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# On a class of unitary operators on the Bergman space of the right half plane

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**Abstract:** In this paper, we introduce a class of unitary operators defined on the Bergman space $L_a^2(\mathbb{C}_+)$ of the right half plane $\mathbb{C}_+$ and study certain algebraic properties of these operators. Using these results, we then show that a bounded linear operator $S$ from $L_a^2(\mathbb{C}_+)$ into itself commutes with all the weighted composition operators $W_a, a \in \mathbb{D}$ if and only if $\widetilde{S}(w) = \langle Sb_{\overline{w}}, b_{\overline{w}}\rangle, w \in \mathbb{C}_+$ satisfies a certain averaging condition. Here for $a = c + id \in \mathbb{D}, f \in L_a^2(\mathbb{C}_+), W_a f = (f \circ t_a)\frac{M'}{M' \circ t_a}, Ms = \frac{1-s}{1+s}, t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$, and $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}}\frac{1+w}{1+\overline{w}}\frac{2\text{Re}w}{(s+w)^2}, w = M\overline{a}, s \in \mathbb{C}_+$. Some applications of these results are also discussed.

**Key words:** Right half plane, Bergman space, unitary operator, automorphism, Toeplitz operators

## 1. Introduction

Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : \text{Re}s > 0\}$ be the right half plane. Let $d\widetilde{A}(s) = dxdy$ be the area measure. Let $L^2(\mathbb{C}_+, d\widetilde{A})$ be the space of complex-valued, square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. Let $L_a^2(\mathbb{C}_+)$ be the closed subspace [2] of $L^2(\mathbb{C}_+, d\widetilde{A})$ consisting of those functions in $L^2(\mathbb{C}_+, d\widetilde{A})$ that are analytic. The space $L_a^2(\mathbb{C}_+)$ is referred to as the Bergman space of the right half plane. The functions $H(s,w) = \frac{1}{(s+\overline{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$ are the reproducing kernel [4] for $L_a^2(\mathbb{C}_+)$. Let $\mathbf{h}_w(s) = \frac{H(s,w)}{\sqrt{H(w,w)}} = \frac{2\text{Re}w}{(s+\overline{w})^2}$. The functions $\mathbf{h}_w, w \in \mathbb{C}_+$ are the normalized reproducing kernels for $L_a^2(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. Define for $f \in L^\infty(\mathbb{C}_+), ||f||_\infty = \text{ess}\sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm. For $\phi \in L^\infty(\mathbb{C}_+)$, we define [6, 8] the Toeplitz operator $\mathcal{T}_\phi$ from $L_a^2(\mathbb{C}_+)$ into $L_a^2(\mathbb{C}_+)$ by $\mathcal{T}_\phi\{ = \mathcal{P}_+(\phi\{)$, where $P_+$ denote the orthogonal projection from $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{C}_+)$; the multiplication operator $\mathcal{M}_\phi$ from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ by $(\mathcal{M}_\phi\{)(\int) = \phi(\int)\{(\int)$. The big Hankel operator $\mathcal{H}_\phi$ from $L_a^2(\mathbb{C}_+)$ into $(L_a^2(\mathbb{C}_+))^\perp$ is defined by $\mathcal{H}_\phi\{ = (\mathcal{I} - \mathcal{P}_+)(\phi\{), \{ \in \mathcal{L}^\in_\dashv(\mathbb{C}_+)$. The little Hankel operator $h_\phi$ is a mapping from $L_a^2(\mathbb{C}_+)$ into $\overline{L_a^2(\mathbb{C}_+)}$ defined by $h_\phi f = \overline{P}_+(\phi f)$, where $\overline{P}_+$ is the orthogonal projection from $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $\overline{L_a^2(\mathbb{C}_+)} = \{\overline{f} : f \in L_a^2(\mathbb{C}_+)\}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D}, dA)$ be the space

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of complex-valued, square-integrable, measurable functions on $\mathbb{D}$ with respect to the normalized area measure $dA(z) = \frac{1}{\pi}dxdy$. Let $L^2_a(\mathbb{D})$ be the space consisting of those functions of $L^2(\mathbb{D}, dA)$ that are analytic. The space $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$ and is called the Bergman space of the open unit disk $\mathbb{D}$. The sequence of functions $\{e_n(z)\}_{n=0}^{\infty} = \{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ form an orthonormal basis for $L^2_a(\mathbb{D})$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L^2_a(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function $K_z$ in $L^2_a(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w).$$

for all $f$ in $L^2_a(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

The function $K(z,w)$ is analytic in $z$ and co-analytic in $w$. Since

$$f(z) = \int_{\mathbb{D}} f(w)K(z,w)dA(w), f \in L^2_a(\mathbb{D}),$$

the function $K(z,w) = \frac{1}{(1-z\overline{w})^2}$, $z,w \in \mathbb{D}$ and is the reproducing kernel [11] of $L^2_a(\mathbb{D})$. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{(1-|a|^2)}{(1-\overline{a}z)^2}$. The function $k_a$ is called the normalized reproducing kernel for $L^2_a(\mathbb{D})$. It is clear that $||k_a||_2 = 1$. Important works on the application of the reproducing kernel were obtained by Karaev et al. [9]. These results in reproducing kernel and Berezin symbols are important in operator theory [7]. Let $P$ denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism $\phi_a$ in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \circ \phi_a)(z) = z$;

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) $\phi_a$ has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all $a$ and $z$ in $\mathbb{D}$. An easy calculation shows that the derivative of $\phi_a$ at $z$ is equal to $-k_a(z)$. It follows that the real Jacobian determinant of $\phi_a$ at $z$ is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)}{|1-\overline{a}z|^4}$. Given $a \in \mathbb{D}$ and $f$ any measurable function on $\mathbb{D}$, we define a function $U_a f$ on $\mathbb{D}$ by $U_a f(z) = k_a(z)f(\phi_a(z))$. In this paper, we introduce a class of unitary operators defined on the Bergman space $L^2_a(\mathbb{C}_+)$ and study certain algebraic properties of these operators. Using these results, we then show that a bounded linear operator $S$ from $L^2_a(\mathbb{C}_+)$ into itself commutes with all the weighted composition operators $W_a, a \in \mathbb{D}$ if and only if $\widetilde{S}(w) = \langle Sb_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$ satisfies certain averaging condition. Some applications of these results are also discussed. The organization of this paper is as follows. In §2, we introduce a class of unitary operators $V_a, a \in \mathbb{D}$ with the help of the automorphisms of $\mathbb{C}_+$. We establish certain algebraic properties of these unitary operators, which are also self-adjoint. In §3, we show that a bounded linear operator $S$ from $L^2_a(\mathbb{C}_+)$ into itself commutes with all the weighted composition operators $W_a, a \in \mathbb{D}$ if and only if $\widetilde{S}(w) = \langle Sb_{\overline{w}}, b_{\overline{w}} \rangle, w \in \mathbb{C}_+$ satisfies certain averaging condition. Further, in §4, we establish certain applications of the main result of the paper involving multiplication, Toeplitz, and Hankel operators.

## 2. The unitary operator $V_a$

In this section, we shall introduce the operator $V_a, a \in \mathbb{D}$ and prove certain elementary properties of the unitary operator $V_a$.

Define $M : \mathbb{C}_+ \to \mathbb{D}$ by $Ms = \frac{1-s}{1+s}$. Then $M$ is one-one and onto, and $M^{-1} : \mathbb{D} \to \mathbb{C}_+$ is given by $M^{-1}(z) = \frac{1-z}{1+z}$. Thus $M$ is its self-inverse. Let $W : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^2}$. Then $W^{-1} : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi} G(Mz) \frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$.

**Lemma 2.1** *If $a \in \mathbb{D}$ and $a = c + id$, $c, d \in \mathbb{R}$, then $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$.*

**Proof** It is not difficult to verify that the map $t_a : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is one-one and onto. The lemma follows. □

**Proposition 2.2** *For $a \in \mathbb{D}$, the following hold:*

*(i)* $(t_a \circ t_a)(s) = s$.

*(ii)* $t_a'(s) = -l_a(s)$, *where* $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$.

**Proof** One can verify (i) and (ii) by direct calculation.

□

For $a \in \mathbb{D}$, define $V_a : L_a^2(\mathbb{C}_+) \to L_a^2(\mathbb{C}_+)$ by $(V_a g)(s) = (g \circ t_a)(s) l_a(s)$. In Proposition 2.3, we show that $V_a$ is a self-adjoint unitary operator that is also an idempotent.

**Proposition 2.3** *For $a \in \mathbb{D}$,*

*(i)* $V_a l_a = 1$.

*(ii)* $V_a^{-1} = V_a, V_a^2 = I$.

*(iii)* $V_a$ *is self-adjoint.*

*(iv)* $V_a$ *is unitary.*

*(v)* $V_a P_+ = P_+ V_a$.

**Proof** We shall first prove (i). If $a \in \mathbb{D}$, then by Proposition 2.2, $t_a'(s) = -l_a(s)$. Therefore

$$\begin{aligned}
(V_a l_a)(s) &= (l_a \circ t_a)(s) l_a(s) \\
&= (-t_a' \circ t_a)(s) l_a(s) \\
&= -(t_a' \circ t_a)(s) l_a(s) \\
&= [-t_a'(t_a(s))] l_a(s) \\
&= -\left[ t_a' \left( \frac{-ids + (1-c)}{(1+c)s + id} \right) \frac{1 - |a|^2}{((1+c)s + id)^2} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-|a|^2}{\left[(1+c)\left(\frac{-ids+(1-c)}{(1+c)s+id}\right)+id\right]^2}\frac{1-|a|^2}{[(1+c)s+id]^2} \\
&= \frac{(1-|a|^2)(1-|a|^2)[(1+c)s+id]^2}{[-ids+1-c-idsc+c-c^2+id(s+cs+id)]^2[(1+c)s+id]^2} \\
&= \frac{(1-|a|^2)^2}{[-ids+1-c-idsc+c-c^2+ids+idsc-d^2]^2} \\
&= \frac{(1-|a|^2)^2}{[1-c^2-d^2]^2} \\
&= \frac{(1-|a|^2)^2}{[1-(c^2+d^2)]^2} \\
&= \frac{(1-|a|^2)^2}{(1-|a|^2)^2} = 1.
\end{aligned}
$$

This proves (i). The assertions in (ii), (iii), and (iv) can be verified by direct calculation. Note that $V_a$ can also be defined from $L^2(\mathbb{C}_+)$. To prove (v), observe that $V_a(L^2_a(\mathbb{C}_+)) \subset L^2_a(\mathbb{C}_+)$ and $V_a(L^2_a(\mathbb{C}_+))^\perp \subset (L^2_a(\mathbb{C}_+))^\perp$. Now let $f \in L^2(\mathbb{C}_+)$ and $f = f_1 + f_2$, where $f_1 \in L^2_a(\mathbb{C}_+)$ and $f_2 \in (L^2_a(\mathbb{C}_+))^\perp$. Hence,

$$
\begin{aligned}
P_+V_af &= P_+V_a(f_1+f_2) \\
&= P_J(V_af_1+V_af_2) \\
&= P_+V_af_1 \\
&= V_af_1 \\
&= V_aP_+f.
\end{aligned}
$$

□

Suppose $a \in \mathbb{D}$ and $w = \frac{1-\overline{a}}{1+\overline{a}} = M\overline{a} \in \mathbb{C}_+$. Define $b_{\overline{w}}(s) = \frac{(-1)}{\sqrt{\pi}}(k_a \circ M)(s)M'(s)$.

**Lemma 2.4** *Let $a \in \mathbb{D}$. For $w_1 \in \mathbb{C}_+, V_ab_{\overline{w}_1} = \alpha b_{t_{\overline{a}}(w_1)}$ for some $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$.*

**Proof** To prove the lemma, we shall first show that for $z_1, z_2 \in \mathbb{D}, U_{z_1}k_{z_2} = \alpha k_{\phi_{z_1}(z_2)}$ for some complex constant $\alpha$ such that $|\alpha| = 1$. Suppose $z_1, z_2 \in \mathbb{D}$. If $f \in L^2_a(\mathbb{D})$, then

$$
\langle f, U_{z_1}K_{z_2}\rangle = \langle U_{z_1}f, K_{z_2}\rangle = (U_{z_1}f)(z_2) = -(f\circ\phi_{z_1})(z_2)\phi'_{z_1}(z_2) = \langle f, (\overline{-\phi'_{z_1}(z_2)})K_{\phi_{z_1}(z_2)}\rangle. \tag{2.1}
$$

Thus $U_{z_1}K_{z_2} = \overline{-\phi'_{z_1}(z_2)}K_{\phi_{z_1}(z_2)}$. Rewriting this in terms of normalized reproducing kernels, we have

$$
U_{z_1}k_{z_2} = \alpha k_{\phi_{z_1}(z_2)} \tag{2.2}
$$

for some complex constant $\alpha$. Since $U_{z_1}$ is unitary and $||k_{z_2}||_2 = ||k_{\phi_{z_1}(z_2)}||_2 = 1$, we obtain that $|\alpha| = 1$.

Let $w_1 \in \mathbb{C}_+$ and define $w_1 = M\overline{a}_1$. Since

$$
t_a(\overline{w}_1) = \frac{-id\overline{w}_1 + (1-c)}{(1+c)\overline{w}_1 + id},
$$

we obtain

$$\overline{t_a(\overline{w}_1)} = \frac{idw_1 + (1-c)}{(1+c)w_1 - id} = t_{\overline{a}}(w_1).$$

Thus,

$$\begin{aligned} V_a b_{\overline{w}_1} &= W U_a k_{a_1} \\ &= \alpha W k_{\phi_a(a_1)} \\ &= \alpha b_{\overline{l}}, \end{aligned}$$

where

$$\begin{aligned} l &= \overline{M\phi_a(a_1)} \\ &= \overline{M\phi_a(M\overline{w}_1)} \\ &= \overline{t_a(\overline{w}_1)} = t_{\overline{a}}(w_1). \end{aligned}$$

□

**Lemma 2.5** *Let $a \in \mathbb{D}$, and $w = M\overline{a}$. Then*

*(i)* $V_a b_{\overline{w}} = \frac{(-1)}{\sqrt{\pi}} M'$.

*(ii)* $V_a \left( \frac{(-1)}{\sqrt{\pi}} M' \right) = b_{\overline{w}}$.

**Proof** Let $a \in \mathbb{D}$. Then, since $U_a k_a = 1$ and $W1 = \frac{(-1)}{\sqrt{\pi}} M'$, we obtain

$$\begin{aligned} V_a b_{\overline{w}} &= W U_a k_a \\ &= W1 \\ &= \frac{(-1)}{\sqrt{\pi}} M'. \end{aligned}$$

Now, to prove (ii), observe that

$$\begin{aligned} V_a \left( \frac{(-1)}{\sqrt{\pi}} M' \right) &= W U_a 1 \\ &= W k_a \\ &= b_{\overline{w}}. \end{aligned}$$

□

Let $\mathcal{L}(\mathcal{L}^{\in}_{\dashv}(\mathbb{C}_+))$ be the space of all bounded linear operators from $L^2_a(\mathbb{C}_+)$ into itself. For $T \in \mathcal{L}(\mathcal{L}^{\in}_{\dashv}(\mathbb{C}_+))$, define the function $\widetilde{T}$ on $\mathbb{C}_+$ as $\widetilde{T}(w) = \langle T b_{\overline{w}}, b_{\overline{w}} \rangle$.

**Theorem 2.6** *Let $S, T \in \mathcal{L}(\mathcal{L}^{\in}_{\dashv}(\mathbb{C}_+))$. If $\widetilde{S}(w) = \widetilde{T}(w)$ for all $w \in \mathbb{C}_+$, then $S = T$.*

**Proof** Let $\widetilde{S}(w) = \widetilde{T}(w)$ for all $w \in \mathbb{C}_+$. Then, for $w = M\overline{a}$, we have

$$\begin{aligned}\langle Sb_{\overline{w}}, b_{\overline{w}}\rangle &= \langle SWk_a, Wk_a\rangle \\ &= \langle W^{-1}SWk_a, k_a\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle Tb_{\overline{w}}, b_{\overline{w}}\rangle &= \langle TWk_a, Wk_a\rangle \\ &= \langle W^{-1}TWk_a, k_a\rangle.\end{aligned}$$

Hence for all $a \in \mathbb{D}$,

$$\langle W^{-1}SWk_a, k_a\rangle = \langle W^{-1}TWk_a, k_a\rangle.$$

This implies

$$\langle (W^{-1}SW - W^{-1}TW)k_a, k_a\rangle = \langle Lk_a, k_a\rangle = 0,$$

where $L = W^{-1}SW - W^{-1}TW$. Hence,

$$\langle LK_a, K_a\rangle = K(a,a)\langle Lk_a, k_a\rangle = K(a,a)\cdot 0 = 0.$$

Define $F(x,y) = \langle LK_{\overline{x}}, K_y\rangle$. The function $F$ is holomorphic in $x$ and $y$ and $F(x,y) = 0$ if $x = \overline{y}$ [5]. It can be verified that such functions must vanish identically. Let $x = u + iv, y = u - iv$. Let $G(u,v) = F(x,y)$. The function $G$ is holomorphic and vanishes if $u$ and $v$ are real. Hence $F(x,y) = G(u,v) \equiv 0$. Thus even $\langle LK_x, K_y\rangle = 0$ for any $x, y$. Since linear combinations of $K_x, x \in \mathbb{D}$, are dense in $L^2_a(\mathbb{D})$ [3], it follows that $L = 0$. That is, $W^{-1}SW = W^{-1}TW$. Hence $S = T$. □

**Corollary 2.7** *Let $S, T \in \mathcal{L}(\mathcal{L}^{\in}_{\dashv}(\mathbb{C}_+))$. Suppose for all $a \in \mathbb{D}$,*

$$\left\langle (V_aSV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle = \left\langle (V_aTV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle.$$

*Then $S = T$.*

**Proof** Let $a \in \mathbb{D}$. Then since $W^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right) = 1$, hence

$$\begin{aligned}\left\langle (V_aSV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle &= \left\langle U_a(W^{-1}SW)U_aW^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right), W^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle \\ &= \left\langle U_a(W^{-1}SW)U_a1, 1\right\rangle \\ &= \left\langle (W^{-1}SW)U_a1, U_a1\right\rangle \\ &= \left\langle (W^{-1}SW)k_a, k_a\right\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\langle (V_aTV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle &= \left\langle U_a(W^{-1}TW)U_aW^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right), W^{-1}\left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle \\ &= \left\langle U_a(W^{-1}TW)U_a1, 1\right\rangle \\ &= \left\langle (W^{-1}TW)U_a1, U_a1\right\rangle \\ &= \left\langle (W^{-1}TW)k_a, k_a\right\rangle.\end{aligned}$$

Thus,

$$\left\langle (V_aSV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle = \left\langle (V_aTV_a)\left(\frac{(-1)}{\sqrt{\pi}}M'\right), \left(\frac{(-1)}{\sqrt{\pi}}M'\right)\right\rangle \text{ for all } a \in \mathbb{D}$$

implies

$$\left\langle (W^{-1}SW - W^{-1}TW)k_a, k_a \right\rangle = 0 \text{ for all } a \in \mathbb{D}.$$

Hence,

$$\left\langle (W^{-1}SW - W^{-1}TW)K_a, K_a \right\rangle = K(a,a)\left\langle (W^{-1}SW - W^{-1}TW)k_a, k_a \right\rangle = K(a,a)\cdot 0 = 0.$$

Proceeding similarly as in Corollary 2.7, we obtain $W^{-1}SW = W^{-1}TW$. Hence $S = T$. □

## 3. Main result

The operators $W_a$ are called weighted composition operators on $L^2_a(\mathbb{C}_+)$. In this section, we shall show that a bounded linear operator $S$ from $L^2_a(\mathbb{C}_+)$ into itself commutes with all the weighted composition operators $W_a, a \in \mathbb{D}$, if and only if $\widetilde{S}$ satisfies a certain averaging condition.

**Theorem 3.1** *A bounded linear operator $S \in \mathcal{L}(L^2_a(\mathbb{C}_+))$ commutes with all the weighted composition operators $W_a, a \in \mathbb{D}$ if and only if*

$$\widetilde{S}(w_1) = \int_{\mathbb{D}} \widetilde{S}(t_{\overline{a}}(w_1))dA(a), \text{ for all } w_1 \in \mathbb{C}_+.$$

**Proof** Suppose

$$\widetilde{S}(w_1) = \int_{\mathbb{D}} \widetilde{S}(t_{\overline{a}}(w_1))dA(a), \tag{3.1}$$

for all $w_1 \in \mathbb{C}_+$. Then, by Lemma 2.4, there exists a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that for all $w_1 \in \mathbb{C}_+$,

$$\begin{aligned}\langle Sb_{\overline{w}_1}, b_{\overline{w}_1}\rangle &= \int_{\mathbb{D}} \left\langle Sb_{\overline{t_{\overline{a}}(w_1)}}, b_{\overline{t_{\overline{a}}(w_1)}} \right\rangle dA(a) \\ &= \int_{\mathbb{D}} \langle \alpha SV_a b_{\overline{w}_1}, \alpha V_a b_{\overline{w}_1}\rangle \, dA(a) \\ &= \int_{\mathbb{D}} \langle V_aSV_a b_{\overline{w}_1}, b_{\overline{w}_1}\rangle \, dA(a) \\ &= \left\langle \left(\int_{\mathbb{D}} V_aSV_a dA(a)\right) b_{\overline{w}_1}, b_{\overline{w}_1} \right\rangle \\ &= \langle \widehat{S} b_{\overline{w}_1}, b_{\overline{w}_1}\rangle,\end{aligned}$$

where $\widehat{S} = \int_{\mathbb{D}} V_aSV_a dA(a)$.

Thus, by Theorem 2.6, $S = \widehat{S}$. Hence for all $f, g \in L^2_a(\mathbb{C}_+), \langle Sf, g\rangle = \langle \widehat{S}f, g\rangle$. Thus the equation (3.1) is equivalent to saying that

$$\int_{\mathbb{D}} \langle SV_af, V_ag\rangle dA(a) = \int_{\mathbb{C}_+} (Sf)(w_1)\overline{g(w_1)} d\widetilde{A}(w_1)$$

for all $f, g \in L_a^2(\mathbb{C}_+)$. Let $w \in \mathbb{C}_+$ and let $w = M\overline{a} = \frac{1-\overline{a}}{1+\overline{a}}$. Since

$$\begin{aligned} b_{\overline{w}}(s) &= \frac{(-1)}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}Ms)^2} \frac{(-2)}{(1+s)^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}Ms)^2} \frac{1}{(1+s)^2}, \end{aligned}$$

we obtain

$$\begin{aligned} b_{\overline{w}}(\overline{w}) &= \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}M\overline{w})^2} \frac{1}{(1+\overline{w})^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-|a|^2)^2} \frac{1}{\left(1+\frac{1-a}{1+a}\right)^2} \\ &= \frac{2}{\sqrt{\pi}} \frac{1}{(1-|a|^2)} \frac{(1+a)^2}{4} \\ &= \frac{1}{2\sqrt{\pi}} \frac{(1+a)^2}{(1-|a|^2)}. \end{aligned}$$

Thus

$$\begin{aligned} b_{\overline{w}}(s) b_{\overline{w}}(\overline{w}) &= \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}Ms)^2} \frac{1}{(1+s)^2} \frac{1}{2\sqrt{\pi}} \frac{(1+a)^2}{(1-|a|^2)} \\ &= \frac{1}{\pi} \frac{1}{(1-\overline{a}Ms)^2} \frac{(1+a)^2}{(1+s)^2} \\ &= \frac{(-1)}{2\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} \frac{(-2)}{(1+s)^2} \\ &= \frac{(-1)}{2\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} M' \\ &= B(s,w) \ (let). \end{aligned}$$

Thus $b_{\overline{w}}(s) = \frac{B(s,w)}{b_{\overline{w}}(\overline{w})}$ and $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$. That is, $b_{\overline{w}}(s) = \frac{B(s,w)}{b_{\overline{w}}(\overline{w})} = \frac{B(s,w)}{\sqrt{B(\overline{w},w)}}$. Note that $W1 = \frac{(-1)}{\sqrt{\pi}} M'$ and therefore $W^{-1}\left(\frac{-M'}{\sqrt{\pi}}\right) = 1$.

That is, $(M' \circ M)M' = (-1)\sqrt{\pi}\frac{(-1)}{\sqrt{\pi}}(M' \circ M)M' = W^{-1}\left(\frac{-M'}{\sqrt{\pi}}\right) = 1$.

From Lemma 2.5, it follows that

$$(b_{\overline{w}} \circ t_a) l_a = \frac{(-1)}{\sqrt{\pi}} M'.$$

This implies

$$b_{\overline{w}}(l_a \circ t_a) = \frac{(-1)}{\sqrt{\pi}} (M' \circ t_a). \tag{3.2}$$

That is,

$$b_{\overline{w}} \left( \frac{l_a \circ t_a}{\frac{(-1)}{\sqrt{\pi}} (M' \circ t_a)} \right) = 1.$$

Thus $(-\sqrt{\pi})b_{\overline{w}}\left(\frac{l_a}{M'}\circ t_a\right)=1$ and therefore $(-\sqrt{\pi})b_{\overline{w}}[(l_a(M'\circ M))\circ t_a]=1$. Hence

$$b_{\overline{w}}[(-\sqrt{\pi})(l_a\circ t_a)(M'\circ M\circ t_a)]=1.$$

This implies $b_{\overline{w}}\in H^{\infty}(\mathbb{C}_+)$ and $\frac{1}{b_{\overline{w}}}\in H^{\infty}(\mathbb{C}_+)$. Further $B(s,M\overline{0})=B(s,M0)=B(s,1)=\frac{(-1)}{2\pi}M'=\frac{(-1)}{2\pi}\frac{(-2)}{(1+s)^2}=\frac{1}{\pi}\frac{1}{(1+s)^2}$. Again $B(Ma,M\overline{0})=\frac{1}{\pi}\frac{1}{(1+Ma)^2}$ and $B(M0,M\overline{0})=\frac{1}{4\pi}$. Now note that $W_af=(f\circ t_a)\frac{M'}{M'\circ t_a}$. Hence $W_aSf=SW_af$ for all $f\in L^2_a(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$ if and only if

$$[(Sf)\circ t_a]\frac{M'}{M'\circ t_a}=S\left[(f\circ t_a)\frac{M'}{M'\circ t_a}\right]$$

for all $a\in\mathbb{D}$ and for all $f\in L^2_a(\mathbb{C}_+)$. That is, if and only if,

$$\left[\frac{(Sf)}{M'}\circ t_a\right]M'=S\left[\left(\frac{f}{M'}\circ t_a\right)M'\right]$$

for all $f\in L^2_a(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. Putting $\frac{f}{b_{\overline{w}}}$ in place of $f$, we obtain $SW_af=W_aSf$ for all $f\in L^2_a(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$ if and only if

$$\left[\left(\frac{1}{M'(w_1)}S\left(\frac{f}{b_{\overline{w}}}\right)\right)\circ t_a\right]M'(w_1)=S\left[\left(\frac{f}{M'b_{\overline{w}}}\circ t_a\right)M'\right](w_1) \tag{3.3}$$

for all $f\in L^2_a(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. Now to prove the necessary part of the theorem, assume that $SW_af=W_aSf$ for all $f\in L^2_a(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. We shall prove that

$$\int_{\mathbb{C}_+}(Sf)(w_1)\overline{g(w_1)}d\widetilde{A}(w_1)=\int_{\mathbb{D}}\langle SV_af,V_ag\rangle dA(a)$$

for all $f,g\in L^2_a(\mathbb{C}_+)$. Note that

$$\begin{aligned}\langle Sf,g\rangle&=\int_{\mathbb{C}_+}(Sf)(z)\overline{g(z)}d\widetilde{A}(z)\\&=\int_{\mathbb{C}_+}\frac{\overline{g(z)}d\widetilde{A}(z)}{M'(z)}S\left(\frac{fM'}{(\frac{(-1)}{2\pi})M'}\right)(z)\left(\frac{-1}{2\pi}M'\right)(z)\\&=\int_{\mathbb{C}_+}\frac{\overline{g(z)}d\widetilde{A}(z)}{M'(z)}S\left(\frac{fM'}{B(.,M0)}\right)(z)B(z,M0).\end{aligned}$$

Hence by the mean-value property for harmonic functions [1], we obtain

$$\begin{aligned}\langle Sf,g\rangle&=\int_{\mathbb{C}_+}\frac{\overline{g(z)}d\widetilde{A}(z)}{M'(z)}\int_{\mathbb{D}}S\left(\frac{fM'\sqrt{B(Ma,M\overline{a})}}{B(.,M\overline{a})}\right)(z)\frac{B(z,M\overline{a})}{\sqrt{B(Ma,M\overline{a})}}dA(a)\\&=\int_{\mathbb{C}_+}\frac{\overline{g(z)}d\widetilde{A}(z)}{M'(z)}\int_{\mathbb{D}}S\left(\frac{fM'}{b_{\overline{w}}}\right)(z)b_{\overline{w}}(z)dA(a).\end{aligned}$$

Using Fubini's theorem [10] and using the identity (3.2), we obtain

$$\langle Sf, g\rangle = \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S\left(\frac{fM'}{b_{\overline{w}}}\right)(z) b_{\overline{w}}(z)\overline{g(z)} d\widetilde{A}(z)\right] dA(a)$$

$$= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S\left(\frac{fM'}{b_{\overline{w}}}\right)(z) \frac{b_{\overline{w}}}{l_a}(z)\overline{g(z)} l_a(z) d\widetilde{A}(z)\right] dA(a)$$

$$= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'(z)} S\left(\frac{fM'}{b_{\overline{w}}}\right)(z) b_{\overline{w}}(l_a \circ t_a)(z)\overline{g(z)} l_a(z) d\widetilde{A}(z)\right] dA(a)$$

$$= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \frac{1}{M'} S\left(\frac{fM'}{b_{\overline{w}}}\right)(z) \left(\frac{-1}{\sqrt{\pi}}\right) (M' \circ t_a)(z)\overline{g(z)(l_a \circ t_a)(z)} |l_a(z)|^2 d\widetilde{A}(z)\right] dA(a)$$

$$= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} \left[\left(\frac{1}{M'} S\left(\frac{fM'}{b_{\overline{w}}}\right)\right) \circ t_a\right](z) \left(\frac{-1}{\sqrt{\pi}}\right) M'(z)\overline{(g \circ t_a)(z) l_a(z)} d\widetilde{A}(z)\right] dA(a).$$

Now observe that by the identity (3.2) we obtain

$$SV_a f = S\left[(f \circ t_a) l_a\right] = S\left[\frac{f \circ t_a}{b_{\overline{w}} \circ t_a} \frac{(-1)}{\sqrt{\pi}} M'\right]$$

$$= S\left[\left(\frac{f}{b_{\overline{w}}} \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right] = S\left[\left(\left(\frac{fM'}{M' b_{\overline{w}}}\right) \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right]$$

$$= \left[\left(\left(\frac{1}{M'} S\left(\frac{fM'}{b_{\overline{w}}}\right)\right) \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right].$$

This last equality follows from (3.3) since $SW_a f = W_a Sf$. Thus for all $f, g \in L^2_a(\mathbb{C}_+)$, we obtain

$$\langle Sf, g\rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g\rangle dA(a).$$

We shall now prove the sufficient part. Suppose $\langle Sf, g\rangle = \int_{\mathbb{D}} \langle SV_a f, V_a g\rangle dA(a)$ for all $f, g \in L^2_a(\mathbb{C}_+)$. We shall show that $SW_a = W_a S$ for all $a \in \mathbb{D}$. We have already verified that

$$SV_a f = S\left[\left(\left(\frac{f}{b_{\overline{w}}}\right) \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right].$$

Hence

$$\int_{\mathbb{D}} \langle SV_a f, V_a g\rangle dA(a) = \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} S\left[\left(\left(\frac{f}{b_{\overline{w}}}\right) \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right](w_1) \overline{(g \circ t_a)(w_1) l_a(w_1)} d\widetilde{A}(w_1)\right] dA(a)$$

$$= \int_{\mathbb{D}} \left[ \int_{\mathbb{C}_+} S\left[\left(\left(\frac{fM'}{M' b_{\overline{w}}}\right) \circ t_a\right) \frac{(-1)}{\sqrt{\pi}} M'\right](w_1) \overline{(g \circ t_a)(w_1) l_a(w_1)} d\widetilde{A}(w_1)\right] dA(a).$$

Using Fubini's theorem [10], we obtain

$$\int_{\mathbb{D}}\langle SV_a f, V_a g\rangle dA(a) = \int_{\mathbb{C}_+}\left[\int_{\mathbb{D}} S\left[\left(\left(\frac{fM'}{M'b_{\overline{w}}}\right)\circ t_a\right)\frac{(-1)}{\sqrt{\pi}}M'\right](w_1)\overline{(g\circ t_a)(w_1)l_a(w_1)}dA(a)\right]d\widetilde{A}(w_1). \tag{3.4}$$

Now

$$\begin{aligned}\langle Sf, g\rangle &= \int_{\mathbb{C}_+}(Sf)(z)\overline{g(z)}d\widetilde{A}(z)\\ &= \int_{\mathbb{C}_+}\overline{g(z)}\frac{1}{M'(z)}S\left(\frac{fM'}{M'}\right)(z)M'(z)d\widetilde{A}(z)\\ &= \int_{\mathbb{C}_+}\overline{g(z)}\frac{1}{M'(z)}S\left(\frac{fM'}{(\frac{-1}{2\pi})M'}\right)(z)\left(\frac{-1}{2\pi}\right)M'(z)d\widetilde{A}(z)\\ &= \int_{\mathbb{C}_+}\overline{g(z)}\frac{1}{M'(z)}S\left(\frac{fM'}{B(.,M0)}\right)(z)B(z,M0)d\widetilde{A}(z).\end{aligned}$$

By mean value property for harmonic functions [1], we obtain

$$\begin{aligned}\langle Sf, g\rangle &= \int_{\mathbb{C}_+}\frac{\overline{g(z)}d\widetilde{A}(z)}{M'(z)}\int_{\mathbb{D}} S\left(\frac{fM'\sqrt{B(Ma, M\overline{a})}}{B(., M\overline{a})}\right)(z)\frac{B(z, M\overline{a})}{\sqrt{B(Ma, M\overline{a})}}dA(a)\\ &= \int_{\mathbb{C}_+}\overline{g(z)}d\widetilde{A}(z)\int_{\mathbb{D}}\left[\frac{1}{M'}S\left(\frac{fM'}{\frac{B(.,w)}{\sqrt{B(\overline{w},w)}}}\right)\right](z)\frac{B(z,w)}{\sqrt{B(\overline{w},w)}}dA(a)\\ &= \int_{\mathbb{C}_+}\overline{g(z)}d\widetilde{A}(z)\int_{\mathbb{D}}\left[\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\right](z)b_{\overline{w}}(z)dA(a).\end{aligned}$$

Using Fubini's theorem [10], we obtain

$$\begin{aligned}\langle Sf, g\rangle &= \int_{\mathbb{D}}\int_{\mathbb{C}_+}\left[\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\right](z)b_{\overline{w}}(z)\overline{g(z)}d\widetilde{A}(z)dA(a)\\ &= \int_{\mathbb{D}}\int_{\mathbb{C}_+}\left[\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\circ t_a\right](z)(b_{\overline{w}}\circ t_a)(z)\overline{(g\circ t_a)(z)}|l_a(z)|^2 d\widetilde{A}(z)dA(a)\\ &= \int_{\mathbb{D}}\int_{\mathbb{C}_+}\left[\left(\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\right)\circ t_a\right](z)\frac{\frac{(-1)}{\sqrt{\pi}}M'(z)}{l_a(z)}\overline{(g\circ t_a)(z)l_a(z)}l_a(z)d\widetilde{A}(z)dA(a)\end{aligned}$$

$$= \int_{\mathbb{D}}\int_{\mathbb{C}_+}\left[\left(\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\right)\circ t_a\right](z)\frac{(-1)}{\sqrt{\pi}}M'(z)\overline{(g\circ t_a)(z)l_a(z)}d\widetilde{A}(z)dA(a) \tag{3.5}$$

From (3.4) and (3.5), it follows that if $\langle Sf, g\rangle = \int_{\mathbb{D}}\langle SV_a f, V_a g\rangle dA(a)$ for all $f, g \in L^2_a(\mathbb{C}_+)$ then

$$S\left[\left(\left(\frac{fM'}{M'b_{\overline{w}}}\right)\circ t_a\right)\left(\frac{-1}{\sqrt{\pi}}\right)M'\right] = \left[\left(\frac{1}{M'}S\left(\frac{fM'}{b_{\overline{w}}}\right)\right)\circ t_a\right]\frac{(-1)}{\sqrt{\pi}}M' \tag{3.6}$$

for all $f \in L^2_a(\mathbb{C}_+)$ and for all $a \in \mathbb{D}$.

Putting $\frac{f}{M'}$ in place of $f$ we obtain (3.6), which holds if and only if

$$S\left[\left(\left(\frac{f}{M'b_{\overline{w}}}\right)\circ t_a\right)\frac{(-1)}{\sqrt{\pi}}M'\right]=\left[\left(\frac{1}{M'}S\left(\frac{f}{b_{\overline{w}}}\right)\right)\circ t_a\right]\frac{(-1)}{\sqrt{\pi}}M'$$

for all $f\in L_a^2(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. Thus if $\langle Sf,g\rangle=\int_{\mathbb{D}}\langle SV_af,V_ag\rangle dA(a)$ for all $f,g\in L_a^2(\mathbb{C}_+)$, then

$$S\left[\left(\left(\frac{f}{M'b_{\overline{w}}}\right)\circ t_a\right)M'\right]=\left[\left(\frac{1}{M'}S\left(\frac{f}{b_{\overline{w}}}\right)\right)\circ t_a\right]M' \tag{3.7}$$

for all $f\in L_a^2(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$.

By (3.5), the identity (3.7) holds if and only if $SW_af=W_aSf$ for all $f\in L_a^2(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. Thus we have proved that if $\langle Sf,g\rangle=\int_{\mathbb{D}}\langle SV_af,V_ag\rangle dA(a)$ for all $f,g\in Ł_a^2(\mathbb{C}_+)$ then $SW_af=W_aSf$ for all $f\in L_a^2(\mathbb{C}_+)$ and for all $a\in\mathbb{D}$. The theorem follows. □

## 4. Applications

In this section, we establish certain applications of the main result of the paper involving multiplication, Toeplitz, and Hankel operators. For $\phi\in L^\infty(\mathbb{C}_+,d\widetilde{A})$ and $T\in\mathcal{L}(\mathcal{L}^{\in}_{\dashv}(\mathbb{C}_+))$, let $\widehat{\phi}(s)=\int_{\mathbb{D}}\phi(t_a(s))dA(a)$ and $\widehat{T}=\int_{\mathbb{D}}V_aTV_a dA(a)$.

**Corollary 4.1** *Let $\phi\in L^\infty(\mathbb{C}_+)$. Then the following hold:*

*(i)* $\widehat{\mathcal{M}_\phi}=\mathcal{M}_{\widehat{\phi}}$.

*(ii)* $\widehat{\mathcal{T}_\phi}=\mathcal{T}_{\widehat{\phi}}$.

*(iii)* $\widehat{\mathcal{H}_\phi}=\mathcal{H}_{\widehat{\phi}}$.

**Proof** From Proposition 2.2, it follows that for given $h\in L^2(\mathbb{C}_+,d\widetilde{A})$ and $g\in L^2(\mathbb{C}_+,d\widetilde{A})$ we have

$$\begin{aligned}\langle\widehat{\mathcal{M}_\phi}g,h\rangle&=\int_{\mathbb{D}}\langle\phi V_ag,V_ah\rangle dA(a)\\&=\int_{\mathbb{D}}dA(a)\int_{\mathbb{C}_+}\phi(s)(V_ag)(s)\overline{(V_ah)(s)}d\widetilde{A}(s)\\&=\int_{\mathbb{D}}dA(a)\int_{\mathbb{C}_+}\phi(s)(g\circ t_a)(s)l_a(s)\overline{(h\circ t_a)(s)}\;\overline{l_a(s)}d\widetilde{A}(s)\\&=\int_{\mathbb{D}}dA(a)\int_{\mathbb{C}_+}(\phi\circ t_a)(s)g(s)\overline{h(s)}(l_a\circ t_a)(s)\overline{(l_a\circ t_a)(s)}\;|l_a(s)|^2d\widetilde{A}(s)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)}\ |(l_a \circ t_a)(s)|^2\ |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} g(s) \overline{h(s)} d\widetilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\
&= \int_{\mathbb{C}_+} \widehat{\phi}(s) g(s) \overline{h(s)} d\widetilde{A}(s) = \langle \mathcal{M}_{\widehat{\phi}}\}, \langle\rangle.
\end{aligned}
$$

This proves (i). To prove (ii), let $h$ and $g$ in $L^2_a(\mathbb{C}_+, d\widetilde{A})$. Then since $(l_a \circ t_a)(s) l_a(s) = s$, we obtain

$$
\begin{aligned}
\left\langle \widehat{\mathcal{T}}_\phi g, h \right\rangle &= \int_{\mathbb{D}} \langle V_a \mathcal{T}_\phi \mathcal{V}_\dashv\}, \langle\rangle\, dA(a) \\
&= \int_{\mathbb{D}} \langle \mathcal{T}_\phi \mathcal{V}_\dashv\}, \mathcal{V}_\dashv\langle\rangle\, dA(a) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle P_+(\phi V_a g), V_a h\rangle\, d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle \phi V_a g, P_+(V_a h)\rangle\, d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \langle \phi V_a g, V_a h\rangle\, d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(V_a g)(s) \overline{(V_a h)(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(g \circ t_a)(s) l_a(s) \overline{(h \circ t_a)(s)}\ \overline{l_a(s)}\ d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) (l_a \circ t_a)(s) \overline{h(s)}\ \overline{(l_a \circ t_a)(s)}\ |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} (l_a \circ t_a)(s) \overline{(l_a \circ t_a)(s)}\ |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} |(l_a \circ t_a)(s)|^2 |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} g(s) \overline{h(s)} d\widetilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\
&= \int_{\mathbb{C}_+} \widehat{\phi}(s) g(s) \overline{h(s)} d\widetilde{A}(s) \\
&= \left\langle \widehat{\phi} g, h \right\rangle \\
&= \left\langle \widehat{\phi} g, P_+ h \right\rangle = \left\langle P_+(\widehat{\phi} g), h \right\rangle = \left\langle \mathcal{T}_{\widehat{\phi}}\}, \langle \right\rangle.
\end{aligned}
$$

Therefore, $\widehat{\mathcal{T}}_\phi = \mathcal{T}_{\widehat{\phi}}$. This proves (ii). Now we shall establish (iii). It is not difficult to see that for $a \in \mathbb{D}, V_a(L_a^2(\mathbb{C}_+)) \subset L_a^2(\mathbb{C}_+)$ and $V_a((L_a^2(\mathbb{C}_+))^\perp) \subset (L_a^2(\mathbb{C}_+))^\perp$. Further, from Proposition 2.3, it follows that for $g \in L_a^2(\mathbb{C}_+)$ and $h \in (L_a^2(\mathbb{C}_+))^\perp$, we have

$$
\begin{aligned}
\left\langle \widehat{\mathcal{H}}_\phi g, h \right\rangle &= \int_{\mathbb{D}} \langle V_a \mathcal{H}_\phi \mathcal{V}_\dashv \}, \langle\rangle \lceil \mathcal{A}(\dashv) \\
&= \int_{\mathbb{D}} \langle \mathcal{H}_\phi \mathcal{V}_\dashv \}, \mathcal{V}_\dashv \langle\rangle \lceil \mathcal{A}(\dashv) \\
&= \int_{\mathbb{D}} \langle (I - P_+)(\phi V_a g), V_a h \rangle dA(a) \\
&= \int_{\mathbb{D}} \langle \phi V_a g, (I - P_+)(V_a h) \rangle dA(a) \\
&= \int_{\mathbb{D}} \langle \phi V_a g, V_a h \rangle dA(a) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(V_a g)(s)\overline{(V_a h)(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} \phi(s)(g \circ t_a)(s) l_a(s) \overline{(h \circ t_a)(s) l_a(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s)(l_a \circ t_a)(s) \overline{h(s)(l_a \circ t_a)(s)} |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{D}} dA(a) \int_{\mathbb{C}_+} (\phi \circ t_a)(s) g(s) \overline{h(s)} d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} g(s) \overline{h(s)} d\widetilde{A}(s) \int_{\mathbb{D}} (\phi \circ t_a)(s) dA(a) \\
&= \int_{\mathbb{C}_+} \widehat{\phi}(s) g(s) \overline{h(s)} d\widetilde{A}(s) \\
&= \left\langle \widehat{\phi} g, h \right\rangle \\
&= \left\langle \widehat{\phi} g, (I - P_+) h \right\rangle \\
&= \left\langle (I - P_+)(\widehat{\phi} g), h \right\rangle \\
&= \left\langle H_{\widehat{\phi}} g, h \right\rangle .
\end{aligned}
$$

Hence $\widehat{\mathcal{H}}_\phi = \mathcal{H}_{\widehat{\phi}}$. □

**Corollary 4.2** *Let $a \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{C}_+)$. Then the following hold:*

*(i)* $V_a \mathcal{M}_\phi \mathcal{V}_\dashv = \mathcal{M}_{\phi \circ \sqcup_\dashv}$.

*(ii)* $V_a \mathcal{T}_\phi \mathcal{V}_\dashv = \mathcal{T}_{\phi \circ \sqcup_\dashv}$.

*(iii)* $V_a\mathcal{H}_\phi V_{-a} = \mathcal{H}_{\phi\circ t_{-a}}$.

*(iv)* $V_a h_\phi V_a = h_{\phi\circ t_a}$.

**Proof** We first prove (i). Note that since $(l_a\circ t_a)(s)l_a(s) = s$, we have for $f\in L^2_a(\mathbb{C}_+)$,

$$\begin{aligned}
V_a\mathcal{M}_\phi V_{-a}f &= V_a\mathcal{M}_\phi[(f\circ t_{-a})l_{-a}]\\
&= V_a[\phi(f\circ t_a)l_a]\\
&= (\phi\circ t_a)f(l_a\circ t_a)l_a\\
&= (\phi\circ t_a)f\\
&= \mathcal{M}_{\phi\circ t_{-a}}f.
\end{aligned}$$

This proves (i). To prove (ii), let $f\in L^2_a(\mathbb{C}_+)$. Then we have

$$\begin{aligned}
V_a\mathcal{T}_\phi V_{-a}f &= V_a\mathcal{T}_\phi[(f\circ t_{-a})l_{-a}]\\
&= V_aP_+[\phi(f\circ t_a)l_a]\\
&= P_+V_a[\phi(f\circ t_a)l_a]\\
&= P_+[(\phi\circ t_a)f(l_a\circ t_a)l_a]\\
&= P_+[(\phi\circ t_a)f]\\
&= \mathcal{T}_{\phi\circ t_{-a}}f,
\end{aligned}$$

since $(l_a\circ t_a)(s)l_a(s) = s$. This proves (ii). Now to establish (iii), let $f\in L^2_a(\mathbb{C}_+)$. Then

$$\begin{aligned}
V_a\mathcal{H}_\phi V_{-a}f &= V_a\mathcal{H}_\phi[(f\circ t_{-a})l_{-a}]\\
&= V_a(I-P_+)[\phi(f\circ t_a)l_a]\\
&= (I-P_+)V_a[\phi(f\circ t_a)l_a]\\
&= (I-P_+)[(\phi\circ t_a)f(l_a\circ t_a)l_a]\\
&= (I-P_+)[(\phi\circ t_a)f]\\
&= \mathcal{H}_{\phi\circ t_{-a}}f.
\end{aligned}$$

This proves (iii). To prove (iv), let $f\in L^2_a(\mathbb{C}_+)$. Then since $\overline{P}_+ = JPJ$, where $Jf(s) = f(\overline{s})$ for $f\in L^2_a(\mathbb{C}_+)$ and $V_a\overline{P}_+ = \overline{P}_+V_a$, we obtain

$$\begin{aligned}
V_ah_\phi V_af &= V_ah_\phi[(f\circ t_a)l_a]\\
&= V_a\overline{P_+}[\phi(f\circ t_a)l_a]\\
&= \overline{P_+}V_a[\phi(f\circ t_a)l_a]\\
&= \overline{P_+}[(\phi\circ t_a)f(l_a\circ t_a)l_a]\\
&= \overline{P_+}[(\phi\circ t_a)f]\\
&= h_{\phi\circ t_a}f.
\end{aligned}$$

□

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