Reflexivity of vector-valued Köthe–Orlicz sequence spaces

Mohamed Ahmed OULD SIDATY*

Department of Mathematics and Statistics, Faculty of Science, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

Received: 28.11.2016 ● Accepted/Published Online: 25.08.2017 ● Final Version: 08.05.2018

Abstract: Let $E$ be a Banach space, $\lambda$ a perfect sequence space, and $M$ an Orlicz function. Denote by $\lambda(E, M)_w$ the space of all weakly $(M, \lambda)$-summable sequences from $E$ that are the limit of their finite sections. In this paper, we describe the continuous linear functionals on $\lambda(E, M)_w$ in terms of strongly $(N, \lambda^*)$-summable sequences in the dual $E^*$ of $E$, and then we give a characterization of the reflexivity of $\lambda(E, M)$ in terms of that of $\lambda$ and of $E$ and the AK-property.

Key words: Banach spaces, vector-valued sequence spaces, Orlicz function, duality

1. Introduction

In connection with the nuclearity of a locally convex space $E$, Pietsch [13] introduced the spaces $\ell_p(E)$ and $\ell_p(E)$ respectively of weakly $\ell_p$-summable and absolutely $\ell_p$-summable sequences in $E$. This allowed him also to introduce and study absolutely $p$-summing operators. He introduced and studied also the spaces $\lambda(E)$ and $\lambda(E)$ of $\lambda$-summable and weakly $\lambda$-summable sequences in $E$, $\lambda$ being a perfect sequence space in the sense of Köthe endowed with its normal topology.

Later, Rosier considered in [14] the general case where $\lambda$ is no longer equipped with the normal topology, but with a general polar one. He obtained many results, among them a complete description of the dual space of $\lambda(E)$. Florencio and Paul [3] and [4] considered a general polar topology on $\lambda$ and obtained interesting results on $\lambda(E)$. In particular, using the AK property, they represent the elements of the completion $\lambda \bar{\otimes}_e E$ of the injective tensor product $\lambda \otimes_e E$ as weakly $\lambda$– summable sequences in $E$.

In [10], the authors extend to the locally convex setting the definition of the strong summability introduced first by Cohen [1] in the case when $E$ is a normed space. They made use of this notion to describe the continuous dual space of $\lambda(E)$. Many other results on $\lambda(E)$ have been obtained in [11], [9], and [12].

Ghosh and Srivastava in [5] deal with an Orlicz function $M$ to extend the notion of absolute $\lambda$– summability. They introduce and study the space $F(E, M)$ of those sequences $(x_n)_n$ in a Banach space $E$ for which $(M(||x_n||_E/\rho))_n \in F$, for some $\rho > 0$, where $F$ is a normal sequence space. In this paper, we introduce the space $\lambda(E, M)$ of all weakly $(M, \lambda)$-summable sequences $(x_n)_n$ from a Banach space $E$; that is $(\alpha_n a(x_n))_n \in \ell_M$, for all $(\alpha_n) \in \lambda^*$ and $a \in E^*$, where $\ell_M$ is the Orlicz sequence space associated with the Orlicz function $M$. For $M(t) = t$ the spaces $\lambda(E, M)$ and $\lambda(E)$ coincide.

*Correspondence: sidaty1@hotmail.com
2010 AMS Mathematics Subject Classification: 46A17, 46B35, 46A45
2. Definitions and preliminaries

In the sequel, if $V$ is a normed space then $V^*$, $\| \cdot \|_{V^*}$, and $B_{V^*}$ will denote respectively the topological dual, the norm, and the closed unit ball of $V$.

Let $\omega$ denote the vector space of all real or complex sequences for the usual coordinate operations. For all $k \in \mathbb{N}$, $e_k$ will denote the $k$-th unit vector of $\omega$. We mean by a sequence space any linear subspace of $\omega$.

If $\lambda$ is a sequence space, we denote by $\lambda^*$ its $\alpha$-dual defined by

$$\lambda^* = \left\{ (\beta_n) \in \omega : \sum_{n=1}^{\infty} |\alpha_n \beta_n| \text{ converges, for all } (\alpha_n)_n \in \lambda \right\}.$$  

We see that $\lambda \subset \lambda^{**} = (\lambda^*)^*$, and $\lambda$ is said to be perfect if $\lambda = \lambda^{**}$. Throughout this paper, $\lambda$ stands for a Banach perfect sequence space whose norm $\| \cdot \|_{\lambda}$ satisfies

(1) for all $\alpha$ and $\beta$ in $\lambda$, if $\alpha \leq \beta$ then $\|\alpha\|_{\lambda} \leq \|\beta\|_{\lambda}$.

(2) $\lambda$ is an AK-space. This means that every $(\alpha_n)_n \in \lambda$ is the $\|\cdot\|_{\lambda}$-limit of its finite sections $(\alpha_1, \ldots, \alpha_n, 0, \ldots)$, $n \in \mathbb{N}$.

In this case the topological dual of $\lambda$ coincides with its $\alpha-$dual. The norm of $\lambda^*$ is then defined by

$$\|\beta\|_{\lambda^*} = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n \beta_n|, \alpha \in \lambda \text{ and } \|\alpha\|_{\lambda} \leq 1 \right\}.$$  

We assume moreover that $(\lambda^*, \| \cdot \|_{\lambda^*})$ is also an AK-space. In particular, $\lambda$ is a reflexive Banach space.

An Orlicz function is a continuous, convex, nondecreasing function $M$ defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \mu(t)dt.$$  

Define, for $s \geq 0$,

$$\nu(s) = \sup\{ t : \mu(t) \leq s \}.$$  

Then $\nu$ possesses the same properties as $\mu$ and the function $N$ defined by

$$N(x) = \int_0^x \nu(t)dt$$

is an Orlicz function. The functions $M$ and $N$ are called mutually complementary Orlicz functions and satisfy the Young inequality,

$$uv \leq M(u) + N(v), \text{ for } u, v \geq 0. \quad (2.1)$$

The Orlicz sequence space $\ell_M$, introduced in [8], is defined by
\[ \ell_M = \left\{ (\alpha_n) \in \omega, \sum_{n=1}^{\infty} M \left( \frac{|\alpha_n|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}. \]

\( \ell_M \) is a Banach space with respect to the norm

\[ \| (\alpha_n) \|_M = \inf \left\{ \rho > 0, \sum_{n=1}^{\infty} M \left( \frac{|\alpha_n|}{\rho} \right) \leq 1 \right\}. \]

For \( M(t) = t^p \), \( 1 \leq p < \infty \), the space \( \ell_M \) coincides with the classical sequence space \( \ell_p \).

An Orlicz function \( M \) satisfies the condition that \( M(\eta x) \leq \eta M(x) \), if \( 0 \leq \eta \leq 1 \). It is said to satisfy \( \Delta_2 \)-condition if there exists a constant \( K > 0 \), such that, for every \( x \geq 0 \), \( M(2x) \leq KM(x) \). In this case, \( \ell^*_M = \ell_N \) (see e.g. [6], Corollary 4.2).

3. The space \( \lambda(E,M) \)

Let \( E \) stand for a Banach space and \( \omega(E) \) denote the linear space of all \( E \)-valued sequences. Define the space \( \lambda(E,M) \) of weakly \((M,\lambda)-\)summable sequences of \( E \) by

\[ \lambda(E,M) = \{ x = (x_n)_n \subset E : \text{ for all } a \in E^*, (\alpha_n)_n \in \lambda^*, (\alpha_n a(x_n)) \in \ell_M \}. \]

We have

**Theorem 3.1** With the usual coordinate operations, \( \lambda(E,M) \) is a vector space on which

\[ \| x \|_{\lambda(E,M)} = \sup \left\{ \| (\alpha_n a(x_n)) \|_M : a \in B_{E^*}, \alpha \in B_{\lambda^*} \right\} \]

\[ = \sup_{a \in B_{E^*}, \alpha \in B_{\lambda^*}} \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M(|\alpha_n a(x_n)|/\rho) \leq 1 \right\}, \]

for \( x = (x_n)_n \in \lambda(E,M) \), defines a norm.

**Proof** For all \( a \in E^* \) and \( (\alpha_n)_n \in \lambda^* \), define \( \varphi_{a,\alpha} : \omega(E) \to \omega \) by \( \varphi_{a,\alpha}(x) = (\alpha_n a(x_n)) \), for all \( x = (x_n)_n \in \lambda(E,M) \).

It is clear that \( \varphi_{a,\alpha} \) is linear and that

\[ \lambda(E,M) = \bigcap_{(a,\alpha) \in E^* \times \lambda^*} \varphi_{a,\alpha}^{-1}(\ell_M). \]

Thus, \( \lambda(E,M) \) is a vector space.

For the second part of the theorem, we prove only that \( \| x \|_{\lambda(E,M)} \) is finite. Fix \( x = (x_n)_n \in \lambda(E,M) \) and define the family of linear mappings \( f_a : E^* \to \ell_M \) by \( f_a(a) = (\alpha_n a(x_n)) \), for \( a = (\alpha_n) \in B_{\lambda^*} \). Then \( f_a \) is linear and continuous by the closed graph theorem. The same argument shows that \( g_a(\alpha) = f_a(a) : \lambda^* \to \ell_M \) is continuous for every \( a \in E^* \). On the other hand, for every \( a \in E^* \),

\[ \sup_{\alpha \in B_{\lambda^*}} \| f_a(a) \|_M = \sup_{\alpha \in B_{\lambda^*}} \| g_a(\alpha) \|_M = \| g_a \|_{\mathcal{L}(\lambda^*,\ell_M)} < \infty. \]
By the uniform boundedness principle, 
\[
\sup \{ (\alpha_n a(x_n)) \|M \) : a \in B_{E^*}, \alpha \in B_{\lambda^*} \} = \sup_{a \in B_{\lambda^*}} \| f_\alpha \|_{\mathcal{L}(E^*, \tau_M)} < \infty,
\]
and \( \|x\|_{\lambda(E,M)} \) is finite. \( \square \)

We establish now the continuity of the projections.

**Lemma 3.2** For \( k \in \mathbb{N} \), let \( \pi_k \) denote the projection from \( \lambda(E,M) \) on \( E \) defined by
\[
\pi_k(x) = x_k, \text{ for all } x = (x_n) \in \lambda(E,M).
\]

Then \( \pi_k \) is linear and continuous.

**Proof** Fix \( k \in \mathbb{N} \), \( a \in B_{E^*} \), and \( (\alpha_n)_n \in B_{\lambda^*} \) with \( \alpha_k > 0 \). Let \( \kappa = 1/(\alpha_k e_k \|M \) \). For all \( x = (x_n) \in \lambda(E,M) \), we have
\[
\alpha_k |a(x_k)||e_k|_{\|M \) = \|\alpha_k a(x_k) e_k\|_{\|M \) \leq \|a(\alpha_n x_n)\|_{\|M \) \leq \|(x_n)_n\|_{\lambda(E,M)},
\]
Thus, \( \|x\|_{\|E \) \leq \kappa \|x\|_{\lambda(E,M)} \) for all \( x = (x_n) \in \lambda(E,M) \) and \( \pi_k \) is continuous. \( \square \)

**Theorem 3.3** The normed space \( \lambda(E,M) \) is complete and \( E \) is isomorphic to a closed linear subspace of it.

**Proof** Consider a nonzero \( \alpha = (\alpha_n)_n \in \lambda \). We will show that
\[
\|(\alpha_n t)_n\|_{\lambda(E,M)} \leq M(1) \|\alpha\|_{\|t\|_{\|E \)}} \text{ for all } \alpha = (\alpha_n)_n \in \lambda \text{ and } t \in E. \quad (3.1)
\]
The inequality is obvious if \( t = 0 \). Suppose that \( t \neq 0 \) and set \( \rho_0 = M(1) \|\alpha\|_{\|t\|_{\|E \)}} \). If \( \beta = (\beta_n)_n \in \lambda^* \) with \( \|\beta\|_{\lambda^*} \leq 1 \) and \( a \in E^* \) with \( \|a\|_{\lambda^*} \leq 1 \), then by the convexity of \( M \),
\[
\sum_{n=1}^\infty M \left( \frac{|\alpha_n \beta_n a(t)|}{\rho_0} \right) \leq \sum_{n=1}^\infty \frac{|\alpha_n \beta_n| a(t)}{\rho_0} M(1) \leq 1.
\]
Thus, \( \|(\beta_n \alpha_n a(t))_n\|_{\|M \) \leq \rho_0 \). However,
\[
\|(\alpha_n t)_n\|_{\lambda(E,M)} = \sup_{a \in B_{E^*}, \alpha \in B_{\lambda^*}} \inf \left\{ \rho > 0 : \sum_{n=1}^\infty M(|\alpha_n a(x_n)|/\rho) \leq 1 \right\}
\[
\leq \rho_0 = M(1) \|\alpha\|_{\|t\|_{\|E \)}}.
\]
For a fixed \( \gamma = (\gamma_n)_n \in \lambda \), with \( \gamma \neq 0 \) the mapping \( t \in E \to (\gamma_n t)_n \in \lambda(E,M) \) is well defined, injective, and continuous by (3.1). Let \( (t_k)_k \) be a sequence in \( E \) such that \( (\gamma t_k)_k \) converges in \( \lambda(E,M) \) to \( x = (x_n)_n \). For every \( m \in \mathbb{N} \) with \( \gamma_m \neq 0 \), the sequence \( (t_k)_k \) converges to \( \frac{1}{\gamma_m} x_m \), by Lemma 3.2. If \( t \) denotes the limit of \( (t_k)_k \) then \( x_n = t \) if \( \gamma_n \neq 0 \) and \( x_n = 0 \) otherwise, and so \( x = \gamma t \), and the range of \( E \) is closed in \( \lambda(E,M) \).

Let \( x^k = (x^k_n), k = 1, 2, \ldots \), be a Cauchy sequence in \( \lambda(E,M) \). For a fixed \( n \in \mathbb{N} \), by Lemma 3.2, \( x^k_n, k = 1, 2, \ldots \), is a Cauchy sequence in \( E \); let \( x_n \in E \) be its limit. We will prove that \( x = (x_n)_n \in \lambda(E,M) \).
and that \((x^k)_k\) converges to \(x\). Fix \(\alpha = (\alpha_n) \in \lambda^*\) and \(a \in E^*\). It is clear that the mapping \(\varphi_{\alpha,a} : y = (y_n) \in \lambda(E,M) \to (\alpha_n a(y_n)) \in \ell_M\) is linear and continuous. Thus, \(\varphi_{\alpha,a}(x^k) = (\alpha_n a(x^k_n)), k = 1, 2, \ldots,\) is a Cauchy sequence in the Banach space \(\ell_M\). Let \(\beta = (\beta_n)\) be its limit in \(\ell_M\). For every \(n \in \mathbb{N}\), we have

\[
\alpha_n a(x_n) = \alpha_n a \left( \lim_{k \to \infty} x_n^k \right) = \lim_{k \to \infty} \alpha_n a(x_n^k) = \beta_n.
\]

Hence, \((\alpha_n a(x_n)) = \beta \in \ell_M\). Thus, \(x \in \lambda(E,M)\). It remains to show that \((x^k)_k\) converges to \(x\).

For \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that, for all \(q \geq p \geq N\), \(\alpha = (\alpha_n) \in B_{\lambda^*}\) and \(a \in B_{E^*}\), there exists \(0 < \rho < \varepsilon\) that satisfies

\[
\sup_{K \in \mathbb{N}} \sum_{n=1}^{K} M(\|\alpha_n a(x_n^q - x_n^p)\|/\rho) = \sum_{n=1}^{\infty} M(\|\alpha_n a(x_n^q - x_n^p)\|/\rho) \leq 1.
\]

Since \(M\) is continuous, letting \(q \to \infty\), we get \(\sum_{n=1}^{K} M(\|\alpha_n a(x_n^p - x_n)\|/\varepsilon) \leq 1\) for \(K \geq N\); and then

\[
\|x^p - x\|_{\lambda(E,M)} = \sup_{a \in B_{E^*}, a \in B_{\lambda^*}} \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M(\|\alpha_n a(x_n^p - x_n)\|/\rho) \leq 1 \right\} \leq \varepsilon,
\]

for every \(p \geq N\). This completes the proof. \(\square\)

4. The space \(\lambda(E,M)\)

A sequence \((x_n)_n\) is said to be strongly \((M,\lambda)\)-summable in \(E\), if for every \((a_n)_n \in \lambda^*(E^*,N)\), one has \((a_n(x_n))_n \in \ell_1\). The space of these sequences will be denoted \(\lambda(E,M)\).

That is

\[
\lambda(E,M) = \{ x = (x_n)_n \subset E : \text{for all} \ a = (a_n)_n \in \lambda^*(E^*,N), (a_n(x_n))_n \in \ell_1 \}.
\]

If we endow \(\lambda(E,M)\) with the standard coordinate operations \(\lambda(E,M)\) is a vector space over \(\mathbb{K}\) that contains the finite sequences of \(E\). Indeed, if \(a = (a_n)_n \in \lambda^*(E^*,N)\), the map \(\varphi_a\) from \(\omega(E)\) into \(\omega\) defined by \(\varphi_a(x) = (a_n(x_n))\), for all \(x = (x_n) \in \lambda(E,M)\) is linear such that

\[
\lambda(E,M) = \bigcap_{a \in \lambda^*(E^*,N)} \varphi_a^{-1}(\ell_1).
\]

Although many properties of the spaces \(\lambda(E,M)\) and \(\lambda(E,M)\) are similar, the techniques of their proofs are different.

Next, we define a norm on \(\lambda(E,M)\).

**Theorem 4.1** For \(x = (x_n)_n \in \lambda(E,M)\) set

\[
\|x\|_{\lambda(E,M)} = \sup \left\{ \sum_{n=1}^{\infty} |a_n(x_n)| : a = (a_n)_n \in \lambda^*(E^*,N), \|a\|_{\lambda^*(E^*,N)} \leq 1 \right\}.
\]

Then \(\|x\|_{\lambda(E,M)}\) defines a norm on \(\lambda(E,M)\).
Theorem

sequence \((x_φ^f)\) and define the family of linear mappings \(φ_x : \lambda^*(E^*, N) \to Ŗ_1\) by \(φ_x(a) = (a_n(x_n)), \) for all \(a = (a_n) \in \lambda^*(E^*, N)\).

Then \(φ_x\) is linear and it is easy to check that the graph of \(φ_x\) is closed. As \(\lambda^*(E^*, N)\) is a Banach space by Theorem 3.3, \(φ_x\) is continuous. This proves that \(\|x\|\lambda(E,M)\) is finite.

The other properties of the norm derive from that of \(\| \cdot \|_M\) and the supremum.

Next, we establish the continuity of the projections.

Lemma 4.2 For \(k \in \mathbb{N}\), let \(π_k\) denote the projection from \(λ(E,M)\) on \(E\) defined by

\[π_k(x) = x_k, \text{ for all } x = (x_n) \in λ(E, M).\]

Then \(π_k\) is linear and continuous.

Proof Fix \(k \in \mathbb{N}\). Let \(x^* \in E^*\) such that \(\|x^*\|_{E^*} \leq 1\). Set

\[δ_k = \sup \{\|α_k e_k\|_N : α = (α_n) \in λ \text{ and } \|α\|_λ \leq 1\}.\]

Define \(a = 1/δ_k x^* e_k\). It is easy to check that \(a \in \lambda^*(E^*, N)\) with \(\|a\|_{\lambda^*(E^*, N)} \leq 1\) and \(|x^*(x_k)| \leq δ_k\|x\|_{λ(E,M)}\), for every \(x = (x_n) \in λ(E,M)\). Since this is satisfied for any \(x^* \in E^*\) such that \(\|x^*\|_{E^*} \leq 1\), we have

\[\|x_k\|_E \leq δ_k\|x\|_{λ(E,M)}, \text{ for every } x = (x_n) \in λ(E,M).\]

This shows the continuity of \(π_k\).

Theorem 4.3 The normed space \(λ(E,M)\) is complete and \(E\) is isomorphic to a closed linear subspace of it.

Proof Fix \(p \in \mathbb{N}\) and define \(θ_p : E \to λ(E,M)\) by \(θ_p(t) = t e_p\) for every \(t \in E\). It is clear that \(θ\) is linear and injective. Suppose that \(\|t\|_E < 1\) and choose \(α = (α_n) \in λ \) with \(\|α\|_λ \leq 1\) and \(α_p > 0\). Let \(κ = 1/\|α_p e_p\|_N\) and \(a \in \lambda^*(E^*, N)\) with \(\|a\|_{λ^*(E^*, N)} \leq 1\). Then we have

\[\|a_p(t)\|\alpha_p e_p\|_N = \|α_p a_p(t) e_p\|_N \leq 1.\]

However, \(\|t e_p\|_{λ(E,M)} = \sup \{\|a_p(t)\| : \|a\|_{λ^*(E^*, N)} \leq 1\} \leq κ\). This means that \(\|t e_p\|_{λ(E,M)} \leq κ\|t\|_E\) for every \(t \in E\) and then \(θ\) is continuous. On the other hand, it is easy to check that \(\|t e_p\|_{λ(E,M)} \geq \|t\|_E\|e_p\|_λ\) for every \(t \in E\) and \(θ\) is open.

For the completeness of \(λ(E,M)\), let \(x^k = (x_n^k), k = 1,2,\ldots\) be a Cauchy sequence in \(λ(E,M)\). For a fixed \(n \in \mathbb{N}\), by Lemma 4.2, the sequence \(x_n^k, k = 1,2,\ldots\) is Cauchy in \(E\) and then converges to an \(x_n \in E\). Set \(x = (x_n)\). We will prove that \(x \in λ(E,M)\) and that \((x^k)_k\) converges to \(x\) in \(λ(E,M)\). Let \(X\) denote the unit ball of \(λ^*(E^*, N)\). For every \(k \in \mathbb{N}\), let \(f_k : X \to Ŗ_1\) be defined by \(f_k(a) = (a_n(x_n^k))\) for all \(a = (a_n) \in X\). Since \((f_k)_k\) is a uniformly Cauchy sequence in \(λ(E,M)\) and \(λ(E,M)\) is a Banach space, \((f_k)_k\) must converge uniformly on \(X\) to a function \(f : X \to Ŗ_1\). Let \(α = (a_n)_n \in X\) and \(α = (α_n)_n = f(a)\). Then \(α \in Ŗ_1\). On the other hand, for every \(n \in \mathbb{N}\), the sequence \(a_n(x_n^k), k = 1,2,\ldots\) converges to \(a_n(x_n)\). However, \(f_k(a) = (a_n(x_n^k))_n\) converges to \(f(a) = α\). This gives \((a_n(x_n))_n \in Ŗ_1\) and then \(x \in λ(E,M)\). Since \((f_k)_k\) converges uniformly on \(X\) to \(f\), the sequence \((x^k)_k\) converges in \(λ(E,M)\) to \(x\).
Theorem 5.2

5. Dual space of $\lambda(E, M)$

If $x = (x_n) \in \omega(E)$ then we denote by $x^{(k)} = (x_1, x_2, \ldots, x_k, 0 \ldots)$ the sequence of the finite sections of $x$. If $x \in \lambda(E, M)$, then $x^{(k)} \in \lambda(E, M)$ for all $k \in \mathbb{N}$. Using the notation $x^{(k)} = \sum_{n=1}^{k} x_n e_n$, we see that if $x$ is the limit of its finite sections, then

$$x = \lim_{k \to \infty} x^{(k)} = \sum_{n=1}^{\infty} x_n e_n. \quad (5.1)$$

If $\lambda(E, M)_r$ denotes the subspace of $\lambda(E, M)$ of the sequences of $\lambda(E, M)$, which are the limit of their finite sections, then $\lambda(E, M)$ is said to have the AK-property if $\lambda(E, M) = \lambda(E, M)_r$.

We have

**Theorem 5.1** $\lambda(E, M)_r$ is a closed subspace of $\lambda(E, M)$.

**Proof** It is easy to check that if $x = (x_n) \in \lambda(E, M)$ then $\|x^{(k)}\|_{\lambda(E, M)} \leq \|x\|_{\lambda(E, M)}$. Suppose that $x$ is in the closure of $\lambda(E, M)_r$ and $\varepsilon > 0$. There exist $y \in \lambda(E, M)_r$ and $K \in \mathbb{N}$ such that $\|x - y\|_{\lambda(E, M)} < \varepsilon/3$ and $\|y^{(k)} - y\|_{\lambda(E, M)} < \varepsilon/3$ for all $k \geq K$. Now,

$$\|x^{(k)} - x\|_{\lambda(E, M)} \leq \|x^{(k)} - y^{(k)}\|_{\lambda(E, M)} + \|y - y^{(k)}\|_{\lambda(E, M)} + \|x - y\|_{\lambda(E, M)}$$

$$< 2\|x - y\|_{\lambda(E, M)} + \varepsilon/3 < \varepsilon,$$

for all $k \geq K$. Then $x \in \lambda(E, M)_r$.  \qed

The following theorem gives an analogue of a result of [10] given for $M(t) = t$, when $\lambda$ and $E$ are Banach spaces.

**Theorem 5.2** Let $F$ be a continuous linear functional on $\lambda(E, M)$ and, for every $n \in \mathbb{N}$ and $t \in E$, $a_n(t) = F(te_n)$. Then the sequence $(a_n)_n$ is strongly $(\mathcal{N}, \lambda^*)$-summable in $E^*$.

**Proof** Since $F$ is continuous, there exists $\kappa > 0$ such that

$$|F(x)| \leq \kappa \|x\|_{\lambda(E, M)}, \text{ for all } x = (x_n)_n \in \lambda(E, M).$$

Fix $n \in \mathbb{N}$ and $t \in E$. We have

$$|a_n(t)| = |F(te_n)| \leq \kappa \|te_n\|_{\lambda(E, M)} \leq \kappa M(1)\|e_n\|_\lambda \|t\|_E.$$

This means that $(a_n)_n \subset E^*$.

It remains to show that $(a_n)_n \in \lambda^*(E^*, \mathcal{N})$. To this end, let $(f_n)_n \in \lambda(E^{**}, M)$, $k \in \mathbb{N}$, and $\delta > 0$ be given. Then, due to the principle of local reflexivity (cf. [2]), there exists a continuous operator $u_k: \text{span}\{f_1, f_2, \ldots, f_k\} \to E$ such that $\|u_k\|_{E^{**}} \leq 1 + \delta$ and $a_n(u_k f_n) = f_n(a_n)$ for all $n \in \{1, 2, \ldots, k\}$.

Since every $a_n$ is continuous, there exist $0 < \delta_n \leq \frac{\delta}{k(1 + \|e_n\|_\lambda)}$ and $x_n \in E$ such that $\|x_n - u_k f_n\|_E \leq \delta_n$. 917
and $|a_n(x_n - u_k f_n)| \leq \frac{\delta}{k(1 + \|e_n\|_\lambda)}$. Now,

$$\left| \sum_{n=1}^{k} a_n(x_n - u_k f_n) \right| \leq \left| \sum_{n=1}^{k} a_n(x_n - u_k f_n) \right| + \left| \sum_{n=1}^{k} a_n(x_n) \right| \leq \sum_{n=1}^{k} |a_n(x_n - u_k f_n)| + \left| F\left( \sum_{n=1}^{k} x_n e_n \right) \right| \leq \delta + k \left| \sum_{n=1}^{k} x_n e_n \right|_{\lambda(E,M)}.$$

However, for $\alpha = (\alpha_n) \in \lambda^*$, with $\|\alpha\|_{\lambda^*} \leq 1$ and $a \in E^*$, with $\|a\|_{E^*} \leq 1$,

$$\left| \sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n \right|_M \leq \left( \sum_{n=1}^{k} \alpha_n |a(x_n - u_k f_n)| e_n \right) + \left| \sum_{n=1}^{k} \alpha_n a(u_k f_n) e_n \right|_M. \quad (5.2)$$

On one hand,

$$\sum_{n=1}^{k} M(\|\alpha_n a(x_n - u_k f_n)\|/\delta) \leq \sum_{n=1}^{k} M(\|\alpha_n\|/k(1 + \|e_n\|_\lambda)) \leq k \left( \sum_{n=1}^{k} \|\alpha_n\|_\lambda (1/k) \right) \leq k \left( \sum_{n=1}^{k} \|\alpha_n\|_\lambda \right) \leq k M(1/k) \leq M(1).$$

Thus, $\left| \sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n \right|_M \leq \delta$, if $M(1) \leq 1$, and $\left| \sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n \right|_M \leq M(1) \delta$, if $M(1) \geq 1$. Replacing $\delta$ by $M(1)\delta$ if necessary, we may suppose that

$$\left| \sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n \right|_M \leq \delta. \quad (5.3)$$

On the other hand,

$$\left| \sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n \right|_M \leq (1 + \delta) \left| \sum_{n=1}^{k} f_n e_n \right|_{\lambda(E^{**}, M)} \leq (1 + \delta) \left( \|f_n\|_{\lambda(E^{**}, M)} \right). \quad (5.4)$$

Combining (5.3) and (5.4) in (5.2) and taking the supremum on $B_{E^*}$ and $B_{\lambda^*}$, we get $\left| \sum_{n=1}^{k} x_n e_n \right|_{\lambda(E,M)} \leq \delta + (1 + \delta) \left( \|f_n\|_{\lambda(E^{**}, M)} \right).$
We establish now the converse of Theorem ∑satisfy ∆2 condition. Let (\(E, \epsilon\)) be given. It suffices to show that the series \(\sum \left| \epsilon_n f_n(a_n) \right|\) is convergent and that \((\epsilon_n f_n(a_n))_n \in \lambda^*(E^*, M)\).

Further, let \((\epsilon_n)_n\) be such that \(|f_n(a_n)| = \epsilon_n f_n(a_n), n \in \mathbb{N}\). Then \((\epsilon_n f_n)_n \in \lambda(E^{**}, M)\) and

\[
\sum_{n=1}^{k} |f_n(a_n)| = \sum_{n=1}^{k} \epsilon_n f_n(a_n) \leq \delta (\kappa + 1) + \kappa(1 + \delta) \|(f_n)_n\|_{\lambda(E^{**}, M)}.
\]

It follows that \((f_n(a_n))_n \in \ell_1\) and \((\epsilon_n)_n \in \lambda^*(E^*, N)\). \(\square\)

**Remark 5.3** From the preceding proof, since \(\delta\) is arbitrary, one gets

\[
\sum_{n=1}^{\infty} |f_n(a_n)| \leq \kappa \|(f_n)_n\|_{\lambda(E^{**}, M)}, \text{ for all } (f_n)_n \in \lambda(E^{**}, M).
\]

(5.5)

Therefore, \(\|(\epsilon_n)_n\|_{\lambda^*(E^*, N)} \leq \|F\|_{\lambda(E, M)^*}\).

In order to establish the converse of the last result we need the following characterization of weakly \((M, \lambda)\)-summable sequences in \(E^*\).

**Lemma 5.4**

\(\lambda(E^*, M) = \{(a_n)_n \subset E^* : (a_n(a_n x))_n \in \ell_M, \text{ for all } x \in E, (a_n)_n \in \lambda^*\}\)

**Proof** Let \(a = (a_n)_n \in \lambda(E^*, M)\). For all \(x \in E\), the evaluation \(\delta_x(u) = u(x)\) can be regarded as an element of \(E^{**}\). Then, for every \((a_n)_n \in \lambda^*\), \((\alpha_n \delta_x)_n = (a_n(a_n x))_n \in \ell_M\). Conversely, assume that for all \(x \in E\), \((\alpha_n)_n \in \lambda^*, (\alpha_n a_n(x))_n \in \ell_M\) and let \(f \in E^{**}\). We shall use the fact that \(\ell_M\) is perfect, since \(M\) is supposed to satisfy \(\Delta 2\) condition. Let \((\gamma_n)_n \in \ell_M^*\) be given. It suffices to show that the series \(\sum |\gamma_n a_n f(a_n)|\) is convergent. Choose \((\epsilon_n)_n\) so that \(\epsilon_n f(\gamma_n a_n a_n) = |f(\gamma_n a_n a_n)|\) for all \(n\) and set

\[A = \left\{ \sum_{n=1}^{p} \epsilon_n \gamma_n a_n a_n : p \in \mathbb{N} \right\}.\]

For all \(p \in \mathbb{N}\) and all \(x \in E\), one has

\[
\sum_{n=1}^{p} |\epsilon_n \gamma_n a_n a_n(x)| \leq \sum_{n=1}^{\infty} |\gamma_n a_n a_n(x)|,
\]

which is finite since \((\alpha_n a_n(x))_n \in \ell_M\). The set \(A\) is then weak* -bounded in \(E^*\), and so \(A\) is weakly bounded in \(E^*\). Hence there exists \(\rho_f > 0\) such that \(\sum_{n=1}^{p} \epsilon_n \gamma_n a_n f(a_n) \leq \rho_f\), for all \(p \in \mathbb{N}\). This proves that the series \(\sum |\gamma_n a_n f(a_n)|\) is convergent and that \((\alpha_n f(a_n))_n \in \ell_M\). \(\square\)

We establish now the converse of Theorem 5.2.

919
Theorem 5.5 For every $a = (a_n)_n \in \lambda^*\langle E^*, N \rangle$, the mapping

$$f_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)$$

defines a continuous linear functional on $\lambda(E, M)$.

**Proof** Let $a = (a_n)_n \in \lambda^*\langle E^*, N \rangle$ and $x = (x_n)_n \in \lambda(E, M)$. We have $(\delta_n)_n \subset E^{**}$, where $\delta_n$ is the evaluation $u \mapsto u(x_n)$ at $x_n$, $u \in E^*$. Thanks to lemma 5.4, since $(\alpha_n \delta_n(u))_n \in \ell_M$, for every $(\alpha_n)_n \in \lambda^*$, we have $(\delta_n)_n \subset \lambda(E^{**}, M)$. Hence $\sum |\delta_n(a_n)|$ converges and $f_a$ is well defined.

Next consider the map $\varphi_a$ defined from $\lambda(E, M)$ into $\ell_1$ by $\varphi_a((f_n)_n) = (f_n(a_n))_n$. Then $\varphi_a$ is well defined. Moreover, suppose that $(x^i)_{i \in N} \in \lambda(E, M)$ converges to $x := (x_n)_n$ and $(\varphi_a(x^i))_i$ converges in $\ell_1$ to $(\alpha_n)_n$. By the continuity of the projections (Lemma 3.2), $(x^i)_{i \in N}$ converges to $x_n$ for every $n \in N$ and then $(a_n(x^i))_{i \in N}$ converges to $a_n(x_n)$ as well. It follows that $(a_n(x_n))_n = (\alpha_n)_n$, showing that the graph of $\varphi_a$ is closed and then that $\varphi_a$ is continuous, since $\lambda(E, M)$ is a Banach space (Theorem 6.4). Then there exists $c > 0$ so that

$$\sum_{n=1}^{\infty} |a_n(x_n)| \leq c\|x_n\|_{\lambda(E, M)},$$

for all $(x_n)_n \in \lambda(E, M)$.

This shows that $f_a$ is continuous on $\lambda(E, M)$. \hfill \square

We now obtain the promised characterization of continuous linear functionals on $\lambda(E, M)_r$.

Theorem 5.6 The following equality holds algebraically and topologically

$$(\lambda(E, M)_r)^* = \lambda^*\langle E^*, N \rangle.$$

(5.6)

**Proof** Consider the mapping $\varphi : a \mapsto f_a$ from $\lambda^*\langle E^*, N \rangle$ to $(\lambda(E, M)_r)^*$ defined in Theorem 5.5. $\varphi$ is clearly linear. Suppose that there exists $a = (a_n)_n \in \lambda^*\langle E^*, N \rangle$ such that $f_a(x) = 0$, for every $x = (x_n)_n \in \lambda(E, M)_r$. Fix $k \in N$ and $t \in E$. We have $a_k(t) = f_a(te_n) = 0$, which means that $a_k = 0$. Since $k$ was arbitrary, $a = (a_n)_n = 0$ and $\varphi$ is one to one. Conversely, if $f \in (\lambda(E, M)_r)^*$ then let $a = (a_n)_n \in \lambda^*\langle E^*, N \rangle$ as defined in Theorem 5.2. If $x = (x_n)_n \in \lambda(E, M)_r$, then $x = \sum_{n=1}^{\infty} x_n e_n$ by (5.1). As $f$ is continuous, $f(x) = \sum_{n=1}^{\infty} f(x_n e_n) = \sum_{n=1}^{\infty} a_n(x_n)$, which gives $\varphi(a) = f$ and $\varphi$ is onto, and (5.6) holds algebraically. Since $\varphi^{-1}$ is defined between Banach spaces (Theorems 3.3 and 4.3), and is continuous by (5.5), $\varphi$ is an isomorphism by the open mapping theorem. \hfill \square

6. Reflexivity of $\lambda(E, M)$

In the sequel, we denote by $\lambda(E, M)_r$ the subspace of $\lambda(E, M)$ formed by the sequences of $\lambda(E, M)$, which are the limit of their finite sections.

The proof of the following theorem is along the same lines as that of Theorem 5.2; we give it for the sake of completeness.

Theorem 6.1 Let $G$ be a continuous linear functional on $\lambda(E, M)$ and, for every $n \in N$ and $t \in E$, $a_n(t) = G(te_n)$. Then the sequence $(a_n)_n$ is weakly $(N, \lambda^*)$-summable in $E^*$.  

920
Hence, the series $k$ for every $\sum_{i=1}^{k}$. Fix

**Remark 6.2** From the preceding proof, since $\varepsilon$ is arbitrary, one gets

$$\sum_{n=1}^{\infty} |\beta_n \alpha_n a_n(u)| \leq 2\eta, \text{ for all } (\alpha_n)_n \in B_{\lambda}, (\beta_n)_n \in B_{\ell_M}, u \in B_E.$$
Theorem 6.3 For every $a = (a_n)_n \in \lambda^*(E^*, N)$, the mapping

$$g_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)$$

defines a continuous linear functional on $\lambda(E, M)$.

Proof Let $a = (a_n)_n \in \lambda^*(E^*, N)$. Then, for every $x \in \lambda(E, M)$, $(a_n(x_n))_n \in \ell_1$, by the definition of $\lambda(E, M)$. Therefore $g_a$ is well defined. Suppose that $(x^i)_i \in \ell_1 \subseteq \lambda(E, M)$ converges to $x := (x_n)_n$ and $(g_a(x^i))_i$ converges in $\ell_1$ to $(a_n)_n$. By the continuity of the projections (Lemma 4.2), $(x_n^i) \in \ell_1$ converges to $x_n$ for every $n \in \mathbb{N}$ and then $(a_n(x^i))_i \in \ell_1$ converges to $a_n(x_n)$ as well. It follows that $(a_n(x_n))_n = (a_n)_n$ and that the graph of $\varphi_a$ is closed. This shows that $\varphi_a$ is continuous. Hence $g_a$ is continuous on $\lambda(E, M)$. \hfill \Box

We now state the characterization of continuous linear functionals on $\lambda(E, M)_r$.

Theorem 6.4 The following equality holds algebraically and topologically

$$(\lambda(E, M)_r)^* = \lambda^*(E^*, N). \quad (6.1)$$

Proof Consider the mapping $\psi : a \mapsto g_a$ from $\lambda^*(E^*, N)$ to $(\lambda(E, M)_r)^*$ defined in Theorem 6.3. It is clear that $\psi$ is linear. Suppose that there exists $a = (a_n)_n \in \lambda^*(E^*, N)$ such that $g_a(x) = 0$, for every $x = (x_n)_n \in \lambda(E, M)_r$. Fix $k \in \mathbb{N}$ and $t \in E$. We have $a_k(t) = g_a(te_n) = 0$, which means that $a_k = 0$. Since $k$ was arbitrary, $a = (a_n)_n = 0$ and $\psi$ is one to one.

Conversely, let $g \in (\lambda(E, M)_r)^*$ and $a = (a_n)_n \in \lambda^*(E^*, N)$ as defined in Theorem 6.1. If $x = (x_n)_n \in \lambda(E, M)_r$, then $x = \sum_{n=1}^{\infty} x_n e_n$. As $g$ is continuous, $g(x) = \sum_{n=1}^{\infty} g(x_n e_n) = \sum_{n=1}^{\infty} a_n(x_n)$, and $\psi(a) = g$. Thus $\psi$ is onto. The equality (6.1) holds algebraically.

However, according to Remark 6.2, $\| (a_n)_n|_{\lambda^*(E^*, N)} \leq 2\| g_a \|_{\lambda(E, M)_r}$, and then $\psi$ is open. Since $\psi$ is bijective between Banach spaces (Theorems 3.3, 4.3), $\psi$ is continuous by the open mapping theorem. This finishes the proof. \hfill \Box

We give our main result in the following

Theorem 6.5 If $M$ and $N$ possess the $\Delta_2$-condition, then $\lambda(E, M)$ is reflexive if and only if the following assertions hold:

(i) $E$ is reflexive,

(ii) $\lambda(E, M)$ is an AK-space,

(iii) $\lambda^*(E^*, N)$ is an AK-space.

Proof If $\lambda(E, M)$ is reflexive, then $E$ is reflexive as a closed subspace of $\lambda(E, M)$, by Theorem 3.3. Hence, (i) holds.

By [7, 23.5(10)] and our Theorem 5.1, $\lambda(E, M)_r$ is also reflexive as a closed subspace of $\lambda(E, M)$. It is then weakly quasi-complete by [7, 23.5(2)]. Thus, $\lambda(E, M)_r$ is weakly sequentially complete.

Let $x = (x_n)_n \in \lambda(E, M)$. Then the sequence $(x^{(k)})_{k \in \mathbb{N}}$ consisting of the finite sections of $x$ is contained in
$\lambda(E, M)_r$ and is weakly Cauchy in it. In fact, let $a$ be in $(\lambda(E, M)_r)^*$. By Theorem 5.5, the series $\sum a_n(x_n)$ converges, and $\langle (x^{(k)}, a) \rangle_k = (\sum_{n=1}^{k} a_n(x_n))_k$ is then a Cauchy sequence; hence $(x^{(k)})_{k \in \mathbb{N}}$ converges weakly to a limit $y = (y_n)_n \in \lambda(E, M)_r$ and it is obvious that $x = y$ so that (ii) holds.

Now, since $\lambda(E, M)_r$ is reflexive, the same holds for its dual $\lambda^*(E^*, N)$ and the argumentation above still works to infer that (iii) holds.

Conversely, assume that (i), (ii), and (iii) are satisfied. Then, since $\lambda$ and $E$ are reflexive, an application of Theorems 5.6 and 6.4 gives, algebraically and topologically,

$$
(\lambda(E, M))^{**} = (\lambda(E, M)_r)^{**}, \quad \text{(by (ii))}
$$

$$
= (\lambda^* (E^*, N))^{**} = (\lambda^* (E^*, N)_r)^*, \quad \text{(by (iii))}
$$

$$
= \lambda^{**}(E^{**}, M), \quad \text{(by Theorem 6.4)}
$$

$$
= \lambda(E, M), \quad \text{(by (i)).}
$$

Then $\lambda(E, M)$ is reflexive.

\[\square\]

Acknowledgment

The author thanks the Deanship of Scientific Research at Al Imam Mohammad Ibn Saud Islamic University in the Kingdom of Saudi Arabia to have supported this project number 331217, in 2013. We are also indebted to the referee for relevant suggestions that have improved the quality of the paper.

References


