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Hyperplanes, parallelism, and related problems in Veronese spaces

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Abstract: We determine hyperplanes in Veronese spaces associated with projective spaces and polar spaces, and we analyze the geometry of parallelisms induced by these hyperplanes. We also discuss whether or not parallelisms on Veronese spaces associated with affine spaces can be imposed.

Key words: Veronese space, projective space, affine space, polar space, partial linear space, affine partial linear space, hyperplane, correlation

1. Introduction

The term 'Veronese space' refers, primarily (and historically), to the structure of prisms in a projective space with 'double hyperplanes' as the points (see [6, 15]); after that it refers to an algebraic variety that represents this structure (cf., e.g., [1]) and as such it was generalized in recent decades and its geometry was studied and developed (see, e.g., [2, 12, 13, 16]).

A task to find a synthetic approach to ('original') Veronese spaces was undertaken by the group of geometers around Tallini in the 1970s and the results are presented in [3, 5, 6, 14]. The way in which Veronese spaces were presented in those papers, where the points are two-element sets with repetitions with the elements being projective points, turned out to be fruitfully generalizable to an abstract construction, which associates with an arbitrary partial linear space (or even more generally: with an arbitrary incidence structure) another partial linear space (incidence structure, resp.); this construction and its basic properties were presented in [9]. The point universe of the constructed Veronese space $\mathbf{V}_k(\mathfrak{M})$ consists of the k-element sets with repetitions with the elements in the universe of the underlying 'starting' structure $\mathfrak{M}$ (cf. definitions (2) and (1)).

The construction discussed belongs to the family of, informally speaking, 'multiplying' a given structure somehow in the spirit of 'manifold theory': the structure $\mathbf{V}_k(\mathfrak{M})$ can be covered by a family of copies of $\mathfrak{M}$ so as through each point of it there pass k copies of $\mathfrak{M}$. Any two distinct copies of $\mathfrak{M}$ in this covering do not share more than a single point. The abstract schemas (in fact: partial linear spaces of particular type) of the coverings induced by the construction of a Veronese space were studied in more detail in [11].

In this paper we develop to some extent the theory of Veronese spaces associated with spaces with some additional structures (like a parallelism) and we discuss possibilities to introduce parallelism in the Veronese spaces and related questions, all in a sense referring to a broad problem: "hyperplanes and parallelisms". The main approach is determining a hyperplane in a Veronese space associated with a linear space that contains

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hyperplanes (say: with a projective space) and deleting this hyperplane. The obtained reduct cannot be presented as a Veronese space (3.19). Nevertheless we could succeed in characterizing the hyperplanes in ('classical') Veronese spaces associated with projective spaces (3.6) and prove that (similarly to the case when a hyperplane is deleted from a projective space) from the obtained partially affine partial linear space the underlying Veronese space can be recovered (3.17). We find also interesting examples of hyperplanes in Veronese spaces associated with polar spaces and we generalize our construction to this class also. This completes the set of our main results.

In the Appendix we discuss another approach: defining the Veronese space associated with a structure with a parallelism (say: with an affine space). There is no simple way to introduce a parallelism on the defined 'Veronese product' (4.2). In both approaches we obtain a structure that can be covered by several copies of an affine space. At the end of this paper we formulate several open problems. They are out of the scope of our main reasoning but they seem closely related to it.

2. Basics

2.1. Incidence structures, partial linear spaces

In the paper we consider incidence structures, i.e. structures of the form $\mathfrak{W} = \langle S, \mathcal{B}, \mid \rangle$, where $\mid \subseteq S \times \mathcal{B}$ and $\mathcal{B} \neq \emptyset$.

In the context of (general) incidence structures the binary joinability relation (adjacency relation) $\sim = \sim_{\mathcal{B}}$ defined over an incidence structure $\langle S, \mathcal{B}, \mid \rangle$ plays a fundamental role:

$$a \sim b \iff (\exists B \in \mathcal{B})[a, b \mid B].$$

An incidence structure is *connected* when the relation (nonoriented graph) $\sim$ is connected. For blocks L, M the formula $L \sim M$ means that they intersect each other.

An incidence structure as above is *a partial linear space* (a PLS) if any two blocks that are $\mid$-related to two distinct points coincide, and any block is on at least three points (most of the time, in the literature it is assumed that a block of a PLS has at least two points, but in this paper we need a stronger requirement). In that case the incidence structure $\mathfrak{W} = \langle S, \mathcal{B}, \mid \rangle$ is isomorphic to the structure $\langle S, \mathcal{B}^!, \in \rangle$, where $\mathcal{B}^! = \big\{\{x \in S \colon x \mid B\} \colon B \in \mathcal{B}\big\}$; this second approach to partial linear spaces will be preferred in this paper.

A *subspace* of a partial linear space is a set X of its points such that each line that meets X in at least two points is entirely contained in X. A subspace X is *strong* (or *singular*, or *linear*) if any two of its points are adjacent.

A *linear space* is a partial linear space in which any two points are adjacent. Two classes of linear spaces are of primary interest in geometry: affine and projective spaces, and structures associated with them are of primary interest in this paper.

2.2. Veronese spaces

Letting X be a nonempty set, we write $\mathfrak{y}_k(X)$ for the set of k-element sets with repetitions (multisets) with elements in X, i.e. the set of all the functions $f \colon X \longrightarrow \mathbb{N}$ such that $|f| := \sum_{x \in X} f(x) = k$. Then, directly by the definition, $f \in \mathfrak{y}_{|f|}(X)$. If $f \in \mathfrak{y}_k(X)$ then the set $\{x \in X \colon f(x) \neq 0\} =: \operatorname{supp}(f)$ is finite and $\sum_{x \in X} f(x) = \sum_{x \in \operatorname{supp}(f)} f(x)$; in such a case we write $f = \sum_{x \in \operatorname{supp}(f)} f(x) \cdot x$. Finally, set

$\mathfrak{w}_k(X) = \bigcup\{\mathfrak{y}_l(X) \colon l < k\}$. Let $\mathfrak{W} = \langle S, \mathcal{B}, \mid \rangle$ be an incidence structure and k be an integer. We set

$$\mathcal{B}^{\textcircled{k}} = \big\{\{e + rx \colon x \mid B\} \colon 0 < r \leq k,\ e \in \mathfrak{y}_{k-r}(S),\ B \in \mathcal{B}\big\}. \tag{1}$$

Then we define (cf. [9])

$$\mathbf{V}_k\mathfrak{W} \quad := \quad \langle \mathfrak{y}_k(S), \mathcal{B}^{\textcircled{k}}, \in \rangle. \tag{2}$$

The structure $\mathbf{V}_k\mathfrak{W}$ will be called the *(k th) Veronese space associated with* $\mathfrak{W}$ (or *over* $\mathfrak{W}$). We also say that it is a Veronese space *of level* k.

The following simple observations are quoted from [4] and [9].

Fact 2.1 *Let $\mathfrak{W} = \langle S, \mathcal{B}, \mid \rangle$ be an incidence structure. Assume that $\mathfrak{W}$ satisfies the extensionality principle:*

$$B_1, B_2 \in \mathcal{B} \wedge (\forall x)[x \mid B_1 \iff x \mid B_2] \implies B_1 = B_2. \tag{3}$$

Let k', r be nonnegative integers, $r \geq 1$. Fix $e \in \mathfrak{y}_{k'}(S)$ and define the maps

$$\mu_r \colon \mathfrak{y}_k(S) \ni f \quad \longmapsto \quad rf \in \mathfrak{y}_{rk}(S), \tag{4}$$

$$\tau_e \colon \mathfrak{y}_k(S) \ni f \quad \longmapsto \quad e + f \in \mathfrak{y}_{k+k'}(S). \tag{5}$$

Then μ_r is an embedding of $\mathbf{V}_k\mathfrak{W}$ into $\mathbf{V}_{rk}(\mathfrak{W})$ and τ_e is an embedding of $\mathbf{V}_k\mathfrak{W}$ into $\mathbf{V}_{k'+k}(\mathfrak{W})$.

Moreover, the identification

$$^* \colon S \ni x \longmapsto 1 \cdot x (= x^*) \in \mathfrak{y}_1(S) \tag{6}$$

is an isomorphism of $\mathfrak{W}$ and $\mathbf{V}_1(\mathfrak{W})$.

Fact 2.2 *The structure*

$$\mathbf{V}_k(S) := \mathbf{V}_k(\langle S, \{S\}, \in \rangle) \tag{7}$$

is a partial linear space.

The lines of $\mathbf{V}_k(S)$ will be called *leaves of* $\mathbf{V}_k\mathfrak{W}$.

The following is folklore (see [4, 9]):

Proposition 2.3 *A Veronese space associated with a partial linear space is a partial linear space.*

Fact 2.4 (see [9], Prop. 2.9) *Let B be a block of the Veronese space $\mathbf{V}_k\mathfrak{W}$ associated with an incidence structure $\mathfrak{W}$ and $\mathrm{T}(B)$ be the leaf of $\mathbf{V}_k\mathfrak{W}$, which contains B. If a point e is adjacent to at least three points on B then $e \in \mathrm{T}(B)$.*

Note that a leaf $e + rS$ of a Veronese space can be identified with $e \in \mathfrak{w}_k(S)$. Indeed, a leaf of $\mathbf{V}_k\mathfrak{W}$ uniquely associated with e has the form $e + (k - |e|)S$.

Each leaf $e+(k-|e|)S$ of $\mathbf{V}_k\mathfrak{W}$ is a subspace of $\mathbf{V}_k\mathfrak{W}$ isomorphic to $\mathfrak{W}$ under the map $x \longmapsto e+(k-|e|)\cdot x$.

2.3. Veblen and net configurations in Veronese spaces

A *Veblen configuration* consists of four lines, no three concurrent, and any two intersecting each the other. In a more explicit way we call *a Veblen configuration* a 4-tuple of lines L_1, L_2, M_1, M_2 and a point p such that $p \mid L_1, L_2$, $p \nmid M_1, M_2$, $L_i \sim M_j$ for $i, j = 1, 2$, and $M_1 \sim M_2$. An *incomplete Veblen configuration* is the family as above, where the condition $M_1 \sim M_2$ is not assumed.

A *net configuration* consists of four lines L_1, L_2, L_3, L_4 and four points p_1, p_2, p_3, p_4 such that $p_i \mid L_i, L_{i+1 \mod 4}$, and $p_i \not\sim p_{i+2 \mod 4}$ for $i = 1, 2, 3, 4$. In plain words, it is a quadrangle in which diagonals do not exist. In the literature the condition $L_i \not\sim L_{i+2 \mod 4}$ (= 'opposite sides do not intersect') is frequently added; in the configurations arising via affinizations such a requirement is too restrictive.

Together with these two configurations two configurational axioms are considered: the *Veblen condition* and the *net axiom*. The Veblen condition states that every incomplete Veblen configuration closes, i.e. if p, L_1, L_2, M_1, M_2 is an incomplete Veblen configuration defined above then $M_1 \sim M_2$. The net axiom states that if L_1, L_2, L_3, L_4 is a quadrangle defined above, $M_1 \sim L_1, L_3$, $M_2 \sim L_2, L_4$ then $M_1 \sim M_2$. A partial linear space that satisfies the Veblen condition is called *veblenian*. Explicit forms of the Veblen configurations contained in a Veronese space are shown in [9, Lem. 3.1]. Let us recall this characterization.

Fact 2.5 *Let $\mathfrak{M}_0$ be a partial linear space and $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$. A Veblen configuration contained in $\mathfrak{M}$ arises as a result of a natural embedding of one of the following figures:*

(i) *a Veblen configuration contained in $\mathfrak{M}_0$,*

(ii) *the set $\{a + m : a \in A\}$ for a 4-subset A of a line m of $\mathfrak{M}_0$,*

(iii) *the set $\{a + m : a \in A\} \cup \{2m\}$ for a 3-subset A of a line m of $\mathfrak{M}_0$.*

From this one derives, in particular, that the Veblen axiom is not, generally, preserved under Veronese products. Let $\kappa(\mathfrak{M}_0) = \kappa$ be the size of a line of $\mathfrak{M}_0$. Note that $\mathbf{V}_k(\mathfrak{M}_0)$ contains a Veblen subconfiguration of the form (ii) iff $\kappa \geq 4$ and it contains a Veblen subconfiguration of the form (iii) iff $\kappa \geq 3$. However, one particular case 'behaves' more regularly. With the classification of triangles in Veronese spaces given in [4, Fact 4.1] and standard computations we directly justify:

Fact 2.6 *If a partial linear space $\mathfrak{M}_0$ is veblenian then $\mathbf{V}_2(\mathfrak{M}_0)$ is veblenian as well.*

Combining 2.5 and 2.4, we obtain:

Lemma 2.7 *Let p, L_1, L_2, M_1, M_2 form an incomplete Veblen configuration in a Veronese space associated with a partial linear space $\mathfrak{M}_0$. Assume, moreover, that points in one of the following pairs, $(L_1 \cap M_1,\ L_2 \cap M_2)$ and $(L_1 \cap M_2,\ L_2 \cap M_1)$, are collinear. Then the lines L_1, L_2, M_1, M_2 are contained in a leaf, and so if $\mathfrak{M}_0$ is veblenian this configuration closes.*

Analogously, explicit forms of a realization of the net configuration in a Veronese space were also established in [9]. It follows that the net axiom is not, generally, preserved under Veronese 'products'. In the sequel we shall concentrate on a special case of Veronese spaces, namely on the structures $\mathbf{V}_k(\mathfrak{M}_0)$ with $k = 2$ (i.e. on those originally considered in the history). In this case more 'regular' figures appear. Clearly, if L_1, L_2, L_3, L_4 form a quadrangle with $\mathrm{T}(L_1) = \mathrm{T}(L_3)$ then the quadrangle in question is obtained by an embedding of a quadrangle in $\mathfrak{M}_0$. If $\mathfrak{M}_0$ is a linear space (and this case is primarily studied in this paper) and $\mathrm{T}(L_1) = \mathrm{T}(L_2)$ then such a quadrangle has a diagonal. Consequently, searching for a quadrangle without diagonals we can restrict ourselves to the case where $\mathrm{T}(L_1)$, $\mathrm{T}(L_2)$, $\mathrm{T}(L_3)$, and $\mathrm{T}(L_4)$ are pairwise distinct. Such a quadrangle without diagonals will be called *proper* (note: this definition makes sense only in structures in which the notion of a 'leaf' was introduced). The following is just a matter of direct (though quite tedious) verification. Let $\mathfrak{M}_0$ be a linear space and $\mathfrak{M} = \mathbf{V}_2(\mathfrak{M}_0)$.

Lemma 2.8 *Let L_1, K_1, L_2, K_2 be lines of $\mathfrak{M}$ in four pairwise distinct leaves. These lines yield a quadrangle without diagonals iff one of the following holds, up to permutations: of the pairs $(L_1, L_2), (K_1, K_2)$, and of lines in each of these pairs:*

(i) there are lines m, n of $\mathfrak{M}_0$ and points $a_1, b_1 \in n$, $a_2, b_2 \in m$ such that

$$L_1 = a_1 + m,\ L_2 = b_1 + m,\ K_1 = a_2 + n,\ \text{and}\ K_2 = b_2 + n;$$

(ii) there are three lines m, n, l of $\mathfrak{M}_0$ and points a, b, c such that $a, b \in n$, $a, c \in m$, $b, c \in l$ and

$$K_1 = 2n,\ K_2 = c + n,\ L_1 = a + m,\ \text{and}\ L_2 = b + l.$$

The vertices of the respective quadrangles are $a_1 + a_2$, $a_1 + b_2$, $a_2 + b_1$, $b_1 + b_2$ in (i), and $2a$, $a + c$, $2b$, $b + c$ in (ii).

Lemma 2.9 *Let L_1, L_2 be opposite sides of a proper quadrangle in $\mathfrak{M}$. Let K be a line of $\mathfrak{M}$ crossing L_1, L_2 such that $\mathrm{T}(K) \neq \mathrm{T}(L_1), \mathrm{T}(L_2)$. Then one of the following holds:*

(i) $L_1 = a + m$, $L_2 = b + m$ for $a \neq b$ and a line m of $\mathfrak{M}_0$; and $K = x + \overline{a, b}$ for some $x \in m$ or $K = 2m$, when $a, b \in m$,

(ii) $L_1 = 2n$, $L_2 = c + n$, $K = x + \overline{x, c}$ for $x \in n$, $x \neq c$,

(iii) $L_1 = a + m_1$, $L_2 = b + m_2$ for $a \neq b$ and distinct lines m_1, m_2 of $\mathfrak{M}_0$ that share a point c. Then $K = 2\overline{a, b}$ or $K = c + \overline{a, b}$.

Finally, gathering together the possibilities listed in 2.8 and 2.9, we conclude with:

Proposition 2.10 *If K_1, K_2 are two lines of $\mathfrak{M}$ that cross two other lines L_1, L_2 so as L_1, K_1, L_2, K_2 is a proper quadrangle, L_3 crosses K_1, K_2, and K_3 crosses L_1, L_2, then K_3, L_3 share a point.*

Loosely (and not really strictly) speaking: $\mathbf{V}_2(\mathfrak{M}_0)$ satisfies the net axiom.

3. Hyperplanes in Veronese spaces

The main problems of this paper concentrate around 'affinizations' of Veronese structures. These problems concern the following question: *what are hyperplanes in Veronese structures, if they exist?*

A set X of points of a partial linear space $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ is *l-transversal* if it meets every line of $\mathfrak{M}$. Clearly, S is l-transversal; a proper l-transversal subspace is called a *hyperplane*.

A subspace X of a partial linear space is called *spiky* when through each point on X there goes a line that is not contained in X (so, it meets X in the given point only), and X is *flappy* when through each line contained in X there passes a plane not contained in X.

A hyperplane $\mathcal{H}$ of a partial linear space $\mathfrak{M}$ determines a parallelism $\|_{\mathcal{H}}$ on the lines not contained in $\mathcal{H}$ defined by the formula

$$L_1 \|_{\mathcal{H}} L_2 \iff L_1 \cap \mathcal{H} = L_2 \cap \mathcal{H}.$$

Set $\mathcal{L}^{\propto} = \{L \setminus \mathcal{H} \colon L \in \mathcal{L},\ L \not\subseteq \mathcal{H}\}$. Recall that we have assumed $|L| \geq 3$ for every $L \in \mathcal{L}$. Then each $l \in \mathcal{L}^{\propto}$ uniquely determines $\bar{l} \in \mathcal{L}$ such that $l \subset \bar{l}$ and it makes sense to define the parallelism $\|_{\mathcal{H}}$ on $\mathcal{L}^{\propto}$ by the condition $l_1 \|_{\mathcal{H}} l_2 \iff \overline{l_1} \|_{\mathcal{H}} \overline{l_2}$. Let us write

$$\mathfrak{M} \setminus \mathcal{H} = \langle S \setminus \mathcal{H}, \mathcal{L}^{\propto}, \|_{\mathcal{H}} \rangle.$$

Then $\mathfrak{M} \setminus \mathcal{H}$ is a partially affine partial linear space. Note that it is not necessarily an affine partial linear space. If $\mathcal{H}$ is spiky then the points of $\mathcal{H}$ can be interpreted in terms of $\mathfrak{M} \setminus \mathcal{H}$ as the equivalence classes of the parallelism $\|_{\mathcal{H}}$ (comp. definitions in [10]).

Let $\mathfrak{M}_0 = \langle S, \mathcal{L}_0 \rangle$ be a partial linear space; let $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$ and $\mathfrak{M}^* = \mathbf{V}_k(S)$ be the structure of the leaves of $\mathfrak{M}$. Let us pass to our main goal: determine the hyperplanes in $\mathfrak{M}$.

The first solution that comes to mind, though natural, is defective:

Remark 3.1 *If $\mathcal{H}_0$ is a hyperplane of $\mathfrak{M}_0$, then, clearly, the set $\mathfrak{y}_k(\mathcal{H}_0)$ is a subspace of $\mathfrak{M}$ but it is not l-transversal for $k > 1$.*

Proof Let $L \in \mathcal{L}_0$ be not contained in $\mathcal{H}_0$, and $b \in L$ such that $b \notin \mathcal{H}_0$. Then $(b + L) \cap \mathfrak{y}_k(\mathcal{H}_0) = \emptyset$. □

Let $\mathcal{H}$ be a hyperplane of $\mathfrak{M}$. Then, for each leaf $S' = e + (k - |e|)S$, the intersection $S' \cap \mathcal{H}$ is an l-transversal subspace of S', so it is determined by an l-transversal subspace $\mathcal{H}_e$ of $\mathfrak{M}_0$. Write $\mathcal{H}$ for the set of all the hyperplanes of $\mathfrak{M}_0$. Thus, $\mathcal{H}$ determines via the formula $\mathfrak{h}(e) = \{x \in S \colon e + (k - |e|)x \in \mathcal{H}\}$ a function

$$\mathfrak{h} \colon \mathfrak{w}_k(S) \longrightarrow \mathcal{H} \cup \{S\} \tag{8}$$

such that

$$\mathcal{H} = \bigcup \{e + (k - |e|)\mathfrak{h}_e \colon e \in \mathfrak{w}_k(S)\} \quad \text{(we write } \mathfrak{h}_e = \mathfrak{h}(e)\text{)}. \tag{9}$$

Moreover, $\mathcal{H}$ is an l-transversal set in $\mathfrak{M}^*$. The following is a standard exercise.

Lemma 3.2 *For every function as in (8) the set $\mathcal{H}$ defined by (9) is l-transversal in $\mathfrak{M}$.*

In what follows we shall give a series of interesting (we believe) examples of hyperplanes in Veronese spaces associated with ‘classical’ geometries.

3.1. First example: hyperplanes in Veronese spaces associated with projective spaces

At the beginning of this part let us recall properties characteristic for projective and affine spaces. The projective spaces are the veblenian linear spaces. Affine spaces satisfy *the Tamaschke Bedingung* (*if a line parallel to one side of a triangle crosses a second side then it crosses the third side as well*) and the *parallelogram completion condition* (*if of two pairs of parallel lines three intersections of lines in pairs of nonparallel lines exist, then the fourth intersection point exists as well*).

Let $\varkappa$ be a quasi-correlation in a projective space $\mathfrak{P} = \langle S, \mathcal{L} \rangle$ over a field with odd characteristic. That means there is a nonzero reflexive form ξ on the vector space $\mathbb{V}$ coordinatizing $\mathfrak{P}$ such that $\langle u \rangle \in \varkappa(\langle v \rangle)$ is equivalent to $\xi(u,v)=0$ for any nonzero vectors u,v of $\mathbb{V}$. Consider $\mathfrak{M}=\mathbf{V}_2(\mathfrak{P})$ (the primary example of a Veronese space: the Veronese variety, cf. [6]) and define a function $\mathfrak{h}$ on $\mathfrak{w}_2(S)$ as required in (8). First, we set

$$\mathfrak{h}(x)=\varkappa(x) \ \text{ for } x\in S. \tag{10}$$

In accordance with (8), either $\mathfrak{h}(0)=S$ or $\mathfrak{h}(0)=h_0$ for some hyperplane h_0 of $\mathfrak{P}$.

Lemma 3.3 *Suppose that* $\mathfrak{h}(0)=h_0$ *and assume that* $h_0 \nsubseteq \{a\colon a\in\varkappa(a)\}$. *Let* $\mathcal{H}$ *be defined by* (9). *Then* $\mathcal{H}$ is not *a subspace of* $\mathfrak{M}$.

Proof Let $a\in h_0$, $a\notin\varkappa(a)$, and let $q\in\varkappa(a)\setminus h_0$. Then $2a, a+q\in a+\overline{a,q}\cap\mathcal{H}$. Supposing $a+\overline{a,q}\subseteq\mathcal{H}$, then there exists a point $q'\in\overline{a,q}$, $q\neq q'$ with $q'\in\varkappa(a)$, so (contradictory) $a\in\varkappa(a)$. □

Accordingly, to 'extend' $\varkappa$ to a function $\mathfrak{h}$ that determines a hyperplane in $\mathfrak{M}$ we must put

$$\mathfrak{h}(0)=S. \tag{11}$$

Let $\mathcal{H}$ be defined by (9).

Lemma 3.4 *If* $\varkappa$ *is symplectic (all the points of* $\mathfrak{P}$ *are self-conjugate), then* $\mathcal{H}$ *determined by a function* $\mathfrak{h}$ *defined in* (10) *and* (11) *coincides with* $\bigcup\{x+\mathfrak{h}(x)\colon x\in S\}$.

Proof Evident, as $x\in\varkappa(x)$ yields, in accordance with the definition, $2x=x+x\in\mathcal{H}$, for each point x of $\mathfrak{P}$. □

Lemma 3.5 *If* $\varkappa$ *is symplectic, then* $\mathcal{H}$ *is a proper subspace of* $\mathfrak{M}$ *and therefore it is a hyperplane of* $\mathfrak{M}$.

Proof Let $L=a+L_0$ with $|L\cap\mathcal{H}|\geq 2$, and $L_0\in\mathcal{L}$. Then there are at least two points $a+x_1, a+x_2$ in $L\cap\mathcal{H}$ and either $a\neq x_1,x_2$ or $a=x_1\neq x_2$. If $a\neq x_1,x_2$ then $x_1,x_2\in\varkappa(a)$, which gives $L_0\subseteq\varkappa(a)$. Consequently, $L\subseteq\mathcal{H}$. If $a=x_1\neq x_2$, then $x_2\in\varkappa(a)$; if $\varkappa$ symplectic then $x_1\in\varkappa(a)$ as well and the claim follows. □

Now we are in a position to characterize all the hyperplanes in Veronese spaces of level two associated with projective spaces.

Theorem 3.6 *The set* $\mathcal{H}$ *is a hyperplane in* $\mathbf{V}_2(\mathfrak{P})$ *iff* $\mathcal{H}$ *is defined by* (9), *where* $\mathfrak{h}$ *is defined by* (10) *and* (11) *for some symplectic quasi-correlation* $\varkappa$ *in* $\mathfrak{P}$.

Proof The right-to-left implication follows directly from 3.5. Let $\mathcal{H}$ be a hyperplane of $\mathbf{V}_2(\mathfrak{P})$. Consider the binary relation $\perp$ on the points of $\mathfrak{P}$ defined by the condition $x\perp y \iff x+y\in\mathcal{H}$. Clearly, $\perp$ is symmetric. Set $\mathfrak{h}(x)=\{y\colon x\perp y\}$ for each point x of $\mathfrak{P}$. By definition, $x+\mathfrak{h}(x)=(x+S)\cap\mathcal{H}$ is l-transversal in $x+S$, so $\mathfrak{h}(x)=S$ or $\mathfrak{h}(x)$ is a hyperplane in $\mathfrak{P}$. As $\mathcal{H}$ is a proper subspace, for at least one x we have $\mathfrak{h}(x)\neq S$. From [7] we deduce that $\perp$ can be characterized by the formula $\langle u\rangle\perp\langle v\rangle \iff \xi(u,v)=0$ for a sesquilinear form ξ defined on $\mathbb{V}$. Let $\varkappa$ be the quasi-correlation of $\mathfrak{P}$ determined by ξ. Note that $\mathfrak{y}_2(S)=\bigcup\{x+S\colon x\in S\}$, so $\mathcal{H}=\bigcup\{x+\mathfrak{h}(x)\colon x\in S\}$, i.e. $\mathcal{H}$ is defined by (9) with $\mathfrak{h}$ satisfying (10). Recall that $\mathfrak{h}(0)=\{x\in S\colon 2x\in\mathcal{H}\}$. From 3.3 we deduce (formally) that either $\mathfrak{h}(0)=S$ or $\mathfrak{h}(0)=h_0$ is a hyperplane and this hyperplane is contained in the set of $\varkappa$-self-conjugate points. In both cases we conclude with $2S \subseteq \mathcal{H}$, which gives $a \perp a$ for each a. Consequently, $\varkappa$ is symplectic. $\square$

From now on we assume that $\varkappa$ is symplectic. Denote

$$\mathfrak{A} := \mathfrak{M} \setminus \mathcal{H},$$

the corresponding affine reduct. From 3.5 it follows that $\mathfrak{A}$ is a partially affine partial linear space.

Lemma 3.7 *The point set of $\mathfrak{A}$ consists of all the multisets $x+y$ with $x, y \in S$ and $x \notin \varkappa(y)$ (so: $x \neq y$); therefore, this point set can be identified with a subset of $\wp_2(S)$.*

Lemma 3.8 *The hyperplane $\mathcal{H}$ determined by a symplectic polarity is spiky, but it is not flappy.*

Proof Take a point $e = x+y$ with $x \perp y$, $x \neq y$. Then $y \in x^\perp$. There is a line L_0 through y not contained in $x^\perp$. Setting $L = x + L_0$, then $e \in L$ and L is not contained in $\mathcal{H}$. Indeed, suppose $x + z \in \mathcal{H}$ for $y \neq z \in L_0$; then $z \in x^\perp$, so $L_0 \subseteq x^\perp$.

Take any line L_0 of $\mathfrak{M}$; then $L = 2L_0 \subseteq \mathcal{H}$. On the other hand, any plane of $\mathfrak{M}$ that contains $2L_0$ is contained in $\mathrm{T}(L) = 2S \subseteq \mathcal{H}$, which yields our second claim. $\square$

It is rather easy to observe the following:

Fact 3.9 *The maximal strong subspaces of $\mathfrak{M}$ are the leaves of $\mathfrak{M}$. Consequently, the maximal strong subspaces of $\mathfrak{A}$ are affine spaces of the form $(x+S) \setminus \varkappa(x)$, $x \in S$ (cf. [4, Fact 2.2], [9, Prop. 2.11]).*

As a direct consequence of 3.9 and 2.6 (cf. [10, Lem. 2.4]) we obtain:

Lemma 3.10 *(i) Each Veblen subconfiguration 'with diagonals' (i.e. each projective quadrangle) of $\mathfrak{M}$ is contained in a leaf.*

(ii) The relation of Veblen parallelism is properly definable in $\mathfrak{A}$ by the formula (15), i.e. lines are in the relation $\|^\circ$ when they are on a (affine) plane and are parallel on that plane.

As an important by-product of 2.6 we get nearly immediately the following:

Proposition 3.11 *The structure $\mathfrak{A}$ satisfies the Tamaschke Bedingung and the parallelogram completion condition.*

The framework proposed admits some degenerations. Namely, we cannot expect that $\varkappa$ is nondegenerate, and thus it may happen that $\mathcal{H} \supseteq 2S$ contains some leaves of the form $x+S$ as well. Also, lines contained in these leaves cannot be extended to "proper" planes of $\mathfrak{A}$ and $\mathcal{H}$ is 'more nonflappy than expected'. In essence, this happens in every odd dimension of $\mathbb{V}$.

In what follows we assume that $\varkappa$ *is nondegenerate*, and then $\dim(\mathbb{V})$ is even. Let us examine the structure of the parallelism $\|_{\mathcal{H}}$ and that of the horizon determined by it. We begin with an evident observation:

Lemma 3.12 *Let $e \in \mathcal{H}$. Set $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$. Then one of the following two possibilities holds:*

- For any two lines L_1, L_2 of $\mathfrak{A}$ that pass through e there is a plane of $\mathfrak{A}$ that contains them; in consequence, $L_1 \parallel^\circ L_2$*: this happens when* $e = 2x$ *with* $x \in S$*. In this case the two leaves of* $\mathfrak{M}$ *through* e *are* $x + S$ *and* $2S \subseteq \mathcal{H}$ *and thus* L_1, L_2 *are determined by two lines, both two in* $x + S$*.*
- There are two lines L_1, L_2 through e such that $L_1 \not\parallel^\circ L_2$ and any L_3 through e (i.e. any L_3 with $L_3 \parallel L_1$) satisfies $L_3 \parallel^\circ L_1$ or $L_3 \parallel^\circ L_2$*: this happens when* $e = x + y$ *with* $x \neq y$*, and we take* $L_1 \subseteq x + S$*,* $L_2 \subseteq y + S$ *(comp. 3.8).*

This allows us to distinguish two types of directions in $\mathfrak{A}$; directions of the first type ($[L]_\parallel$ when $L \parallel L'$ iff $L \parallel^\circ L'$ for each line L') correspond to the elements of one totally deleted leaf $2S$. In any case, a direction of $\mathfrak{A}$ uniquely corresponds to a point in $\mathcal{H}$; in what follows we shall frequently identify corresponding two objects.

For a direction $\mathfrak{a}$ (an equivalence class under *a parallelism* $\parallel'$) and a set of points X write the following:

$$\mathfrak{a} \mid X \text{ (in words: } \mathfrak{a} \text{ is incident with } X\text{) when } \mathfrak{a} = [L]_{\parallel'} \text{ for a line } L \subseteq X.$$

Remark 3.13 Note that though the two relations $\parallel$ and $\parallel^\circ$ *do not coincide*, the relation $\parallel^\circ$ *is a partial parallelism*, so it also determines its directions (equivalence classes). Formally speaking, a point $x + y \in \mathcal{H} \setminus 2S$ determines *two distinct* $\parallel^\circ$-directions.

Let us stress the fact that the distinction formulated in 3.12 refers entirely to $\parallel_\mathcal{H}$-directions; loosely speaking $\mathfrak{a} \in 2S$ iff it is incident with exactly one leaf, and other $\parallel$-directions are incident with two leaves. However, *each* $\parallel^\circ$*-direction is incident with exactly one leaf*! ○

$\mathcal{H}$ is spiky, so its points can be identified with the equivalence classes of $\parallel_\mathcal{H}$. However, $\mathcal{H}$ is not flappy, so lines on $\mathcal{H}$ cannot be identified, in general, with directions of planes of $\mathfrak{A}$ and the standard way to recover $\mathfrak{M}$ from $\mathfrak{A}$ fails. This recovering is still possible though, only the recovering procedure must be complicated a bit.

Lemma 3.14 *The lines of the horizon of* $\mathfrak{A}$ *that are contained in a leaf of the form* $x + S$ *are definable in* $\mathfrak{A}$*.*

Proof First, we note that the class of planes of $\mathfrak{A}$ can be defined in $\mathfrak{A}$. Indeed, let Δ be a triangle with the sides L_1, L_2, L_3 and the vertices e_1, e_2, e_3, $e_i \not\mid L_i$ for $i = 1, 2, 3$, such that $e_1 \sim e_0 \mid L_1$ for some $e_0 \neq e_2, e_3$. Then the set

$$\pi(L_1, L_2, L_3) \quad := \quad \bigcup \{L \colon L \parallel L_1 \wedge L \sim L_2, L_3\} \tag{12}$$

is a plane in $\mathfrak{A}$, i.e. $\pi(L_1, L_2, L_3) = a + A$ for a point a of $\mathfrak{P}$ and a plane A of the affine reduct $\mathfrak{P} \setminus \mathfrak{h}(a)$, and each plane of $\mathfrak{A}$ has a form as in (12). Let $\mathcal{P}$ be the class of planes. The collinearity of the required form is defined by

$$\mathbf{L}([L_1]_\parallel, [L_2]_\parallel, [L_3]_\parallel) \iff (\exists A \in \mathcal{P})(\exists L_1', L_2', L_3' \subseteq A)\ [\wedge_{i=1}^{3} L_i' \parallel L_i]. \tag{13}$$

This argument closes the reasoning. □

Remark 3.15 In the ordinary affine geometry the formula (12) defines a plane for every triangle L_1, L_2, L_3. In the case of Veronese spaces we must be cautious. Indeed, *if lines* L_0, L_1, L_2, L_3 *yield in* $\mathfrak{M}$ *a Veblen figure of the form 2.5(ii) or 2.5(iii) then* $\pi(L_1, L_2, L_3) = \mathfrak{y}_2(m)$ *and, clearly, the latter is not a plane.* ○

Though the lines on $2S$ are not "improper lines" of affine planes, we can also recover these lines in terms of $\mathfrak{A}$.

Lemma 3.16 *The lines of the horizon of* $\mathfrak{A}$ *that are contained in the leaf* $2S$ *are definable in terms of* $\mathfrak{A}$*.*

Proof Let $\mathcal{S}$ stand for the class of the maximal strong subspaces of $\mathfrak{A}$; it is definable. From 3.9 $\mathcal{S} = \{a + (S \setminus \varkappa(a)) \colon a \in S\}$. Thus, with each line L of $\mathfrak{A}$ we have a definable set $\mathrm{T}(L)$ with $L \subseteq \mathrm{T}(L) \in \mathcal{S}$. In particular, the notion of a proper quadrangle can be expressed in terms of $\mathfrak{A}$. We have for $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in 2S$

$$\mathbf{L}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \iff \exists L_1, L_2, L_3 \Big(\exists L', L'', M', M'' \colon \text{- a proper quadrangle in } \mathfrak{A}$$

$$\big(L_1, L_2, L_3 \sim L', L'' \wedge \wedge_{i=1}^{3} \mathfrak{a}_i \mathrel{\big|} \mathrm{T}(L_i)\big)\Big).$$

The claim is immediate after 2.8 and 2.9. □

As an important consequence we obtain now the following:

Theorem 3.17 *The underlying Veronese space* $\mathfrak{M}$ *can be recovered from its affine reduct* $\mathfrak{A}$*.*

Note 3.18 Slightly rephrasing the proof of 3.16 with the help of 2.10 one can prove that *the parallelism* $\parallel_{\mathcal{H}}$ *can be defined in terms of the incidence structure of* $\mathfrak{A}$*.* Indeed, two lines are parallel either when they are in one leaf: then their parallelism coincides with $\parallel^{\circ}$. Or they are in distinct leaves of $\mathfrak{M}$: then they can be completed to a proper net; if they do not intersect each other, their common point must lie in $\mathcal{H}$.

As we already noted, the leaves of $\mathfrak{A}$ carry the structure of an affine space. Particularly, if $\mathfrak{P}$ is the n-dimensional projective space $PG(n, q)$ over the q-element field and n, q are odd, then the leaves of $\mathfrak{A}$ have the structure of the n-dimensional affine space $AG(n, q)$ over the q-element field. However, the geometry of affine reducts of Veronese spaces associated with projective spaces and the geometry of Veronese spaces associated with affine spaces are essentially distinct.

Theorem 3.19 *Let* $\mathcal{H}$ *be a hyperplane in* $\mathbf{V}_2(\mathfrak{P})$*. Then*

$$\mathbf{V}_2(\mathfrak{P}) \setminus \mathcal{H} \ncong \mathbf{V}_k(\mathfrak{A}_0)$$

for every affine space $\mathfrak{A}_0$ *and every integer* k*.*

Proof Let $x + S$ be a leaf of $\mathfrak{M} = \mathbf{V}_2(\mathfrak{P})$. Then the corresponding leaf of $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$ is $S' := (x + S) \setminus \mathcal{H}$; let $\mathfrak{A}'_0$ be the restriction of $\mathfrak{A}$ to S'. Each leaf of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'_0$ and there pass exactly two leaves through each point of $\mathfrak{A}$. Suppose an isomorphism exists. Then $k = 2$ and $\mathfrak{A}_0 = \mathfrak{A}'_0$. To close the proof it suffices to observe that the Veronese space $\mathbf{V}_2(\mathfrak{A}_0)$ associated with an affine space satisfies the net condition. The reduct $\mathfrak{A}$ does not satisfy this condition: two lines through $x + y \in \mathcal{H}$, one in $x + S$ and the second in $y + S$, can be completed to a net on its proper points, and clearly they do not cross each other in $\mathfrak{A}$. □

3.2. Generalization: hyperplanes in Veronesians with level $k > 2$ associated with projective spaces

The construction of a hyperplane determined by a symplectic form can be applied to Veronesians of level greater than 2 as well. Let $\eta \colon V^k \longrightarrow F$ be a k-linear nondegenerate alternating form, defined on a vector space $\mathbb{V}$ with V being its set of vectors, and let F be the set of scalars. Recall two basic properties of η:

a) $\eta(v_1,\dots,v_k)=0$ yields $\eta(v_{\sigma(1)},\dots,v_{\sigma(k)})=0$ for every permutation σ of $\{1,\dots,k\}$,

b) if $v_i=v_j$ for $1\le i<j\le k$ then $\eta(v_1,\dots,v_k)=0$.

For a family $q_1=\langle v_1\rangle,\dots,q_k=\langle v_k\rangle$ of points of the projective space $\mathfrak{P}=\mathbf{P}_1(\mathbb{V})=\langle S,\mathcal{L}\rangle$ over $\mathbb{V}$ we write $\perp_\eta(q_1,\dots,q_k)$ when $\eta(v_1,\dots,v_k)=0$. From the property a) of η we get that the relation $\perp=\perp_\eta$ is fully symmetric.

Define

$$\mathcal{H}:=\{q_1+q_2+\ldots+q_k\colon q_1,\dots,q_k \text{ - points of } \mathfrak{P},\ \perp(q_1,\dots,q_k)\}. \tag{14}$$

Fact 3.20 *The set $\mathcal{H}$ defined by (14) is a (nondegenerate) hyperplane in $\mathbf{V}_k(\mathfrak{P})$.*

From the property b) of η, the points of $\mathfrak{A}=\mathbf{V}_k(\mathfrak{P})\setminus\mathcal{H}$, i.e. the set $\mathfrak{y}_k(S)\setminus\mathcal{H}$ are k-subsets of S (that we denote by $\wp_k(S)$) and therefore the affine reduct $\mathfrak{A}$ can be characterized as follows:

the points of $\mathfrak{A}$: $\{\{q_1,\dots,q_k\}\in\mathcal{P}_k(S)\colon \not\perp(q_1,\dots,q_k)\}$,

the lines of $\mathfrak{A}$: $\big\{\{\{q_1,\dots,q_{k-1},x\}\colon x\in L, \not\perp(q_1,\dots,q_{k-1},x)\}\colon L\in\mathcal{L}, L\not\subseteq(q_1,\dots,q_{k-1})^\perp\big\}$.

The structures obtained are more 'Grassmannians' than 'Veronesians': they are defined on sets *without repetitions*! The problem to enter deeper into geometry of such reducts of Grassmannians is certanly interesting, but it is not the goal of this paper.

3.3. More sophisticated example: Veronese spaces associated with polar spaces

Let us begin with three rather evident observations:

Fact 3.21 *Let $\mathcal{H}$ be a hyperplane in a partial linear space $\mathfrak{N}=\langle S,\mathcal{L}\rangle$ and let $S_0\subseteq S$. Set $\mathcal{L}[S_0]=\{L\in\mathcal{L}\colon L\subseteq S_0\}$. If $\mathcal{L}_0\subseteq\mathcal{L}[S_0]$, then $\mathcal{H}\cap S_0$ is an l-transversal subset in $\langle S_0,\mathcal{L}_0\rangle$.*

In the notation of Fact 3.21 we write $\mathfrak{N}[S_0]=\langle S_0,\mathcal{L}[S_0]\rangle$. Generally, $\mathfrak{N}[S_0]$ need not be a partial linear space, but for some 'nonsense reasons' only: its line set may be empty, it may have isolated points, etc.

Fact 3.22 *Let $\mathfrak{M}_0=\langle S_0,\mathcal{L}_0\rangle$ be a partial linear space and let $S_0'\subseteq S_0$. Then $\mathbf{V}_k(\langle S_0',\mathcal{L}[S_0']\rangle)$ and $\mathbf{V}_k(\langle S_0,\mathcal{L}_0\rangle)[\mathfrak{y}_k(S_0')]$ coincide.*

Proof The point sets of both structures are equal: just from definitions.

Let L be a line of $\mathbf{V}_k(\langle S_0',\mathcal{L}[S_0']\rangle)$. Thus, $L=e+(k-|e|)L_0$, where $L_0\in\mathcal{L}[S_0']\subseteq\mathcal{L}_0$, so $L_0\subseteq S_0'$, and $e\in\mathfrak{y}_k(S_0')\subseteq\mathfrak{y}_k(S_0)$. Finally, L is a line of $\mathbf{V}_k(\langle S_0,\mathcal{L}_0\rangle)$ and $L\subseteq\mathfrak{y}_k(S_0')$. Conversely, let $L=e+(k-|e|)L_0$ be a line of $\mathbf{V}_k(\langle S_0,\mathcal{L}_0\rangle)$. Assume that $L\subseteq\mathfrak{y}_k(S_0')$. Then $e\in\mathfrak{y}_k(S_0')$ and $L_0\subseteq S_0'$, $L_0\in\mathcal{L}_0$ and thus L is a line of $\mathbf{V}_k(\langle S_0',\mathcal{L}[S_0']\rangle)$. □

Fact 3.23 *If $\mathfrak{M}'=\langle S,\mathcal{L}'\rangle$ and $\mathfrak{M}''=\langle S,\mathcal{L}''\rangle$ are partial linear spaces such that $\mathcal{L}''\subseteq\mathcal{L}'$ then the line set of $\mathbf{V}_k(\mathfrak{M}'')$ is a subset of the line set of $\mathbf{V}_k(\mathfrak{M}')$.*

Let us quote the standard models of polar spaces (in what follows *a polar space* will always mean one of the below). Let ϖ be a polarity in $\mathfrak{P} = \langle S, \mathcal{L} \rangle$; let $Q_0(\varpi) = \{p\colon p \in \varpi(p)\}$ be the set of points of $\mathfrak{P}$ that are self-conjugate under ϖ and $Q_1(\varpi) = \{L\colon L \subseteq \varpi(L)\}$ be the set of self-conjugate lines. Then the polar space determined by ϖ is the structure $\mathbf{Q}(\varpi) := \langle Q_0(\varpi), Q_1(\varpi) \rangle$ provided that ϖ is symplectic or ϖ is quadratic with index at least 2, i.e. $Q_2(\varpi) \neq \emptyset$. In corresponding cases we have $\mathbf{Q}(\varpi) = \langle S, Q_1(\varpi) \rangle$ and $\mathbf{Q}(\varpi) = \mathfrak{P}[Q_0(\varpi)]$. Let us recall also that an *affine polar space* is an affine reduct of a polar space obtained by deleting a hyperplane of it. In our approach this hyperplane can be always considered as the restriction to Q_0 of a hyperplane of $\mathfrak{P}$.

The following will be needed, which is known in the literature.

Fact 3.24 *A maximal strong subspace of a polar space (of an affine polar space) is a projective (affine, resp.) subspace of $\mathfrak{P}$ (of the affine reduct of $\mathfrak{P}$).*

Any two planes of a polar space and of an affine polar space can be joined by a sequence of planes where each two consecutive planes share a line (polar spaces and affine polar spaces are strongly connected*, cf.* [8]*).*

Let us consider the structure $\mathfrak{M} := \mathbf{V}_k(\mathbf{Q}(\varpi))$. Then, as an immediate consequence of 3.23 and 3.22 resp., we have:

Lemma 3.25

(i) When ϖ is symplectic, the line set of $\mathfrak{M}$ is a subset of the line set of $\mathbf{V}_k(\mathfrak{P})$,

(ii) $\mathfrak{M} = \mathbf{V}_k(\mathfrak{P})[\mathfrak{y}_k(Q_0(\varpi))]$ when ϖ is quadratic.

Let $\mathcal{H}$ be a hyperplane of $\mathbf{V}_k(\mathfrak{P})$ determined by a polarity $\varkappa$. From 3.21 and 3.25 we obtain:

Fact 3.26 *The set $\mathcal{H} \cap \mathfrak{y}_k(Q_0(\varpi))$ is a hyperplane in $\mathfrak{M}$.*

We have no proof, but it seems that:

Conjecture 3.27 *Those of the form in 3.26 are the only possible hyperplanes in $\mathfrak{M}$.*

In the sequel we write simply $\mathfrak{M} \setminus \mathcal{H}$ instead of $\mathfrak{M} \setminus (\mathcal{H} \cap \mathfrak{y}_k(Q_0(\varpi)))$.

Lemma 3.28 *The leaves of $\mathfrak{M}$ and their restrictions in $\mathfrak{M} \setminus \mathcal{H}$ are definable in the internal geometry of $\mathfrak{M}$ (of $\mathfrak{M} \setminus \mathcal{H}$, resp.).*

Proof In the first step we consider strong subspaces of $\mathfrak{M}$ and $\mathfrak{M} \setminus \mathcal{H}$. Clearly, they have form $e + (k - |e|)X$ ($(e + (k - |e|)X) \setminus \mathcal{H} = e + (k - |e|)A$, resp.), where X is a strong subspace of $\mathbf{Q}(\varpi)$ (A is a strong subspace of the affine polar space $\mathbf{Q}(\varpi) \setminus \varkappa(e)$ resp.).

Therefore, in the second step we can define the class $\mathcal{P}$ of planes of $\mathfrak{M}$ (of $\mathfrak{M} \setminus \mathcal{H}$).

In the third step we define a point-to-point relation γ: $a \,\gamma\, b$ when a and b can be joined by a sequence of elements of $\mathcal{P}$ where each two consecutive elements share a line.

In the last step we note that the leaves in question are the equivalence classes under the relation γ. □

Let $\varkappa$ be symplectic and $k = 2$. Applying the techniques of the previous subsection we can prove now the following:

Theorem 3.29 *The underlying Veronese space $\mathfrak{M}$ can be defined in terms of its affine reduct $\mathfrak{M} \setminus \mathcal{H}$.*

4. Appendix: multiplying a parallelism

In this additional part we shall discuss the following problem: *how do we extend a parallelism from a given structure to a Veronese space associated with it?*

Let $\langle S_0, \mathcal{L}_0 \rangle$ be a partial linear space. A relation $\parallel \subseteq \mathcal{L}_0 \times \mathcal{L}_0$ is called a *partial parallelism* (*a preparallelism*) if it is an equivalence relation such that distinct parallel lines are disjoint. If $\parallel$ is a preparallelism as above then the structure $\langle S_0, \mathcal{L}_0, \parallel \rangle$ is called *a partially affine partial linear space*; the relation $\parallel$ is called *a parallelism* and the respective structure is called *an affine partial linear space* when the following form of the (affine) Euclid axiom holds: *the equivalence class* $[L]_\parallel$ (the *direction*) *of each line* L *covers* S_0. The most celebrated class of affine partial linear spaces constitutes affine spaces. Recall that the natural parallelism of an affine space coincides with the relation (so-called *Veblen parallelism*) $\parallel^\circ$ defined by the formula

$$\begin{aligned} L_1 \parallel^\circ L_2 \quad \iff \quad & L_1 = L_2 \vee \big(L_1 \not\sim L_2 \ \ \& \ \text{there are two lines } L', L'' \text{ through a point } p \\ & \text{such that } p \not\mid L_1, L_2 \wedge L_1, L_2 \sim L', L' \\ & \& \text{ there are collinear points } a_1 \in L_1 \cap L',\ a_2 \in L_2 \cap L''\big). \end{aligned} \tag{15}$$

Intuitively speaking: $L_1 \parallel^\circ L_2$ when L_1, L_2 are on a plane and either coincide or have no common point.

Most of the time (e.g., in the case of affine spaces and their Segre products, cf. [10]) (15) is equivalent to a simpler formula with the condition that *there are collinear points* $a_1 \in L_1 \cap L'$, $a_2 \in L_2 \cap L''$ on its right-hand side omitted. Here, we must handle this more complex formula since in a Veronese space not every triangle determines a plane.

Let us start with the following approach, which may seem, at first look, natural (especially taking into account similarities between Veronese and Segre products). Let $\mathfrak{A}_0 = \langle S_0, \mathcal{L}_0, \parallel_0 \rangle$ be a partial affine partial linear space. We define on the lines of $\mathbf{V}_k(\langle S_0, \mathcal{L}_0 \rangle)$ the relation $\parallel$ by the formula

$$B_1 \parallel B_2 \iff \text{there are } e_1, r_1, r_2, L_1, L_2 \in \mathcal{L}_0 \text{ such that } B_1 = e_1 + r_1 L_1 \wedge B_2 = e_2 + r_2 L_2 \wedge L_1 \parallel L_2. \tag{16}$$

Then we set

$$\mathbf{V}_k(\mathfrak{A}_0) = \langle \mathfrak{y}_k(S), \mathcal{L}^{\circledast}, \parallel \rangle.$$

Remark 4.1 *Let* $\mathfrak{A}_0$ *be an affine partial linear space,* $k > 1$*, and* $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$*. Then* $\parallel$ *defined by* (16) *is an equivalence relation and each of its equivalence classes covers the point set of* $\mathfrak{A}$*. If a line* L *of* $\mathfrak{A}$ *is given then in each of the leaves through a point* f *there passes exactly one line parallel to* L *with respect to the parallelism just defined.*

Since there are k *leaves through* f*, the relation* $\parallel$ is not *a parallelism in* $\langle \mathfrak{y}_k(S), \mathcal{L}^{\circledast} \rangle$ *and therefore* $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$ is not a partially affine partial linear space.

Proof It is evident that $\parallel$ defined by (16) is an equivalence relation in $\mathcal{L}^{\circledast}$. To justify the 'negative' part note, first, that an equivalence class $[e + rL]_\parallel$ is determined uniquely by the line L of $\mathfrak{A}_0$, and each leaf is isomorphic to $\mathfrak{A}_0$. □

If $\mathfrak{M}_0$ is a partial linear space on $\boldsymbol{v}_0$ points and $\boldsymbol{b}_0$ lines with the corresponding point and line ranks $\boldsymbol{r}_0$ and $\boldsymbol{\kappa}_0$, then the parameters of $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$ are as follows: $\boldsymbol{v}_{\mathfrak{M}} = \binom{\boldsymbol{b}_0 + k - 1}{k}$, $\boldsymbol{r}_{\mathfrak{M}} = k \cdot \boldsymbol{r}_0$, $\boldsymbol{\kappa}_{\mathfrak{M}} = \boldsymbol{\kappa}_0$, and $\boldsymbol{b}_{\mathfrak{M}} = \binom{\boldsymbol{v}_0 + k - 1}{k - 1} \cdot \boldsymbol{b}_0$.

Theorem 4.2 *Let $k > 1$. There is no finite affine partial linear space $\mathfrak{A}_0$ with fixed size of directions of its lines such that $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$ admits a parallelism, the directions of lines of $\mathfrak{A}$ have a constant size, and each leaf of $\mathfrak{A}$ is an* affine subspace *of $\mathfrak{A}$ (i.e. each of its leaves is closed under this parallelism).*

Proof Suppose, to the contrary, that there are such $\mathfrak{A}$ and $\mathfrak{A}_0$; let n be the number of points of $\mathfrak{A}_0$. Denote by $\boldsymbol{\delta}_0 = \boldsymbol{\delta}_{\mathfrak{A}_0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_{\mathfrak{A}}$ the corresponding sizes of directions. Let $\boldsymbol{\kappa}$ be the line size of $\mathfrak{A}_0$, and thus of $\mathfrak{A}$ as well. When we observe that a direction must cover the point set of $\mathfrak{A}$ we get

$$\boldsymbol{\delta}_0 \cdot \boldsymbol{\kappa} = n \text{ and } \boldsymbol{\delta} \cdot \boldsymbol{\kappa} = \binom{n+k-1}{k}.$$

Since no two leaves of $\mathfrak{A}$ have a line in common, a direction of $\mathfrak{A}$ is a union of directions considered in each of the leaves, and each of these leaves is isomorphic to $\mathfrak{A}_0$. This gives us the relation $\boldsymbol{\delta} =$ the number of leaves $\cdot\, \boldsymbol{\delta}_0$, which yields $\binom{n+k-1}{k} = n\binom{n+k-1}{k-1}$, so $k = 1$. □

In particular, 4.2 yields a rather strange result: the structure $\mathbf{V}_k(AG(m,q))$ can be covered by a family of subspaces each one isomorphic to the affine space $AG(m,q)$ and therefore a natural parallelism intrinsically definable exists in each of the covering subspaces:

Remark 4.3 *Let $\mathfrak{A}_0$ be an affine space and let L_1, L_2 be lines of $\mathbf{V}_k(\mathfrak{A}_0)$. Then $L_1 \parallel^\circ L_2$ iff there are lines l_1, l_2 parallel in $\mathfrak{A}_0$ such that and $L_i = e + (k - |e|)l_i$ for some e and $i = 1, 2$.*

The proof of 4.3 is immediate after (15) and 2.4. The union of the parallelisms in leaves is a partial parallelism and there is no 'global' parallelism that agrees in a natural way with the parallelisms on leaves, definable on the whole line set of $\mathbf{V}_k(AG(m,q))$.

However, Theorem 4.2 does not mean that $\mathbf{V}_k(\mathfrak{A}_0)$ does not admit *some* parallelism, but this problem is not addressed in this paper.

There are more open problems that have arisen throughout this paper. We believe they are interesting in their own right and worth considering. Let us formulate the main ones.

Problem 4.4 *The claim of 3.17 seems valid also for Veronese spaces associated with projective spaces, where a quasi-correlation $\varkappa$ is degenerate, since* $\dim(\mathrm{Rad}(\varkappa)) = 1$. *Does 3.17 remain true when $\varkappa$ is degenerate and* $\dim(\mathrm{Rad}(\varkappa)) > 1$*?*

Problem 4.5 *If $\varkappa = \varpi$ then the point set of $\mathbf{V}_2(\mathfrak{P})$ and of $\mathbf{V}_2(\mathfrak{Q})$ (we write $\mathfrak{Q} = \mathbf{Q}(\varpi)$) coincide. The distinction concerns lines only. How do we characterize $\mathbf{V}_2(\mathfrak{P}) \setminus \mathcal{H}$ and $\mathbf{V}_2(\mathfrak{Q}) \setminus \mathcal{H}$?*

Problem 4.6 *How do we characterize the geometry of the partially affine partial linear space $\langle S', \mathcal{L}', \parallel^\circ \rangle$, where $\langle S', \mathcal{L}', \parallel_{\mathcal{H}} \rangle = \mathfrak{A}$, i.e. of the reduct $\mathbf{V}_2(\mathfrak{P})$ equipped with the parallelism imitating the affine one?*

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