Some series involving the Euler zeta function

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Abstract: In this paper, using the Boole summation formula, we obtain a new integral representation of n-th quasi-periodic Euler functions $E_n(x)$ for $n = 1, 2, \ldots$. We also prove several series involving Euler zeta functions $\zeta_E(s)$, which are analogues of the corresponding results by Apostol on some series involving the Riemann zeta function $\zeta(s)$.

Key words: Hurwitz-type Euler zeta functions, Euler zeta functions, Euler polynomials, Boole summation formula, quasi-periodic Euler functions

1. Introduction
The Hurwitz-type Euler zeta function is defined as follows

$\zeta_E(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s}$

for complex arguments $s$ with $\text{Re}(s) > 0$ and $a$ with $\text{Re}(a) > 0$, which is a deformation of the well-known Hurwitz zeta function

$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$

for $\text{Re}(s) > 1$ and $\text{Re}(a) > 0$. Note that $\zeta(s, 1) = \zeta(s)$, the Riemann zeta function. The series (1) converges for $\text{Re}(s) > 0$ and it can be analytically continued to the complex plane without any pole. For further results concerning the Hurwitz-type Euler zeta function, we refer to the recent works in [10] and [14]. Let $a = 1$ in (1); it reduces to the Euler zeta function

$\zeta_E(s) = \zeta_E(s, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$

for $\text{Re}(s) > 0$, which is also a special case of Witten zeta functions in mathematical physics (see [20, p. 248, (3.14)]). In fact, it is shown that the Euler zeta function $\zeta_E(s)$ is summable (in the sense of Abel) to $(1 - 2^{1-s})\zeta(s)$ for all values of $s$. Several properties of $\zeta_E(s)$ can be found in [3, 10, 12, 16]. For example, in the form on [1, p. 811], the left-hand side is the special values of the Riemann zeta functions at positive integers,
and the right-hand side is the special values of Euler zeta functions at positive integers. In number theory, the Hurwitz-type Euler zeta function (1) represents the partial zeta function in one version of Stark’s conjecture of cyclotomic fields (see [15, p. 4249, (6.13)]). The corresponding $L$-functions (the alternating $L$-series) have also appeared in a decomposition of the $(S, \{2\})$-refined Dedekind zeta functions of cyclotomic fields (see [12, p. 81, (3.8)]). Recently, using Log Gamma functions, Can and Dağlı proved a derivative formula of these $L$-functions (see [8, Eq. (4.13)]).

The Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

for $|t| < \pi$ (see, for details, [11, 21, 27]). They are the special values of (1) at nonpositive integers (see [10, p. 520, Corollary 3], [9, p. 761, (2.3)], [14, p. 2983, (3.1)], [29, p. 41, (3.8)] and (46) below). The integers $E_n = 2^n E_n(1/2), n \in \mathbb{N} = \mathbb{N} \cup \{0\}$, are called Euler numbers. For example, $E_0 = 1, E_2 = -1, E_4 = 5$, and $E_6 = -61$. The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler’s famous book, Institutiones Calculi Differentials (1755, pp. 487-491 and p. 522). Notice that the Euler numbers with odd subscripts vanish, that is, $E_{2m+1} = 0$ for all $m \in \mathbb{N}$.

For $n \in \mathbb{N}_0$, the $n$-th quasi-periodic Euler functions are defined by

$$\mathcal{E}_n(x+1) = -\mathcal{E}_n(x)$$

(5) for all $x \in \mathbb{R}$, and

$$\mathcal{E}_n(x) = E_n(x) \text{ for } 0 \leq x < 1$$

(6) (see [7, p. 661]). For arbitrary real numbers $x$, $[x]$ denotes the greatest integer not exceeding $x$ and $\{x\}$ denotes the fractional part of real number $x$; thus

$$\{x\} = x - [x].$$

(7)

Then, for $r \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we have

$$\mathcal{E}_n(x) = (-1)^{[x]} E_n(\{x\}), \quad \mathcal{E}_n(x+r) = (-1)^r \mathcal{E}_n(x)$$

(8) (see [4, (1.2.9)] and [7, (3.3)]). For further properties of the quasi-periodic Euler functions, we refer to [4, 7, 8, 13].

In this paper, we obtain a new integral representation of $n$-th quasi-periodic Euler functions $\mathcal{E}_n(x)$ as follows.

**Theorem 1.1** Let $n \in \mathbb{N}_0$ and let $\mathcal{E}_n(x)$ be the $n$-th quasi-periodic Euler functions. Then for $x > 0$

$$\mathcal{E}_n(x) = (-1)^n n! \frac{1}{\pi i} \int_{(c)} \frac{\Gamma(s)}{\Gamma(s+n+1)} \zeta_E(-s-n)x^{-s}ds,$$  

where $(c)$ denotes the vertical straight line from $c - i\infty$ to $c + i\infty$ with $0 < c < 1$ and $\Gamma(s)$ denotes the Euler gamma function.

**Remark 1.2** We remark that this theorem is an analogue of a result by Li et al. on Riemann zeta functions (see [19, Proposition 1]).
Furthermore, we also obtain the following two theorems on series involving Euler zeta functions $\zeta_E(s)$. They are the analogues of the corresponding results of Apostol [2] on some series involving the Riemann zeta function.

**Theorem 1.3** Let $\binom{-s}{r}$ denote the binomial symbol defined through the Euler gamma function $\Gamma(s)$ as follows

$$\binom{-s}{r} = (-1)^r \frac{(s+r-1)!}{r!},$$

where $s \in \mathbb{C}$ and $r \in \mathbb{N}$. Then the following identities hold:

1. For $k$ odd and $k > 1$, we have

$$\zeta_E(s)\left(1-k^{-s}\right) = \frac{1}{2} \sum_{h=1}^{k-1} \frac{(-1)^{h-1}}{h^s} + \sum_{r=1}^{\infty} \frac{\binom{-s}{2r}}{k^{s+2r}} \frac{\zeta_E(s+2r) E_{2r}(k)}{2}.$$

2. For $k$ odd and $k > 1$, we have

$$\sum_{h=1}^{k-1} \frac{(-1)^h}{h^s} = \sum_{r=0}^{\infty} \frac{\binom{-s}{2r+1}}{k^{s+2r+1}} \left(E_{2r+1}(k) + E_{2r+1}(0)\right).$$

**Theorem 1.4** Let $\mu$ be the Möbius function. Then for $k$ odd and $k > 1$, we have

$$\zeta_E(s) \sigma_d \mu d^{-s} = 2 \sum_{r=0}^{\infty} \frac{\binom{-s}{2r}}{k^{s+2r}} \zeta_E(s+2r) k^{-2r+s} H(2r, k) - H(-s, k),$$

where

$$H(\alpha, k) = \sum_{\substack{h=1 \\ (h, k)=1}}^{\left[\frac{k}{2}\right]} (-1)^h h^\alpha \quad (\alpha \in \mathbb{C})$$

is the alternating sum of the $\alpha$-th power of those integers not exceeding $\left[\frac{k}{2}\right]$ that are relatively prime to $k$.

**Remark 1.5** The evaluations of series involving Riemann zeta function $\zeta(s)$ and related functions have a long history that can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783) (see, for details, [26, Chapter 3]). Ramaswami [24] presented numerous interesting recursion formulas that can be employed to get the analytic continuation of Riemann zeta function $\zeta(s)$ over the whole complex plane. Apostol [2] also gave some formulas involving the Riemann zeta function $\zeta(s)$; some of them are generalizations of Ramaswami’s identities. For more results, we refer to, e.g., Apostol [2], Choi and Srivastava [26], Landau [18], Murty and Reece [23], Ramaswami [24], and Srivastava [25].

### 2. Proof Theorem 1.1

To derive Theorem 1.1, we need the following lemmas.

In this section, we first present the Boole summation formula as follows:
Lemma 2.1 ([8, Boole summation formula]) Let $\alpha, \beta$, and $l$ be integers such that $\alpha < \beta$ and $l > 0$. If $f^{(l)}(t)$ is absolutely integrable over $[\alpha, \beta]$, then

$$2 \sum_{n=\alpha}^{\beta-1} (-1)^n f(n) = \sum_{r=0}^{l-1} \frac{E_r(0)}{r!} \left( (-1)^{\beta-1}(\beta) + (-1)^{\alpha}(\alpha) \right)$$

$$+ \frac{1}{(l-1)!} \int_{\alpha}^{\beta} E_{l-1}(-t) f^{(l)}(t) dt,$$

where $E_n(t)$ is the $n$-th quasi-periodic Euler functions defended by (6) and (8).

Remark 2.2 The alternating version of Euler–MacLaurin summation formula is the Boole summation formula (see, for example, [8, Theorem 1.2] and [21, 24.17.1–2]), which is proved by Boole [5], but a similar one may be known by Euler as well (see [22]). Recently, Can and Dağılı derived a generalization of the above Boole summation formula involving Dirichlet characters (see [8, Theorem 1.3]).

A proof of Lemma 2.1 can be found, for example, in [6, Section 5] and [8, Theorem 1.3].

Using the Boole summation formula (see Lemma 2.1 above), we obtain the following formula.

Lemma 2.3 The integral representation

$$\zeta_E(-u, a) = \frac{1}{2} \sum_{r=0}^{l-1} \binom{u}{r} E_r(0) a^{u-r}$$

$$+ \frac{1}{2(l-1)!} \frac{\Gamma(u+1)}{\Gamma(a+1-l)} \int_0^{\infty} E_{l-1}(-t)(t+a)^{u-1} dt,$$

holds true for all complex numbers $u$ and $\text{Re}(a) > 0$, where $l$ is any natural number subject only to the condition that $l > \text{Re}(u)$.

Proof The proof from Lemma 2.1 is exactly like the proof given by Can and Dağılı [8, Theorem 1.4] when $\chi = \chi_0$, where $\chi_0$ is the principal character modulo 1, and so we omit it.

Proof of Theorem 1.1 Putting $a = 1$ and $u = s$ in Lemma 2.3, by (3), we find that

$$2\zeta_E(-s) = \sum_{r=0}^{l-1} \binom{s}{r} E_r(0) + \frac{1}{(l-1)!} \frac{\Gamma(s+1)}{\Gamma(s+1-l)} \int_1^{\infty} E_{l-1}(1-t)t^{s-l} dt.$$  \hfill (9)

By Dirichlet’s test in analysis (e.g., [17, p. 333, Theorem 2.6]), the integral on the right-hand side of the above equation converges absolutely for $\text{Re}(s) < l$ and the convergence is uniform in every half-plane $\text{Re}(s) \leq l - \delta$, $\delta > 0$, and so $\zeta_E(-s)$ is an analytic function of $s$ in the half-plane $\text{Re}(s) < l$. Since

$$E_{l-1}(1-t) = (-1)^{l-1} E_{l-1}(t) \quad (t \in \mathbb{R})$$  \hfill (10)
(see [8, (2.7)] with \( \chi = \chi_0 \) and [13, (2.7)]), for \( \Re(s) > l - 1 \), we have
\[
\int_0^1 \mathcal{E}_{l-1}(1-t)t^{s-l}dt = (-1)^{l-1} \int_0^1 \mathcal{E}_{l-1}(t)t^{s-l}dt
\]
\[
= (-1)^{l-1} \int_0^1 \mathcal{E}_{l-1}(t)t^{s-l}dt
\]
\[
= (-1)^{l-1} \sum_{m=0}^{l-1} \binom{l-1}{m} E_m(0) \frac{1}{s-m},
\]
and thus the expression
\[
\frac{1}{(l-1)!} \Gamma(s+1-l) \int_0^1 \mathcal{E}_{l-1}(1-t)t^{s-l}dt = \sum_{k=0}^{l-1} \binom{s}{k} E_k(0),
\]
is valid for \( \Re(s) > l - 1 \). Therefore by (9) and (12), for \( l - 1 < \Re(s) < l \), we have
\[
2\zeta_E(-s) = \frac{1}{(l-1)!} \Gamma(s+1-l) \int_0^1 \mathcal{E}_{l-1}(1-t)t^{s-l}dt
\]
\[
= (-1)^{l-1} \frac{1}{(l-1)!} \Gamma(s+1-l) \int_1^\infty \frac{\mathcal{E}_{l-1}(t)}{t^{s-l}}dt.
\]
Replacing \( s \) by \( s + l - 1 \) in (13), for \( 0 < \Re(s) < 1 \), we have
\[
\int_0^\infty \mathcal{E}_{l-1}(t)t^{s-1}dt = \frac{2(-1)^{l-1}(l-1)!\Gamma(s)}{\Gamma(s+l)} \zeta_E(1-s-l).
\]
Finally, by Mellin’s inversion formula (see, e.g., [11, p. 49] and [19, p. 1127]), we obtain
\[
\mathcal{E}_{l-1}(t) = 2(-1)^{l-1}(l-1)! \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s)}{\Gamma(s+l)} \zeta_E(1-s-l)t^{-s}ds,
\]
where \((c)\) denotes the vertical straight line from \( c - i\infty \) to \( c + i\infty \) with \( 0 < c < 1 \) and \( t > 0 \). Thus the proof of Theorem 1.1 is completed.

3. Proofs of Theorem 1.3 and Theorem 1.4
In this section, we prove Theorem 1.3 and Theorem 1.4 by a method similar to that used by Apostol in [2].

First we need the following lemmas.

**Lemma 3.1** Let \( a \) be a complex number with a positive real part. The Hurwitz-type Euler zeta function satisfies the following:

1. Difference equation: For \( k \in \mathbb{N} \),
\[
(-1)^{k-1}\zeta_E(s, a + k) + \zeta_E(s, a) = \sum_{h=0}^{k-1} (-1)^h (a + h)^{-s}.
\]
2. Distribution relation: For an odd positive integer $k$,

$$\zeta_E(s, ka) = k^{-s} \sum_{r=0}^{k-1} (-1)^r \zeta_E \left( s, a + \frac{r}{k} \right).$$

**Proof** From the definition of $\zeta_E(s, a)$, it is easy to show that $\zeta_E(s, a + 1) + \zeta_E(s, a) = a^{-s}$. We can rewrite this identity as

$$\zeta_E(s, a + h + 1) + \zeta_E(s, a + h) = (a + h)^{-s}, \quad (14)$$

where $h \in \mathbb{N}_0$. Taking the alternating sum on both sides of the above identity as $h$ ranges from 0 to $k - 1$, we have

$$(-1)^{k-1} \zeta_E(s, a + k) + \zeta_E(s, a) = \sum_{h=0}^{k-1} (-1)^h (a + h)^{-s},$$

which completes the proof of Part 1.

Part 2 can be derived directly from the definition of $\zeta_E(s, a)$ (see (1) above).

**Lemma 3.2** The following identities hold:

1. Let $a \in \mathbb{R}$ and $a > 0$. Then

$$\zeta_E(s, x + a) = \sum_{r=0}^{\infty} \binom{-s}{r} \zeta_E(s + r, a)x^r, \quad |x| < a,$$

in which we understand $0^0 = 1$ if $r = 0$, and $0^r = 0$ otherwise.

2. Let $|x| < a + 1$ with $a \in \mathbb{R}$ and $a > 0$. Then

$$\zeta_E(s, a + 1 - x) = \sum_{r=0}^{\infty} (-1)^{r-1} \binom{-s}{r} \{\zeta_E(s + r, a) - a^{-s-r}\} x^r.$$

**Remark 3.3** Part 1 of Lemma 3.2 (and then (4.8) and (4.9) below) is a special case of [23, Theorem 2.4]. Part 2 of Lemma 3.2, when $a = 1$, is similar to Eq. (18) in a 2001 book by Srivastava and Choi [26, p. 147].

**Proof of Lemma 3.2** Note that for $|x| < a$

$$\zeta_E(s, x + a) - \zeta_E(s, a) = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{(n + x + a)^s} - \frac{1}{(n + a)^s} \right\}. \quad (15)$$

Writing the summand as

$$\frac{1}{(n + x + a)^s} - \frac{1}{(n + a)^s} = \frac{1}{(n + a)^s} \left( \left( 1 + \frac{x}{n + a} \right)^{-s} - 1 \right)$$

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and using the binomial theorem,

\[
\frac{1}{(n + x + a)^s} - \frac{1}{(n + a)^s} = \frac{1}{(n + a)^s} \left( \sum_{r=0}^{\infty} \binom{-s}{r} \left( \frac{x}{n + a} \right)^r - 1 \right)
\]

\[
= \frac{1}{(n + a)^s} \sum_{r=1}^{\infty} \binom{-s}{r} \left( \frac{x}{n + a} \right)^r.
\]

(16)

The right side of (15), by (16), is

\[
\sum_{r=1}^{\infty} \binom{-s}{r} x^r \sum_{n=0}^{\infty} \binom{(-1)^n}{n+a} \sum_{r=1}^{\infty} \binom{-s}{r} \zeta_E(s + r, a)x^r,
\]

(17)

where \(a > 0\). By using (15) and (17), we obtain the first part.

For the second part, note that from the binomial theorem we have

\[
(a - x)^{-s} = a^{-s} (1 - \frac{x}{a})^{-s} = a^{-s} \sum_{r=0}^{\infty} \binom{-s}{r} \left( \frac{x}{a} \right)^r
\]

(18)

for \(|x| < a\). Setting \(h = 0\) and replacing \(a\) by \(a - x\) in (14), we get

\[
\zeta_E(s, a - x + 1) + \zeta_E(s, a - x) = (a - x)^{-s}.
\]

(19)

If we replace \(x\) by \(-x\) in Part 1 and use (18) and (19), we get

\[
\sum_{r=0}^{\infty} (-1)^r \binom{-s}{r} \{ \zeta_E(s + r, a) - a^{-s-r} \} x^r = \zeta_E(s, a - x) - (a - x)^{-s}
\]

\[
= -\zeta_E(s, a + 1 - x).
\]

Thus the result follows.

**Lemma 3.4** Suppose \(k\) is an odd positive integer. Then we have

\[
\zeta_E(s) (1 - k^{-s}) = \sum_{r=1}^{\infty} (-1)^r \binom{-s}{r} \frac{\zeta_E(s + r) E_r(k) + E_r(0)}{k^{s+r}}.
\]

**Proof** Suppose \(k\) is an odd positive integer. If we take \(a = 1\) and \(x = -h/k, 0 \leq h \leq k - 1\) in Part 1 of Lemma 3.2, multiply by \((-1)^h\), and sum over \(h\), then we have

\[
\sum_{h=0}^{k-1} (-1)^h \zeta_E \left( s, 1 - \frac{h}{k} \right) = \sum_{r=0}^{\infty} (-1)^r \binom{-s}{r} \frac{\zeta_E(s + r) k^r}{k^{s+r}} \sum_{h=0}^{k-1} (-1)^h h^r,
\]

(20)

in which we understand \(0^r = 1\) if \(r = 0\), and \(0^r = 0\) otherwise. Note that for an odd positive integer \(k\) we have

\[
\left\{ 1, 1 - \frac{1}{k}, \ldots, 1 - \frac{k-1}{k} \right\} = \left\{ 1, \frac{2}{k}, \ldots, \frac{k-1}{k} \right\}.
\]

(21)
If we put \( a = 1/k \) in Part 2 of Lemma 3.1 and use (21), we get

\[
\sum_{h=0}^{k-1} (-1)^h \zeta_E \left( s, 1 - \frac{h}{k} \right) = \sum_{h=0}^{k-1} (-1)^h \zeta_E \left( s, \frac{1}{k} + \frac{h}{k} \right) = k^s \zeta_E(s, 1) = k^s \zeta_E(s).
\]

Hence, by (20) and (22), we have

\[
\zeta_E(s) = \sum_{r=0}^{\infty} (-1)^r \left( \frac{-s}{r} \right) \zeta_E(s + r) \sum_{h=0}^{k-1} (-1)^h h^r
\]

\[
= \zeta_E(s)k^{-s} + \sum_{r=1}^{\infty} (-1)^r \left( \frac{-s}{r} \right) \frac{\zeta_E(s + r)}{k^{s+r}} \sum_{h=0}^{k-1} (-1)^h h^r
\]

for odd \( k \). Moreover, it is easily seen that

\[
\sum_{h=0}^{k-1} (-1)^h h^r = \frac{E_r(k) + E_r(0)}{2} \quad \text{for odd } k
\]

(see [21, Equation 24.4.8] and [27, Theorem 2.1]). Thus, the proof is completed by (23) and (24).

Lemma 3.5 Suppose \( k \) is an odd positive integer with \( k > 1 \). Then we have

\[
\zeta_E(s) \left( 1 - k^{-s} \right) = \sum_{h=1}^{k-1} \frac{(-1)^{h-1}}{h^s} + \sum_{r=1}^{\infty} \left( \frac{-s}{r} \right) \frac{\zeta_E(s + r)}{k^{s+r}} \frac{E_r(k) + E_r(0)}{2}.
\]

Proof Suppose \( k \in \mathbb{N} \). If we take \( a = 1 \) and \( x = h/k, 0 \leq h \leq k-1 \) in Part 1 of Lemma 3.2, multiply by \((-1)^h\), and sum over \( h \), then we have

\[
\sum_{h=0}^{k-1} (-1)^h \zeta_E \left( s, 1 + \frac{h}{k} \right) = \sum_{r=0}^{\infty} \left( \frac{-s}{r} \right) \frac{\zeta_E(s + r)}{k^r} \sum_{h=0}^{k-1} (-1)^h h^r
\]

\[
= \sum_{r=1}^{\infty} \left( \frac{-s}{r} \right) \frac{\zeta_E(s + r)}{k^r} \frac{E_r(k) + E_r(0)}{2} + \zeta_E(s).
\]

Now, setting \( a = 1 \) in Part 1 of Lemma 3.1, we obtain

\[
(-1)^{k-1} \zeta_E(s, k + 1) + \zeta_E(s, 1) = \sum_{h=0}^{k-1} (-1)^h (h + 1)^{-s} \quad (k \in \mathbb{N}),
\]

which is equivalent to

\[
(-1)^k \zeta_E(s, k) + \zeta_E(s) = \sum_{h=1}^{k-1} (-1)^{h-1} h^{-s}
\]
for $k \geq 2$. We set $a = 1$ in Part 2 of Lemma 3.1 and use (26); then the first term of (25) equals

$$\sum_{h=0}^{k-1} (-1)^h \zeta_E \left( s, 1 + \frac{h}{k} \right) = k^s \zeta_E(s, k)$$

$$= k^s \left( \zeta_E(s) - \sum_{h=1}^{k-1} (-1)^{h-1} h^{-s} \right)$$

(27)

for odd $k > 1$, and so by combining (25) and (27) we obtain the result.

Now we give proofs of Theorem 1.3 and Theorem 1.4, respectively.

**Proof of Theorem 1.3** It needs to be noted that

$$E_k(0) = 0$$

if $k$ is even ([27, p. 5, Corollary 1.1(ii)]). Using the above identity, adding Lemma 3.4 and Lemma 3.5, we obtain Part 1 of Theorem 1.3. Subtracting Lemma 3.5 from Lemma 3.4, we have Part 2 of Theorem 1.3.

**Proof of the Theorem 1.4** For $\alpha \in \mathbb{C}$, we introduce the alternating sum

$$H(\alpha, k) = \sum_{(h,k)=1}^{\left[ \frac{k}{2} \right]} (-1)^h h^\alpha.$$

From now on, let $k$ denote an odd integer and $k \geq 1$. By taking $a = 1$ and $x = h/k, (h,k) = 1$ in Part 1 of Lemma 3.2, $1 \leq h \leq \left[ \frac{k}{2} \right]$, multiplying by $(-1)^h$, and summing over $h$, we obtain

$$\sum_{h=1}^{\left[ \frac{k}{2} \right]} (-1)^h \zeta_E \left( s, 1 + \frac{h}{k} \right) = \sum_{r=0}^{\infty} \left( -\frac{s}{r} \right) \zeta_E(s + r) k^{-r} H(r, k).$$

(28)

Similarly, we have

$$\sum_{h=1}^{\left[ \frac{k}{2} \right]} (-1)^{h-1} \zeta_E \left( s, 1 - \frac{h}{k} \right) = -\sum_{r=0}^{\infty} (-1)^r \left( -\frac{s}{r} \right) \zeta_E(s + r) k^{-r} H(r, k).$$

(29)

Setting $h = 0$ in (14), the left-hand side of (28) equals

$$- \sum_{h=1}^{\left[ \frac{k}{2} \right]} (-1)^h \zeta_E \left( s, \frac{h}{k} \right) + k^s H(-s, k).$$

(30)
If \( k \) is odd, \((k - 1)/2\) is an integer and so we get

\[
\frac{k - 1}{2} = \left\lfloor \frac{k}{2} \right\rfloor \Leftrightarrow \frac{k}{2} = \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \\
\quad \Leftrightarrow k = 2 \left\lfloor \frac{k}{2} \right\rfloor + 1 \\
\quad \Leftrightarrow k - \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor + 1.
\]

(31)

Hence

\[
1 - \frac{\left\lfloor \frac{k}{2} \right\rfloor + 1}{k} = 1 - \frac{\left\lfloor \frac{k}{2} \right\rfloor - 1}{k} = \frac{\left\lfloor \frac{k}{2} \right\rfloor + 2}{k} = \ldots = 1 - \frac{1}{k} = \frac{k - 1}{k},
\]

which leads easily to the required

\[
- \sum_{h=1 \atop (h,k)=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^h \xi_E \left( s, 1 - \frac{h}{k} \right) = \sum_{h=\left\lfloor \frac{k}{2} \right\rfloor + 1 \atop (h,k)=1}^{k} (-1)^h \xi_E \left( s, \frac{h}{k} \right),
\]

that is,

\[
\sum_{h=1 \atop (h,k)=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^h \left\{ \xi_E \left( s, \frac{h}{k} \right) - \xi_E \left( s, 1 - \frac{h}{k} \right) \right\} = \sum_{h=1 \atop (h,k)=1}^{k} (-1)^h \xi_E \left( s, \frac{h}{k} \right).
\]

(32)

Now subtracting (28) from (29), from (30) and (32), we have

\[
\sum_{h=1 \atop (h,k)=1}^{k} (-1)^h \xi_E \left( s, \frac{h}{k} \right) = k^s H(-s,k) - 2 \sum_{r=0}^{\infty} \left( \frac{-s}{2r} \right) \xi_E(s + 2r)k^{-2r} H(2r,k).
\]

(33)

By the definition of the Möbius functions, for \( n \in \mathbb{N} \), we have

\[
\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}
\]

(see [2, p. 25, Theorem 2.1]). Recalling from Part 2 of Lemma 3.1 that

\[
\xi_E(s,ka) = k^{-s} \sum_{r=0}^{k-1} (-1)^r \xi_E \left( s, a + \frac{r}{k} \right),
\]

(34)

and letting \( a = 1/k \) in (34), we obtain

\[
\xi_E(s) = k^{-s} \sum_{r=0}^{k-1} (-1)^r \xi_E \left( s, \frac{r + 1}{k} \right) = k^{-s} \sum_{r=1}^{k} (-1)^{r-1} \xi_E \left( s, \frac{r}{k} \right),
\]

(35)
where \( k \) is odd. Hence the left-hand side of (33) may be rewritten as
\[
\sum_{h=1 \atop (h,k)=1}^{k} (-1)^h \zeta_E \left( s, \frac{h}{k} \right) = \sum_{h=1}^{k} (-1)^h \sum_{d|h,k} \mu(d) \zeta_E \left( s, \frac{h}{k} \right) \\
= \sum_{h=1}^{k} (-1)^h \sum_{d|h,k} \mu(d) \zeta_E \left( s, \frac{h}{k} \right) \\
= \sum_{d|k} \mu(d) \sum_{m=1}^{k/d} (-1)^m E \left( s, \frac{md}{k} \right) \\
= \sum_{d|k} \mu(d) \sum_{m=1}^{k/d} (-1)^m \zeta_E \left( s, \frac{m}{k/d} \right) \\
= -k^s \zeta_E(s) \sum_{d|k} \mu(d)d^{-s},
\]
since \( d \) is odd in the case \( k \) is odd. Thus, by combining (33) and (36), the proof of Theorem 1.4 is completed.

4. Some further identities

In the spirit of Euler, by working with the formal power series, we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \zeta_E(-n) t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} (-1)^k k^n \right) \frac{(-1)^n t^n}{n!} \\
= \sum_{k=1}^{\infty} (-1)^k \left( \sum_{n=0}^{\infty} \frac{(-kt)^n}{n!} \right).
\]

The last term of (37) converges to
\[
-\frac{1}{e^t + 1},
\]
Thus, directly from definition (4), (38) may be written
\[
-\frac{1}{e^t + 1} = -\frac{1}{2} \sum_{n=0}^{\infty} E_n(0) \frac{t^n}{n!}.
\]
Applying the reflection formula of Euler polynomials (see [21, 24.4.4]):
\[
E_n(1-x) = (-1)^n E_n(x),
\]
with \( x = 0 \), by (37), (38), and (39), we obtain
\[
\zeta_E(-n) = \frac{(-1)^n}{2} E_n(0) = \frac{1}{2} E_n(1)
\]
for \( n \in \mathbb{N}_0 \), which imply \( \zeta_E(-1) = 1/4, \zeta_E(-2) = 0, \zeta_E(-3) = -1/8, \ldots \) (see [10, p. 520, Corollary 3]). The following identity involving Euler polynomials

\[
E_n(x) = 2x^n - \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_{n-r}(0) x^r \quad (n \in \mathbb{N}_0)
\]  

follows from the known formula (see [11, p. 41, (6)] and [21, 24.4.2])

\[
E_n(x + 1) + E_n(x) = 2x^n \quad (n \in \mathbb{N}_0),
\]

in the case we replace \( E_n(x + 1) \) by \( \sum_{r=0}^{n} \binom{n}{r} E_{n-r}(1) x^r \) in (43), then set \( x = 0 \), and replace \( n \) by \( n - r \) in (40).

Putting \( a = 1 \) and \( s = -n \) in Part 1 of Lemma 3.2, we obtain the result

\[
\zeta_E(-n, x + 1) = \sum_{r=0}^{n} \binom{n}{r} \zeta_E(r - n) x^r, \quad |x| < 1.
\]  

Next setting \( a = x, s = -n \), and \( h = 0 \) in (14), we have

\[
\zeta_E(-n, x + 1) + \zeta_E(-n, x) = x^n.
\]

Combining (44) and (45), we have

\[
\zeta_E(-n, x) = x^n - \sum_{r=0}^{n} \binom{n}{r} \zeta_E(r - n) x^r,
\]

and by (41) and (42), we have

\[
\zeta_E(-n, x) = \frac{1}{2} \left( 2x^n - \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_{n-r}(0) x^r \right) = \frac{1}{2} E_n(x)
\]

for \( n \in \mathbb{N}_0 \) (see [10, p. 520, (3.20)], [16, p. 4, (1.22)], and [29, p. 41, (3.8)]).

For \( a = 1 \), Part 2 of Lemma 3.2 yields

\[
\zeta_E(s, 2 - x) = \sum_{r=0}^{\infty} (-1)^{r-1} \binom{-s}{r} \left\{ \zeta_E(s + r) - 1 \right\} x^r,
\]

where \(|x| < 2\) (cf. [26, p. 146, (18)]). Replacing the summation index \( r \) in (47) by \( r + 1 \), and setting \( x = 1 \), we arrive immediately at an analogue form of (2.3) in [25]:

\[
\sum_{r=1}^{\infty} (-1)^r \binom{-s}{r} \left\{ \zeta_E(s + r) - 1 \right\} + 2\zeta_E(s) = 1.
\]
Letting $x = -1$ in (47) and using (14) with $a = 1, 2$ and $h = 0$, that is, $\zeta_E(s, 3) = \zeta_E(s) + 1/2^s - 1$, we find that

$$\zeta_E(s) = 1 - \frac{1}{2^{s+1}} - \frac{1}{2} \sum_{r=1}^{\infty} \binom{-s}{r} \{\zeta_E(s + r) - 1\},$$

(49)

which provides a companion of Landau's formula (see [18, p. 274, (3)] and [28, p. 33, (2.14.1)]). Setting $x = 1/2$ in (47), and using (14) with $a = 1/2$ and $h = 0$, that is, $\zeta_E(s, 3/2) + \zeta_E(s, 1/2) = 2^s$, we obtain a series representation for $\beta(s)$:

$$\beta(s) = 1 + \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r + s} \binom{-s}{r} \{\zeta_E(s + r) - 1\}$$

$$= 1 + \sum_{r=0}^{\infty} \frac{1}{2^{r+s}} \binom{s + r - 1}{r} \{\zeta_E(s + r) - 1\},$$

(50)

where $\beta(s)$ denotes the Dirichlet beta function defined by (see [1, p. 807, 23.2.21])

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s}.$$  

The above series converges for all $\text{Re}(s) > 0$. Setting $s = 2$ in (50), we deduce

$$\text{Catalan’s constant } G = \beta(2) = 1 + \sum_{r=1}^{\infty} \frac{r}{2^r + 1} \{\zeta_E(r + 1) - 1\},$$

(51)

which is one of the basic constants whose irrationality and transcendence (though strongly suspected) remain unproven (cf. [26, p. 29, (16)]).

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References


