1)$ is an ordered semihypergroup and the mapping $\varphi_1 : S \to S/\rho_1, x \mapsto (x)_{\rho_1}$ is isotope. Since $d \leq e$, we have $(d)_{\rho_1} \preceq_1 (c)_{\rho_1}$. Also, since $c \leq e$, we have $(c)_{\rho_1} \preceq_1 (e)_{\rho_1} = (d)_{\rho_1}$. Then $(d)_{\rho_1} = (c)_{\rho_1}$. Impossible.

2. $\rho_2$ is an order-congruence on $S$. In fact, let $S/\rho_2 = \{x, y, z, w\}$, where $x = \{a\}, y = \{b\}, z = \{c,e\}, w = \{d\}$.

The hyperoperation “$\otimes_2$” and the order “$\preceq_2$” on $S/\rho_2$ are as follows:

<table>
<thead>
<tr>
<th>$\otimes_2$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
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</thead>
<tbody>
<tr>
<td>$x$</td>
<td>${y, w}$</td>
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<tr>
<td>$y$</td>
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<td>$z$</td>
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</tr>
</tbody>
</table>
We give the covering relation \( \prec \) and the figure of \( S/\rho_2 \) as follows:

\[
\prec_2 := \{(x,y), (w,y), (w,z)\}.
\]

Then \((S/\rho_2, \otimes_2, \preceq_2)\) is an ordered semihypergroup and the mapping \( \varphi_2 : S \to S/\rho_2, x \mapsto (x)_{\rho_2} \) is isotone.

Hence \( \rho_2 \) is an order-congruence on \( S \).

**Proposition 3.3** Let \((S, \ast, \preceq)\) be an ordered semihypergroup and \( \rho \) a pseudoorder on \( S \). Then \( \rho^* \) is a strong order-congruence on \( S \), where \( \rho^* = \rho \cap \rho^{-1} \).

**Proof** By Lemma 2.5, \((S/\rho^*, \otimes, \preceq_\rho)\) is an ordered semigroup, where the order relation \( \preceq_\rho \) is defined as follows:

\[
\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.
\]

Also, let \( x, y \in S \) and \( x \preceq y \). Then, since \( \rho \) is a pseudoorder on \( S \), \( (x, y) \in \preceq_\rho \), and thus \((x)_{\rho^*}, (y)_{\rho^*}) \in \preceq_\rho \), i.e. \((x)_{\rho^*} \preceq_\rho (y)_{\rho^*} \). Therefore, \( \rho^* \) is a strong order-congruence on \( S \).

In order to establish the relationship between strong order-congruences and pseudoorders on an ordered semihypergroup, the following lemma is essential.

**Lemma 3.4** Let \((S, \ast, \preceq)\) be an ordered semihypergroup and \( \sigma \) a relation on \( S \). Then the following statements are equivalent:

1. \( \sigma \) is a pseudoorder on \( S \).
2. There exist an ordered semihypergroup \((T, \circ, \preceq)\) and a strong homomorphism \( \varphi : S \to T \) such that

\[
\ker \varphi := \{(a, b) \in S \times S \mid \varphi(a) \preceq \varphi(b)\} = \sigma,
\]

where \( \ker \varphi \) is called the directed kernel of \( \varphi \).

**Proof** (1) \( \Rightarrow \) (2). Let \( \sigma \) be a pseudoorder on \( S \). We denote by \( \sigma^* \) the strong congruence on \( S \) defined by

\[
\sigma^* := \{(a, b) \in S \times S \mid (a, b) \in \sigma, (b, a) \in \sigma\} \subseteq \sigma \cap \sigma^{-1}.
\]

Then, by Lemma 2.5, the set \( S/\sigma^* := \{a \ast \sigma \mid a \in S\} \) with the operation \( (a)_{\sigma^*} \ast (b)_{\sigma^*} = (c)_{\sigma^*}, \forall c \in a \ast b \), for all \( a, b \in S \) and the order

\[
\preceq_\sigma := \{((x)_{\sigma^*}, (y)_{\sigma^*}) \in S/\sigma^* \times S/\sigma^* \mid (x, y) \in \sigma\}
\]

is an ordered semigroup. Let \( T = (S/\sigma^*, \otimes, \preceq_\sigma) \) and \( \varphi \) be the mapping of \( S \) onto \( S/\sigma^* \) defined by \( \varphi : S \to S/\sigma^*, a \mapsto (a)_{\sigma^*} \). Then, by Proposition 3.3, \( \varphi \) is a strong homomorphism from \( S \) onto \( S/\sigma^* \) and clearly, 

\[
\ker \varphi = \sigma.
\]
(2) \(\Rightarrow\) (1). Let \( (S, *, \leq) \) be an ordered semihypergroup. If there exist an ordered semihypergroup \( (T, \circ, \preceq) \) and a strong homomorphism \( \varphi : S \to T \) such that \( \overrightarrow{\ker\varphi} = \sigma \), then \( \sigma \) is a pseudoorder on \( S \). Indeed, let \((a, b) \in \leq\). Then, by hypothesis, \( \varphi(a) \preceq \varphi(b) \). Thus \( (a, b) \in \overrightarrow{\ker\varphi} = \sigma \), and we have \( \leq \subseteq \sigma \). Moreover, let \((a, b) \in \sigma \) and \((b, c) \in \sigma \). Then \( \varphi(a) \preceq \varphi(b) \preceq \varphi(c) \). Hence \( \varphi(a) \preceq \varphi(c) \), i.e. \((a, c) \in \overrightarrow{\ker\varphi} = \sigma \).

Also, if \((a, b) \in \sigma \), then \( \varphi(a) \preceq \varphi(b) \). Since \((T, \circ, \preceq)\) is an ordered semihypergroup, for any \( c \in S \) we have \( \varphi(a) \circ \varphi(c) \preceq \varphi(b) \circ \varphi(c) \). Since \( \varphi \) is a strong homomorphism from \( S \) to \( T \), for every \( x \in a * c \) and \( y \in b * c \), we have

\[
\varphi(x) = \varphi(a) \circ \varphi(c) \preceq \varphi(b) \circ \varphi(c) = \varphi(y).
\]

Then \((x, y) \in \overrightarrow{\ker\varphi} = \sigma\), and thus \( a * c \overrightarrow{\preceq} b * c \). In the same way, it can be shown that \( c * a \overrightarrow{\preceq} c * b \).

\[\square\]

**Theorem 3.5** Let \( (S, *, \leq) \) be an ordered semihypergroup and \( \rho \) a strong congruence on \( S \). Then the following statements are equivalent:

1. \( \rho \) is a strong order-congruence on \( S \).
2. There exists a pseudoorder \( \sigma \) on \( S \) such that \( \rho = \sigma \cap \sigma^{-1} \).
3. There exist an ordered semihypergroup \( T \) and a strong homomorphism \( \varphi : S \to T \) such that \( \rho = \overrightarrow{\ker\varphi} \), where \( \overrightarrow{\ker\varphi} = \{(a, b) \in S \times S \mid \varphi(a) = \varphi(b)\} \) is the kernel of \( \varphi \).

**Proof** (1) \(\Rightarrow\) (2). Let \( \rho \) be a strong order-congruence on \( S \). Then there exist an order relation \( \preceq \) on the factor semigroup \( (S/\rho, \circ) \) such that \( (S/\rho, \circ, \preceq) \) is an ordered semigroup, and \( \varphi : S \to S/\rho \) is a strong homomorphism. Let \( \sigma = \overrightarrow{\ker\varphi} \). By Lemma 3.4, \( \sigma \) is a pseudoorder on \( S \) and it is easy to check that \( \rho = \sigma \cap \sigma^{-1} \).

(2) \(\Rightarrow\) (3). For a pseudoorder \( \sigma \) on \( S \), by Lemma 3.4, there exist an ordered semihypergroup \( T \) and a strong homomorphism \( \varphi : S \to T \) such that \( \sigma = \overrightarrow{\ker\varphi} \). Then we have

\[
\overrightarrow{\ker\varphi} = \overrightarrow{\sigma \cap \sigma^{-1}} = \sigma \cap \sigma^{-1} = \rho.
\]

(3) \(\Rightarrow\) (1). By hypothesis and Lemma 3.4, \( \overrightarrow{\ker\varphi} \) is a pseudoorder on \( S \). Then, by Lemma 2.5, \( \rho = \overrightarrow{\ker\varphi \cap \overrightarrow{\ker\varphi}^{-1}} \) is a strong congruence on \( S \). Thus, by the proof of Lemma 3.4, \( \rho \) is a strong order-congruence on \( S \).

\[\square\]

**Lemma 3.6** (1) For a strong order-congruence \( \rho \) on \( S \), since the order \( \preceq \) such that \( (S/\rho, \circ, \preceq) \) is an ordered semigroup is not unique in general, we have the pseudoorder \( \sigma \) containing \( \rho \) such that \( \rho = \sigma \cap \sigma^{-1} \) is not unique.

(2) If \( \sigma \) is a pseudoorder on an ordered semihypergroup \( S \), then \( \rho = \sigma \cap \sigma^{-1} \) is the greatest strong order-congruence on \( S \) contained in \( \sigma \). In fact, if \( \rho_1 \) is a strong order-congruence on \( S \) contained in \( \sigma \), then \( \rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho \).

**Theorem 3.7** Let \( \rho \) be a strong order-congruence on an ordered semihypergroup \( (S, *, \leq) \). Then the least pseudoorder \( \sigma \) containing \( \rho \) is the transitive closure of relations \( \leq \circ \rho \) (resp. \( \rho \circ \leq \)), that is,

\[
\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n = \bigcup_{n=1}^{\infty} (\rho \circ \leq)^n.
\]
Proof

(1) Let \( \sigma_1 = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n \). Clearly, \( \rho \leq \circ \rho \leq \sigma_1 \). Similarly, since \( \leq \subseteq \leq \circ \rho \), we have \( \leq \subseteq \sigma_1 \).

(2) If \((a, b) \in \sigma_1\), \((b, c) \in \sigma_1\), then there exist \(m, n \in \mathbb{Z}^+\) such that \((a, b) \in (\leq \circ \rho)^m\) and \((b, c) \in (\leq \circ \rho)^n\), where \(\mathbb{Z}^+\) denotes the set of positive integers. Thus \((a, c) \in (\leq \circ \rho)^{m+n} \subseteq \sigma_1\), i.e. \(\sigma_1\) is transitive.

(3) Let \((a, b) \in \sigma_1\) and \(c \in S\). Then there exists \(n \in \mathbb{Z}^+\) such that \((a, b) \in (\leq \circ \rho)^n\), that is, there exist \(a_1, b_1, a_2, b_2, \ldots, a_n \in S\) such that

\[
a \leq a_1 \circ b_1 \leq a_2 \circ b_2 \leq \cdots \leq a_n \circ b_n.
\]

Since \((S, \ast, \leq)\) is an ordered semihypergroup and \(\rho\) is a strong congruence on \(S\), we have

\[
a \ast c \leq a_1 \ast c \circ b_1 \ast c \leq a_2 \ast c \circ b_2 \ast c \leq \cdots \leq a_n \ast c \circ b_n \ast c.
\]

Then, for any \(x \in a \ast c, y \in b \ast c\), there exist \(x_i \in a_i \ast c (i = 1, 2, \ldots, n), y_j \in b_j \ast c (j = 1, 2, \ldots, n-1)\) such that

\[
x \leq x_1 \circ y_1 \leq x_2 \circ y_2 \leq \cdots \leq x_n \circ y_n.
\]

It thus implies that \((x, y) \in (\leq \circ \rho)^n \subseteq \sigma_1\), and we obtain that \(a \ast c \circ \bar{\rho} \ast c \). Similar to the above way, it can be shown that \(c \ast a \circ \bar{\rho} \ast c \ast b\). Thus \(\bigcup_{n=1}^{\infty} (\leq \circ \rho)^n\) is a pseudoorder on \(S\) containing \(\rho\).

Furthermore, since \(\sigma\) is transitive, and \(\rho \leq \sigma, \leq \subseteq \sigma\), we have \(\bigcup_{n=1}^{\infty} (\leq \circ \rho)^n \subseteq \sigma\). Thus, by hypothesis, \(\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n\). In the same way, we can conclude that \(\sigma = \bigcup_{n=1}^{\infty} (\rho \circ \leq)^n\). \(\square\)

Let \(\sigma\) be a pseudoorder on an ordered semihypergroup \((S, \ast, \leq)\). Then, by Theorem 3.5, \(\rho = \sigma \cap \sigma^{-1}\) is a strong order-congruence on \(S\). We denote by \(\rho^\ast\) the canonical epimorphism from \(S/\rho\) onto \(S/\rho\), i.e. \(\rho^\ast : S \rightarrow S/\rho | x \mapsto (x)_\rho\), which is a strong homomorphism. In the following, we give out a homomorphism theorem of ordered semihypergroups by pseudoorders, which is a generalization of Theorem 1 in [14]. In fact, in Theorem 3.8, if our ordered semihypergroup is an ordered semigroup, i.e. the hyperoperation is an ordinary binary operation, we shall obtain Theorem 1 in [14].

**Theorem 3.8** Let \((S, \ast, \leq)\) and \((T, \circ, \leq)\) be two ordered semihypergroups, \(\varphi : S \rightarrow T\) a strong homomorphism.

Then: If \(\sigma\) is a pseudoorder on \(S\) such that \(\sigma \subseteq \ker \varphi\), then there exists the unique strong homomorphism \(f : S/\rho \rightarrow T | (a)_\rho \mapsto \varphi(a)\) such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi} & T \\
\rho^\ast \downarrow & & \downarrow f \\
S/\rho & & 
\end{array}
\]

commutes, where \(\rho = \sigma \cap \sigma^{-1}\). Moreover, \(\operatorname{Im}(\varphi) = \operatorname{Im}(f)\). Conversely, if \(\sigma\) is a pseudoorder on \(S\) for which there exists a strong homomorphism \(f : (S/\rho, \circ, \leq_\sigma) \rightarrow (T, \circ, \leq) \ (\rho = \sigma \cap \sigma^{-1})\) such that the above diagram commutes, then \(\sigma \subseteq \ker \varphi\).
**Proof** Let \( \sigma \) be a pseudoorder on \( S \) such that \( \sigma \subseteq \overrightarrow{ker\varphi} \), \( f : S/\rho \to T \mid (a)_\rho \mapsto \varphi(a) \). Then

1. \( f \) is well defined. Indeed, if \( (a)_\rho = (b)_\rho \), then \( (a, b) \in \rho \subseteq \sigma \). Since \( \sigma \subseteq \overrightarrow{ker\varphi} \), we have \( (\varphi(a), \varphi(b)) \in \preceq \).

Furthermore, since \( (b, a) \in \sigma \subseteq \overrightarrow{ker\varphi} \), we have \( (\varphi(b), \varphi(a)) \in \preceq \). Therefore, \( \varphi(a) = \varphi(b) \).

2. \( f \) is a strong homomorphism and \( \varphi = f \circ \rho^\sharp \). In fact: By Lemma 3.4, there exist an order relation "\( \preceq_\rho \)" on the factor semigroup \( (S/\rho, \otimes) \) such that \( (S/\rho, \otimes, \preceq_\rho) \) is an ordered semigroup and the canonical epimorphism \( \rho^\sharp \) is a strong homomorphism. Moreover, we have

\[
(a)_\rho \preceq_\sigma (b)_\rho \implies (a, b) \in \sigma \subseteq \overrightarrow{ker\varphi} \implies \varphi(a) \preceq \varphi(b) \implies f((a)_\rho) \preceq f((b)_\rho).
\]

Also, let \( (a)_\rho, (b)_\rho \in S/\rho \). For any \( (c)_\rho \in (a)_\rho \otimes (b)_\rho \), we have \( c \in a \star b \). Since \( \varphi \) is a strong homomorphism from \( S \) to \( T \), we have

\[
f((a)_\rho) \circ f((b)_\rho) = \varphi(a) \circ \varphi(b) = \varphi(c) = f((c)_\rho).
\]

Furthermore, for any \( a \in S \), \( (f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a) \), and thus \( \varphi = f \circ \rho^\sharp \).

We claim that \( f \) is a unique strong homomorphism from \( S/\rho \) to \( T \). To prove our claim, let \( g \) be a strong homomorphism from \( S/\rho \) to \( T \) such that \( \varphi = g \circ \rho^\sharp \). Then

\[
f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).
\]

Moreover, \( \text{Im}(f) = \{f((a)_\rho) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = \text{Im}(\varphi) \).

Conversely, let \( \sigma \) be a pseudoorder on \( S \), \( f : S/\rho \to T \) is a strong homomorphism, and \( \varphi = f \circ \rho^\sharp \). Then \( \sigma \subseteq \overrightarrow{ker\varphi} \). Indeed, by hypothesis, we have

\[
(a, b) \in \sigma \iff (a)_\rho \preceq_\sigma (b)_\rho \implies f((a)_\rho) \preceq f((b)_\rho) \implies (f \circ \rho^\sharp)(a) \preceq (f \circ \rho^\sharp)(b) \implies \varphi(a) \preceq \varphi(b) \implies (a, b) \in \overrightarrow{ker\varphi},
\]

where the order \( \preceq_\sigma \) on \( S/\rho \) is defined as in the proof of Lemma 3.4, that is

\[
\preceq_\sigma := \{(x)_\rho, (y)_\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.
\]

\[\square\]

**Corollary 3.9** Let \((S, \ast, \preceq)\) and \((T, \circ, \preceq)\) be two ordered semihypergroups and \( \varphi : S \to T \) a strong homomorphism. Then \( S/\ker\varphi \cong \text{Im}(\varphi) \), where \( \ker\varphi \) is the kernel of \( \varphi \).

**Proof** Let \( \sigma = \overrightarrow{ker\varphi} \) and \( \rho = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1} \). Then, by Theorems 3.5 and 3.8, \( \rho \) is a strong order-congruence on \( S \) and \( f : S/\rho \to T \mid (a)_\rho \mapsto \varphi(a) \) is a strong homomorphism. Moreover, \( f \) is inverse isotonous. In fact, let \( (a)_\rho, (b)_\rho \) be two elements of \( S/\rho \) such that \( f((a)_\rho) \preceq f((b)_\rho) \). Then \( \varphi(a) \preceq \varphi(b) \), and we have \( (a, b) \in \overrightarrow{ker\varphi} \). Thus, by Lemma 3.4, \((a)_\rho, (b)_\rho) \in \preceq_\sigma \), i.e. \((a)_\rho \preceq_\sigma (b)_\rho \). Clearly, \( \rho = \ker\varphi \). By Remark 2.6(1), \( S/\ker\varphi \cong \text{Im}(f) \). Also, by Theorem 3.8, \( \text{Im}(f) = \text{Im}(\varphi) \). Therefore, \( S/\ker\varphi \cong \text{Im}(\varphi) \). \(\square\)

Note that if \( S \) and \( T \) are both ordered semigroups, then Corollary 3.9 coincides with Corollary in [14].
4. Characterizations of (strong) order-congruences on ordered semihypergroups

In the above section, we have characterized the strong order-congruences by the properties of pseudoorders on an ordered semihypergroup. In the current section, we shall give out other characterizations of (strong) order-congruences on ordered semihypergroups. In order to prove the main results in this section, we first introduce the following concept.

**Definition 4.1.** Let \((S, *, \leq)\) be an ordered semihypergroup and \(\rho\) an equivalence relation on \(S\). A finite sequence of the form \((x, a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y)\) of elements in \(S\) is called a \(\rho\)-chain if

1. \((a_1, b_1) \in \rho, (a_2, b_2) \in \rho, \ldots, (a_{n-1}, b_{n-1}) \in \rho, (a_n, y) \in \rho;\)
2. \(x \leq a_1, b_1 \leq a_2, b_2 \leq a_3, \ldots, b_{n-2} \leq a_{n-1}, b_{n-1} \leq a_n.\)

Briefly we write

\[
x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho y.
\]

The number \(n\) is called the length, \(x\) and \(y\) initial and terminal elements, respectively, of the \(\rho\)-chain. A \(\rho\)-chain is called close if its initial and terminal elements are equal, i.e. \(x = y.\)

We denote by \(\rho^{C_{xy}}\) the set of all \(\rho\)-chains with \(x\) as the initial and \(y\) as the terminal elements in the sequel.

**Lemma 4.2.** Let \((S, *, \leq)\) be an ordered semihypergroup and \(\rho\) a congruence on \(S\). Then the following statements are true:

1. \((x, y) \in (\leq \circ \rho)^n\) if and only if there exists a \(\rho\)-chain with length \(n\) in \(\rho^{C_{xy}}\), i.e. \(\rho^{C_{xy}} \neq \emptyset.\)
2. For any \(z \in S\), if \(\rho^{C_{xy}} \neq \emptyset\) for some \(x, y \in S\), then for every \(u \in x * z\), there exists \(v \in y * z\) such that \(\rho^{C_{uv}} \neq \emptyset.\)
3. For any \(z \in S\), if \(\rho^{C_{xy}} \neq \emptyset\) for some \(x, y \in S\), then for every \(u' \in z * x\), there exists \(v' \in z * y\) such that \(\rho^{C_{u'v'}} \neq \emptyset.\)

**Proof**

(1) The proof is straightforward by Definition 4.1 and we omit it.

(2) Let \((x, a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y) \in \rho^{C_{xy}}\) and \(z \in S\). Then

\[
x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho y.
\]

Since \((S, *, \leq)\) is an ordered semihypergroup and \(\rho\) is a congruence on \(S\), we have

\[
x * z \leq a_1 * z \rho b_1 * z \leq a_2 * z \rho b_2 * z \leq \cdots \leq a_{n-1} * z \rho b_{n-1} * z \leq a_n * z \rho y * z.
\]

Then, for any \(u \in x * z\), there exist \(x_i \in a_i * z (i = 1, 2, \ldots, n), y_j \in b_j * z (j = 1, 2, \ldots, n - 1), v \in y * z\) such that

\[
u \leq x_1 \rho y_1 \leq x_2 \rho y_2 \leq \cdots \leq x_{n-1} \rho y_{n-1} \leq x_n \rho v.
\]

It thus implies that \((u, x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}, x_n, v) \in \rho^{C_{uv}}\), i.e. \(\rho^{C_{uv}} \neq \emptyset.\)

(3) It is similar to that of (2) and we omit it.  

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Lemma 4.3 Let \((S, \ast, \leq)\) be an ordered semihypergroup and \(\rho\) a strong congruence on \(S\). If \(\rho^{C_{xy}} \neq \emptyset\) for some \(x, y \in S\), then, for any \(z \in S\), we have \(\rho^{C_{zv}} \neq \emptyset\) and \(\rho^{C_{wy'}} \neq \emptyset\) for every \(u \in x \ast z, v \in y \ast z, u' \in z \ast x, v' \in z \ast y\).

**Proof** The proof is similar to that of Lemma 4.2 with a slight modification. \(\square\)

Lemma 4.4 Let \(S\) be an ordered semihypergroup and \(\rho\) a (strong) congruence on \(S\). If \((x, y) \in \rho\), \((k, z) \in \rho\), then \(\rho^{C_{zk}} \neq \emptyset\) if and only if \(\rho^{C_{xy}} \neq \emptyset\).

**Proof** \((\Rightarrow)\). If \(\rho^{C_{zk}} \neq \emptyset\), by Lemma 4.2(1), there exists \(n \in \mathbb{Z}^+\) such that \((x, k) \in (\leq \rho)^n\). Since \((x, y) \in \rho\), \((z, k) \in \rho\), we have \[y \leq ypz(\leq \rho)^nk \leq kpz,\]
which implies that \((y, z) \in (\leq \rho)^{n+2}\). By Lemma 4.2(1), we have \(\rho^{C_{yz}} \neq \emptyset\). \(\square\)

\((\Leftarrow)\). Similar to the proof of necessity and we omit it.

Now we shall give a characterization of order-congruences on an ordered semihypergroup.

**Theorem 4.5** Let \((S, \ast, \leq)\) be an ordered semihypergroup and \(\rho\) a congruence on \(S\). Then \(\rho\) is an order-congruence on \(S\) if and only if every close \(\rho\)-chain is contained in a single equivalent class of \(\rho\).

**Proof** Let \(\rho\) be an order-congruence on \(S\). Then there exists an order \(\preceq\) on the factor semihypergroup \((S/\rho, \otimes)\) such that \((S/\rho, \otimes, \preceq)\) is an ordered semihypergroup and \(\varphi : S \to S/\rho\) is a homomorphism. For any \(x \in S\), and every close \(\rho\)-chain \((x, a_1, b_1, \ldots, a_n, x)\) in \(\rho^{C_{xy}}\), we have \[x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho x.\]

Then, \[\varphi(x) \preceq \varphi(a_1) = \varphi(b_1) \preceq \varphi(a_2) = \varphi(b_2) \preceq \cdots \preceq \varphi(a_n) = \varphi(x).\]

It implies that \(\varphi(x) = \varphi(a_1) = \varphi(b_1) = \varphi(a_2) = \varphi(b_2) = \cdots = \varphi(a_n)\). Consequently, \((x, a_1, b_1, \ldots, a_n, x)\) is contained in a single \(\rho\)-class.

Conversely, since \(\rho\) is a congruence on \(S\), by Lemma 2.2, \((S/\rho, \otimes)\) is a semihypergroup. We define a relation “\(\preceq\)” on the factor semihypergroup \((S/\rho, \otimes)\) as follows:
\[\preceq := \{(x)_\rho(, y)_\rho \mid \rho^{C_{xy}} \neq \emptyset\}.\]

\((1)\) \(\preceq\) is well defined. In fact, let \(x_1, y_1 \in S\) be such that \((x)_\rho = (x_1)_\rho\), \((y)_\rho = (y_1)_\rho\). If \((x)_\rho \preceq (y)_\rho\), then \(\rho^{C_{xy}} \neq \emptyset\). By Lemma 4.4, we have \(\rho^{C_{x_1y_1}} \neq \emptyset\), and \((x_1)_\rho \preceq (y_1)_\rho\).

\((2)\) \(\preceq\) is an ordered relation on \(S/\rho\).

\((\alpha)\) \(\preceq\) is reflexive. In fact, since for any \(x \in S\), \(x \leq x \rho x\), and we have \(\rho^{C_{xx}} \neq \emptyset\), i.e. \(((x)_\rho, (x)_\rho) \in \preceq\).

\((\beta)\) \(\preceq\) is transitive. Indeed, let \(((x)_\rho, (y)_\rho) \in \preceq\), \(((y)_\rho, (z)_\rho) \in \preceq\). Then we have \(\rho^{C_{yz}} \neq \emptyset\). By Lemma 4.2(1), there exist \(m, n \in \mathbb{Z}^+\) such that \((x, y) \in (\leq \rho)^m, (y, z) \in (\leq \rho)^n\). Then we have \[(x, z) \in (\leq \rho)^m \circ (\leq \rho)^n = (\leq \rho)^{m+n},\]
i.e. \(\rho^{C_{xz}} \neq \emptyset\). Thus \(((x)_\rho, (z)_\rho) \in \preceq\).
(1) \( \leq \) is anti-symmetric. In fact, if \(((x)_\rho, (y)_\rho) \leq,(y)_\rho, (x)_\rho) \leq,\) then \(\rho^{xy} \neq \emptyset, \rho^{yx} \neq \emptyset.\) Similar to the above proof, it can be obtained that \(\rho^{xy} \neq \emptyset, i.e.\) there exists a close \(\rho\)-chain in \(\rho C_{xx}\) containing \(x\) and \(y.\) By hypothesis, \((x)_\rho = (y)_\rho.\)

(3) \(\langle S/\rho, \emptyset, \leq \rangle\) is an ordered semihypergroup. Indeed, let \((x)_\rho \leq (y)_\rho\) and \((z)_\rho \in S/\rho.\) Then \(\rho^{xy} \neq \emptyset.\) By Lemma 4.2(2), for every \(u \in x * z,\) there exists \(v \in y * z\) such that \(\rho^{uv} \neq \emptyset, i.e.\) \((u)_\rho \leq (v)_\rho.\) Thus

\[
(x)_\rho \otimes (z)_\rho = \bigcup_{u \in x * z} (u)_\rho \leq \bigcup_{v \in y * z} (v)_\rho = (y)_\rho \otimes (z)_\rho.
\]

Similarly, it can be shown that \((z)_\rho \otimes (x)_\rho \leq (z)_\rho \otimes (y)_\rho.\)

(4) The mapping \(\varphi : S \to S/\rho \mid x \mapsto (x)_\rho\) is isotone. In fact, let \(x, y \in S\) be such that \(x \leq y.\) Then \((x, y) \leq \circ \rho,\) we have \(\rho^{xy} \neq \emptyset, i.e.\) \((x)_\rho \leq (y)_\rho.\)

Therefore, \(\rho\) is an order-congruence on \(S.\)

Similarly, strong order-congruences on an ordered semihypergroup can be characterized as follows:

**Theorem 4.6** Let \((S, \ast, \leq)\) be an ordered semihypergroup and \(\rho\) a strong congruence on \(S.\) Then \(\rho\) is a strong order-congruence on \(S\) if and only if every close \(\rho\)-chain is contained in a single equivalent class of \(\rho.\)

**Proof** The proof is similar to that of Theorem 4.5 with suitable modification by using Lemma 4.3. □

By Theorem 4.5, we immediately obtain the following corollary:

**Corollary 4.7** If \(\rho\) is an order-congruence on an ordered semihypergroup \(S,\) then every \(\rho\)-class in \(S\) is convex.

**Proof** Let \(\rho\) be an order-congruence on \(S\) and \(I\) a congruence class of \(\rho.\) If \(x \leq y \leq z\) and \(x, z \in I,\) then \((x)_\rho = (z)_\rho.\) Thus we have \(x \leq y \leq z \rho x.\) Hence \((x, y, z, x)\) is a close \(\rho\)-chain and by Theorem 4.5 we have \((x)_\rho = (y)_\rho = (z)_\rho.\) It thus follows that \(y \in I,\) and \(I\) is convex. □

Furthermore, we have the following theorem:

**Theorem 4.8** Let \((S, \ast, \leq)\) be an ordered semihypergroup and \(I\) a hyperideal of \(S.\) Then \(I\) is a congruence class of one order-congruence on \(S\) if and only if \(I\) is convex.

**Proof** The proof is straightforward by Corollary 4.7.

Conversely, let \(\rho_I\) be the Rees congruence induced by \(I\) on \(S.\) By Remark 2.4(1), \(I\) is a congruence class of \(\rho_I.\) Now we define a relation \(\leq_I\) on the factor semihypergroup \((S/\rho_I, \otimes)\) as follows:

\[
(x)_{\rho_I} \leq_I (y)_{\rho_I} \iff (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).
\]

We claim that \(\rho_I\) is an order-congruence on \(S.\) To prove our claim, we first show that \(\leq_I\) is order relation on \(S/\rho_I, i.e.\) \(\leq_I\) is reflexive, anti-symmetric, and transitive.

(1) Let \((x)_{\rho_I}\) be any element of \(S/\rho_I.\) Then, since \(x \leq x,\) we have \((x)_{\rho_I} \leq_I (x)_{\rho_I}.\)
(2) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (x)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq x$ or $y \leq b, b' \leq x$ for some $b, b' \in I$. We consider the following four cases:

Case 1. If $x \leq y$ and $y \leq x$, then $x = y$, and thus $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq x$ for some $b, b' \in I$, then $b' \leq x \leq y \leq b$. Since $I$ is convex and $b, b' \in I$, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I} = I$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq x$. Similar to the proof of Case 2, we have $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq x$ for some $b, b' \in I$. Then $b' \leq x \leq a$ and $a' \leq y \leq b$. Since $I$ is convex, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I}$.

(3) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (z)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq z$ or $y \leq b, b' \leq z$ for some $b, b' \in I$. There are four cases to be considered:

Case 1. If $x \leq y$ and $y \leq z$, then $x \leq z$, and thus $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq z$ for some $b, b' \in I$, then $x \leq y \leq b$ and $b' \leq z$. By the definition of $\preceq_I$, $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq z$. Analogous to the proof of Case 2, we have $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq z$ for some $b, b' \in I$. Then $x \leq a$ and $b' \leq z$. Hence $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Now we show that $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semi hypergroup. Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $z \in S$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$. We consider the following two cases:

Case 1. If $x \leq y$, then $x * z \leq y * z$. Thus for every $u \in x * z$, there exists $v \in y * z$ such that $u \leq v$, and we have $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$. Thus

$$
(x)_{\rho_I} \otimes_I (z)_{\rho_I} = \bigcup_{u \in x * z} (u)_{\rho_I} \preceq_I \bigcup_{v \in y * z} (v)_{\rho_I} = (y)_{\rho_I} \otimes_I (z)_{\rho_I}.
$$

Case 2. Let $x \leq a, a' \leq y$ for some $a, a' \in I$. Then $x * z \leq a * z, a' * z \leq y * z$. Thus for every $u \in x * z$, there exists $b \in a * z$ such that $u \leq b$, and for some $b' \in a' * z$ there exists $v \in y * z$ such that $b' \leq v$. Since $I$ is a hyperideal of $S$ and $a, a' \in I$, we have $b \in a * z \subseteq I, b' \in a' * z \subseteq I$. On the other hand, $u \leq b, b' \leq v$ for some $b, b' \in I$. Hence $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$, and thus $(x)_{\rho_I} \otimes_I (z)_{\rho_I} \preceq_I (y)_{\rho_I} \otimes_I (z)_{\rho_I}$.

Similar to the above way, we can show that $(z)_{\rho_I} \otimes_I (x)_{\rho_I} \preceq_I (z)_{\rho_I} \otimes_I (y)_{\rho_I}$. Therefore, $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semi hyper group.

Furthermore, by the definition of $\preceq_I$, it can be obtained that the canonical epimorphism $\varphi : S \rightarrow S/\rho_I, x \mapsto (x)_{\rho_I}$ is isotone. Thus $\rho_I$ is an order-congruence on $S$. The proof is completed.

By the proof of above theorem, we immediately obtain the following corollary:

**Corollary 4.9** Let $(S, \ast, \preceq)$ be an ordered semi hypergroup and $I$ an ordered hyperideal of $S$. Then $(S/\rho_I, \otimes_I, \preceq_I)$ forms an ordered semi hypergroup and the Rees congruence $\rho_I$ induced by $I$ on $S$ is an order-congruence, where the order relation “$\preceq_I$” on $S/\rho_I$ is defined as follows:

$$(x)_{\rho_I} \preceq_I (y)_{\rho_I} \iff (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).$$
As an application of Corollary 4.9, we can give an answer to the open problem given by Davvaz et al. in [7].

**Open Problem** Is there a congruence relation (also called regular relation in [7]) $\rho$ on an ordered semihypergroup $(S, \ast, \leq)$ for which $S/\rho$ is an ordered semihypergroup?

To solve the above problem, we need only show that $I$ is not a strong congruence on $S$ in general. We illustrate it by the following example.

**Example 4.9** We consider a set $S := \{a, b, c, d, e, f\}$ with the following hyperoperation “$\ast$” and the order “$\leq$”:

$$
\begin{array}{c|cccccc}
* & a & b & c & d & e & f \\
\hline
a & \{a\} & \{a, b\} & \{c\} & \{c, d\} & \{e\} & \{e, f\} \\
b & \{b\} & \{b\} & \{d\} & \{d\} & \{f\} & \{f\} \\
c & \{c\} & \{c, d\} & \{c\} & \{c, d\} & \{c, d\} & \{c, d\} \\
d & \{d\} & \{d\} & \{d\} & \{d\} & \{d\} & \{d\} \\
e & \{e\} & \{e, f\} & \{e\} & \{e\} & \{e, f\} & \{e, f\} \\
f & \{f\} & \{f\} & \{f\} & \{f\} & \{f\} & \{f\} \\
\end{array}
$$

$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (f, f)\}$. We give the covering relation “$\prec$” and the figure of $S$ as follows:

$\prec := \{(a, b), (c, d), (e, f)\}$.

```
    b   d   f
   /   \\/
  a   c    e
```

Then $(S, \circ, \leq)$ is an ordered semihypergroup. Let $I = \{c, d\}$. One can easily verify that $I$ is an ordered hyperideal of $S$. Then $\rho_I = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (c, d), (d, c)\}$. By Lemma 2.3, $\rho_I$ is a congruence on $S$. However, we claim that $\rho_I$ is not a strong congruence on $S$. In fact, since $(e, e) \in \rho_I$, while $e \ast b \not{\rho_I} e \ast b$ does not hold.

As a generalization of Proposition 2.7 in [23], we have the following theorem. The following theorem can be proved using similar techniques as in the proof of Theorem 4.8.

**Theorem 4.10** Let $(S, \ast, \leq)$ be an ordered semihypergroup and $I$ an ordered hyperideal of $S$. We define a relation “$\leq_1$” on $S/\rho_I$ ($=$ $\{\{x\} \mid x \in S \setminus I \cup \{I\}\}$) as follows:

$$
\leq_1 := \{\{I, \{x\}\} \mid x \in S \setminus I \cup \{\{x\}\} \mid x, y \in S \setminus I, x \leq y\} \cup \{\{I, I\}\}.
$$

Then $(S/\rho_I, \ominus_I, \leq_1)$ is an ordered semihypergroup, and $\rho_I$ is an order-congruence on $S$.

**Proposition 4.11** Let $(S, \ast, \leq)$ be an ordered semihypergroup and $I$ an ordered hyperideal of $S$. Then the order relations in Corollary 4.9 and Theorem 4.11 are different. Moreover, $\leq_1 \subseteq \leq_1$. 

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Proof Let \((x)_{\rho_I}, (y)_{\rho_I} \in S/\rho_I\) and \((x)_{\rho_I} \preceq_I (y)_{\rho_I}\). Then \(x \preceq y\) or \(x \leq a, a' \leq y\) for some \(a, a' \in I\). Since \((I) = I\), we have \(x \preceq y\) or \(x \in I\) and \(a' \leq y\) for some \(a' \in I\). The first case implies \((x)_{\rho_I} \preceq_I (y)_{\rho_I}\), and the second case implies \((x)_{\rho_I} \preceq_I (a')_{\rho_I} \preceq_I (y)_{\rho_I}\), i.e. \((x)_{\rho_I} \preceq_I (y)_{\rho_I}\). Thus \(\preceq_I \preceq \preceq_I\).

The following example shows that \(\preceq_I \preceq \preceq_I\) in general.

Example 4.12 We consider a set \(S := \{a, b, c, d\}\) with the following hyperoperation \(*\) and the order \(\preceq\):

\[
\begin{array}{cccc}
* & a & b & c & d \\
\hline
a & \{a, d\} & \{a, d\} & \{a, d\} & \{a\} \\
b & \{a, d\} & \{b\} & \{a, d\} & \{a, d\} \\
c & \{a, d\} & \{a, d\} & \{c\} & \{a, d\} \\
d & \{a\} & \{a, d\} & \{a, d\} & \{d\}
\end{array}
\]

\(\preceq := \{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}\).

We give the covering relation \(\prec\) and the figure of \(S\) as follows:

\[
\begin{array}{ccc}
\prec & \{(a, c), (d, c)\} \\
\hline
a & \hspace{1cm} & b \\
\hspace{1cm} & \hspace{1cm} & c \\
\hspace{1cm} & \hspace{1cm} & \hspace{1cm} & d
\end{array}
\]

Then \((S, *, \preceq)\) is an ordered semihypergroup. It is easy to check that \(I = \{a, d\}\) is an ordered hyperideal of \(S\). Since \(a \preceq b\) and there does not exist \(x \in I\) such that \(x \preceq b\), we have \((a)_{\rho_I} \not\preceq_I (b)_{\rho_I}\). However, by the definition of \(\preceq_I\), we have \((a)_{\rho_I} \preceq_I (b)_{\rho_I}\).

In the following we shall consider the strong order-congruence generated by a strong congruence on an ordered semihypergroup.

Definition 4.13 Let \(\rho\) be a strong congruence on an ordered semihypergroup \(S\). A strong order-congruence \(\sigma\) is called the strong order-congruence generated by \(\rho\) on \(S\), if \(\sigma\) satisfies the following conditions:

(1) \(\rho \subseteq \sigma\).

(2) If there is a strong order-congruence \(\eta\) on \(S\) such that \(\rho \subseteq \eta\), then \(\sigma \subseteq \eta\).

Theorem 4.14 Let \(\rho\) be a strong congruence on an ordered semihypergroup \((S, *, \preceq)\). Then

(1) If we define a relation \(\preceq\) on \(S\) as follows:

\[
(x, y \in S) \quad (x, y) \in \rho \text{ if and only if } \rho^C_{xy} \neq \emptyset,
\]

then \(\rho\) is a pseudoorder on \(S\).

(2) \(R_\rho\) is a relation on \(S\) defined as follows:

\[
(x, y \in S) \quad (x, y) \in R_\rho \iff (x, y) \in \rho \text{ and } (y, x) \in \rho.
\]

Then \(R_\rho\) is the strong order-congruence generated by \(\rho\) on \(S\).
Proof

(1) Let \( x, y \in S \) such that \( x \leq y \). Then there is a \( \rho \)-chain from \( x \) to \( y \): \( (x, y, y) \), i.e. \( \rho^{\curvearrowright} \neq \emptyset \). Thus \( x \leq y \) implies \( x \rho y \), and we have \( \leq \rho \). Suppose that \( (x, y) \in \rho \) and \( (y, z) \in \rho \). Then there exist \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_m \in S \) such that

\[
\begin{align*}
x &\leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho y, \\
y &\leq c_1 \rho d_1 \leq c_2 \rho d_2 \leq \cdots \leq c_{m-1} \rho d_{m-1} \leq d_m \rho z.
\end{align*}
\]

Thus, \( x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho y \leq c_1 \rho d_1 \leq c_2 \rho d_2 \leq \cdots \leq c_{m-1} \rho d_{m-1} \leq d_m \rho z \), which is a \( \rho \)-chain from \( x \) to \( z \). Hence \( (x, z) \in \rho \) and \( \rho \) is transitive. Furthermore, let \( (x, y) \in \rho \) and \( z \in S \). Then \( \rho^{\curvearrowright} \neq \emptyset \). By Lemma 4.3, for every \( u \in x \ast z, v \in y \ast z, v' \in z \ast x, v' \in z \ast y \), we have \( \rho^{\curvearrowright} \neq \emptyset \) and \( \rho^{\curvearrowright} \neq \emptyset \), which imply that \( (u, v) \in \rho \) and \( (u', v') \in \rho \). It thus follows that \( x \ast z \rho y \ast z \) and \( z \ast x \rho z \ast y \). Therefore, \( \rho \) is a pseudoorder on \( S \).

(2) By (1), \( \rho \) is a pseudoorder on \( S \). Since \( R_{\rho} = \rho \cap \rho^{-1} \), by Proposition 3.3 \( R_{\rho} \) is a strong order-congruence on \( S \). We claim that \( R_{\rho} \) is the strong order-congruence generated by \( \rho \) on \( S \). To prove our claim, let \( (x, y) \in \rho \). Since \( \rho \) is a strong congruence on \( S \), we have \( (y, x) \in \rho \). Consequently, \( (x, y) \in R_{\rho} \). Hence \( \rho \subseteq R_{\rho} \). Furthermore, suppose that \( \eta \) is a strong order-congruence on \( S \) and \( \rho \subseteq \eta \). Then \( R_{\rho} \subseteq \eta \). Indeed, let \( (x, y) \in R_{\rho} \). Then \( (x, y) \in \rho \) and \( (y, x) \in \rho \). By definition of \( \rho \), there exist \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_{m-1} \in S \) such that

\[
\begin{align*}
x &\leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho y, \\
y &\leq c_1 \rho d_1 \leq c_2 \rho d_2 \leq \cdots \leq c_{m-1} \rho d_{m-1} \leq d_m \rho x.
\end{align*}
\]

Thus, by \( \rho \subseteq \eta \), we have \( x \leq a_1 \eta b_1 \leq a_2 \eta b_2 \leq \cdots \leq a_{n-1} \eta b_{n-1} \leq a_n \eta y \leq c_1 \eta d_1 \leq c_2 \eta d_2 \leq \cdots \leq c_{m-1} \eta d_{m-1} \leq d_m \eta x \). Since \( \eta \) is a strong order-congruence on \( S \), by Theorem 4.6 we can conclude that the closed \( \eta \)-chain \( \langle x, a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y, c_1, d_1, c_2, d_2, \ldots, c_{m-1}, d_{m-1}, d_m, x \rangle \) is contained in a single equivalence class of \( \eta \). In particular, we have \( (x, y) \in \eta \). Therefore, \( R_{\rho} \) is the strong order-congruence generated by \( \rho \) on \( S \).

By Theorem 4.15, we immediately obtain the following corollary:

**Corollary 4.15** Every strong congruence of an ordered semihypergroup \( S \) is contained in a strong order-congruence of \( S \).

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