Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 703 – 715
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doi:10.3906/mat-1704-63

On the unit index of some real biquadratic number fields

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Received: 15.04.2017 • **Accepted/Published Online:** 18.06.2017 • **Final Version:** 24.03.2018

Abstract: Let $p_1 \equiv p_2 \equiv 1 \pmod 4$ be different prime numbers such that $\left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = -\left(\frac{2}{p_1}\right) = -1$. Put $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ and let $\mathbb{K}$ be a quadratic extension of $\Bbbk$ contained in its absolute genus field $\Bbbk^{(*)}$. Denote by k_j, $1 \leq j \leq 3$, the three quadratic subfields of $\mathbb{K}$. Let $E_{\mathbb{K}}$ (resp. E_{k_j}) be the unit group of $\mathbb{K}$ (resp. k_j). The unit index $\left[E_{\mathbb{K}} : \prod_{j=1}^{3} E_{k_j}\right]$ is characterized in terms of biquadratic residue symbols between 2, p_1 and p_2 or by the capitulation of $\mathfrak{2}$, the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ above 2, in $\mathbb{K}$. These results are used to describe the 2-rank of some CM-fields.

Key words: Unit index, fundamental systems of units, 2-class group, real biquadratic fields, multiquadratic CM-fields

1. Introduction and notations

Let k be a multiquadratic number field of degree 2^n, (i.e., $[k : \mathbb{Q}] = 2^n$) and k_i $(i = 1, \cdots, s)$ be the $s = 2^n - 1$ quadratic subfields of k. Denote by E_k (resp. E_{k_i}) the unit group of k (resp. k_i), i.e. the group of the invertible elements of $\mathcal{O}_k$ (resp. $\mathcal{O}_{k_i}$), the ring of integers of k (resp. k_i). Then the index $q(k) = [E_k : \prod_{i=1}^{s} E_{k_i}]$ is called the *unit index* of k. By Dirichlet's unit theorem, if $2^n = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 is the number of pairs of complex conjugate embeddings of k, then there exist $r = r_1 + r_2 - 1$ units of $\mathcal{O}_k$ that generate E_k (modulo the roots of unity), and these r units are called the *fundamental system of units of* k.

One major problem in algebraic number theory is the computation of the number $q(k)$. For quadratic fields, the problem is easily solved. For some fields $k = \mathbb{Q}(\sqrt{-1}, \sqrt{m})$, where m is a positive square-free integer, Dirichlet [9] showed that $q(k) = 1$ or 2. Over time, Dirichlet's result has been generalized by many mathematicians; see, for example [1, 2, 8, 11–13, 16–20, 26]. For quartic bicyclic fields, Kubota [17] gave a method for finding a fundamental system of units and thus for computing the unit index. Wada [26] generalized Kubota's method, creating an algorithm for computing fundamental units in any given multiquadratic field. However, in general, it is not easy to compute this index.

Let p_1 and p_2 be different primes satisfying the following conditions:

$$p_1 \equiv p_2 \equiv 1 \pmod 4 \text{ and } \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = -\left(\frac{2}{p_1}\right) = -1. \tag{1}$$

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2010 *AMS Mathematics Subject Classification:* 11R27, 11R21, 11R29, 11R11

Put $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2})$ and let $\mathbb{K}$ be a quadratic extension of $\mathbb{k}$ contained in its absolute genus field, i.e. $\mathbb{K}$ equals $\mathbb{K}_1^+ = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2})$, $\mathbb{K}_2^+ = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1})$, or $\mathbb{K}_3^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2})$. The purpose of this paper is to characterize the index $q(\mathbb{K})$ in terms of biquadratic residue symbols between 2, p_1 and p_2, or by the capitulation in $\mathbb{K}$ of $\mathfrak{2}$, the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ above 2. Note that in [5], we dealt with the same problem for $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2})$ assuming $\left(\frac{p_1}{p_2}\right) = -1$ and $p_1 \equiv p_2 \equiv 5 \pmod 8$.

The structure of this paper is as follows. Denote by ϵ_j, $1 \leq j \leq 3$, the fundamental units of the three quadratic subfields of $\mathbb{K}$. In Section 2, we collect some necessary results, and we give the abelian types and the generators of the 2-class groups of $\mathbb{Q}(\sqrt{2p_1p_2})$ and $\mathbb{Q}(\sqrt{2p_1})$. In Section 3, we prove necessary and sufficient conditions for $q(\mathbb{K}_j)$, $1 \leq j \leq 3$, to be equal to 1 (Theorems 3.1 and 3.2). This allows us to characterize the solvability in $\mathbb{K}$, whenever the norms of ϵ_j are equal to -1, of the equation $X^2 - \epsilon_1\epsilon_2\epsilon_3 = 0$ in terms of biquadratic residue symbols between 2, p_1 and p_2, if $\mathbb{K} = \mathbb{K}_1^+$ or $\mathbb{K} = \mathbb{K}_3^+$, and by using the capitulation of the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ above 2 if $\mathbb{K} = \mathbb{K}_2^+$. We end this paragraph by giving some results on units, indices, and the structure of $G = \mathrm{G}al(\mathbb{k}_2^{(2)}/\mathbb{k})$, where $\mathbb{k}_2^{(2)}$ is the second Hilbert 2-class field of $\mathbb{k}$. We then apply these results, in Section 4, to compute the 2-rank of the CM-fields $\mathbb{K}_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, \sqrt{-1})$, $\mathbb{K}_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1}, \sqrt{-1})$, $\mathbb{K}_3 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, \sqrt{-1})$, and $\mathbb{F}^{(*)} = \mathbb{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, i)$.

Let k be a number field and m be a square-free integer. In what follows, we adopt the following notations:

- $h(m)$ (resp. $h(k)$): the 2-class number of $\mathbb{Q}(\sqrt{m})$ (resp k).
- E_k: the unit group of k.
- W_k: the group of roots of unity contained in k, and ω_k denotes its order.
- $Q_k = [E_k : W_kE_{k^+}]$ is Hasse's unit index, if k is a CM-field.
- k^+: the maximal real subfield of k.
- $q(k) = [E_k : \prod_i^s E_{k_i}]$, the unit index of k if k is multiquadratic, where k_i are the s quadratic subfields of k.
- $k^{(*)}$: the genus field of k; that is, the maximal abelian unramified extension of k obtained by composing k and an abelian extension over $\mathbb{Q}$.
- $k_2^{(1)}$: the first Hilbert 2-class field of k; that is, the maximal abelian unramified extension of k such that $[k_2^{(1)} : k]$ is a power of 2.
- $k_2^{(2)}$: the second Hilbert 2-class field of k; that is, the first Hilbert 2-class field of $k_2^{(1)}$.
- $\mathbf{C}l_2(k)$ (resp. $\mathbf{C}l(k)$): the 2-class (resp. class) group of k.
- ϵ_m: the fundamental unit of $\mathbb{Q}(\sqrt{m})$.
- $i = \sqrt{-1}$.

2. Preliminaries

Let p_1 and p_2 be different primes satisfying the conditions (1) and put $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ and $k_1 = \mathbb{Q}(\sqrt{2p_1})$. Let ϵ_j, $1 \leq j \leq 3$, denote the three fundamental units of the three quadratic subfields of any biquadratic bicyclic real number field K.

Lemma 2.1 **([18])** *Assuming $N(\epsilon_1) = N(\epsilon_2) = N(\epsilon_3) = \pm 1$, then the equation $X^2 - \epsilon_1\epsilon_2\epsilon_3 = 0$ has a solution in K if and only if $q(K) = 2$.*

Lemma 2.2 *[4, Corollary 3.6] If $d = 2p_1p_2$, where $p_1 \equiv p_2 \equiv 1 \pmod 4$ are different primes, and at least two of the elements of $\left\{\left(\frac{2}{p_1}\right), \left(\frac{2}{p_2}\right), \left(\frac{p_1}{p_2}\right)\right\}$ are equal to -1, then the norm of the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is -1.*

Lemma 2.3 *Let p_1 and p_2 be different primes satisfying the conditions (1). Then the 2-class group $\mathbf{C}l_2(k_1)$ of k_1 is cyclic of order $h_2(2p_1) = 2^n$, $n \geq 1$. It is generated by the class of P_2, a prime ideal of k_1 above p_2. Moreover, $P_2^{2^{n-1}} \sim \mathfrak{2}$ in $\mathbf{C}l_2(k_1)$, where $\mathfrak{2}$ is the prime ideal of k_1 above 2.*

Proof As $\left(\frac{2p_1}{p_2}\right) = 1$, so p_2 splits in k_1. Put $p_2O_{k_1} = P_2P_2'$ and denote by $\mathfrak{2}$ and P_1 the prime ideals of k_1 above 2 and p_1, respectively. P_1 is not principal in k_1, as otherwise we will get $p_1 = x^2 - 2p_1y^2$, where $x, y \in \mathbb{Q}$; this contradicts the fact that $\left(\frac{p_1}{p_2}\right) = -1$. Similarly, we prove that $\mathfrak{2}$ and P_2 are not principal.

It is well known, under our conditions, that $\mathbf{C}l_2(k_1)$ is cyclic of order 2^n where $n \geq 1$. On the other hand, $\left(\frac{p_2, 2p_1}{p_1}\right) = \left(\frac{p_2}{p_1}\right) = -1$, and then by genus theory $[P_2]$ is not a square in $\mathbf{C}l_2(k_1)$. Thus, $\mathbf{C}l_2(k_1) = \langle [P_2] \rangle$. Finally, since $\mathfrak{2} \sim P_1$ are of order 2, we deduce that $P_2^{2^{n-1}} \sim \mathfrak{2} \sim P_1$. □

Lemma 2.4 *Let p_1 and p_2 be different primes satisfying the conditions (1). Then the 2-class group $\mathbf{C}l_2(\Bbbk)$ of $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ is of type $(2,2)$. It is generated by the classes of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the prime ideals above p_1 and p_2, respectively.*

Proof According to [14] $\mathbf{C}l_2(\Bbbk)$, the 2-class group of $\Bbbk$ is of type $(2,2)$. It is generated by the classes of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the prime ideals above p_1 and p_2, respectively. In fact, $\mathfrak{p}_i$ is of order 2 since $\mathfrak{p}_i^2 = (p_i)$, and it is not principal for all $i \in \{1,2\}$; otherwise, we would get $p_i = \mp(x^2 - 2p_1p_2y^2)$ for some x and y in $\mathbb{Q}$ and this implies the contradiction $\left(\frac{p_i}{p_j}\right) = -1$ where $i \neq j \in \{1,2\}$. Similarly, we show that $\tilde{\mathfrak{2}}$ the prime ideal of $\Bbbk$ above 2 is not principal, too. The same reasoning shows that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ (resp. $\tilde{\mathfrak{2}}$ and $\mathfrak{p}_2$) are independent. As $\tilde{\mathfrak{2}}\mathfrak{p}_1 \sim \mathfrak{p}_2$, so $\tilde{\mathfrak{2}}\mathfrak{p}_1$ is not principal, too. Finally, the classes of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are not squares since $\left(\frac{p_1}{p_2}\right) = -1$. □

3. Main results

Let p_1 and p_2 be different primes satisfying the conditions (1) and put $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$. Letting $\mathbb{F} = \Bbbk(i) = \mathbb{Q}(\sqrt{2p_1p_2}, i)$, then $\mathbb{F}$ admits three unramified quadratic extensions that are abelian over $\mathbb{Q}$, which are $\mathbb{K}_1 = \mathbb{F}(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$, $\mathbb{K}_2 = \mathbb{F}(\sqrt{p_2}) = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1}, i)$, and $\mathbb{K}_3 = \mathbb{F}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$. Let $\mathbb{K}_j^+$ denote the maximal real subfield of $\mathbb{K}_j$ where $1 \leq j \leq 3$.

Theorem 3.1 *Let p_1 and p_2 be different primes satisfying the conditions (1). Then the following assertions are equivalent:*

1. $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1$,

2. $q(\mathbb{K}_1^+) = 1$ *or* $q(\mathbb{K}_3^+) = 1$,

3. $q(\mathbb{K}_2^+) = 1$ *and* $h(2p_1) = 2$.

Proof To prove this theorem, consider Figure 1 below, where $\Bbbk^{(*)}$ denotes the genus field of $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ and $\Bbbk_2^{(1)}$ denotes its Hilbert 2-class field. Since the 2-class group of $\Bbbk$ is of type $(2,2)$ (see Lemma 2.4), and

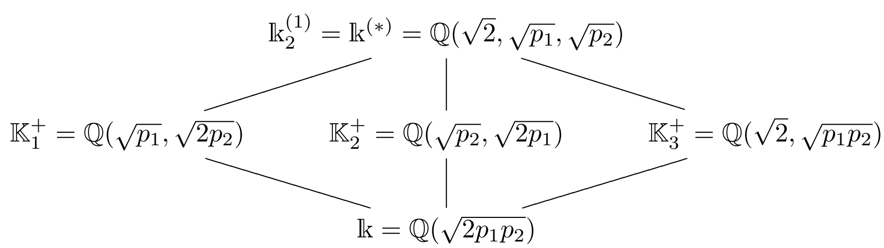

Figure 1. Subfields of $\Bbbk^{(*)}/\Bbbk$.

since also the discriminant of $\Bbbk$ is equal to $d_{\Bbbk} = 8p_1p_2$, then by [7] we have $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1$ if and only if $\Bbbk_1^{(1)} = \Bbbk_2^{(2)}$.

On one hand, according to [15] and [25] the condition $\Bbbk_2^{(1)} = \Bbbk_2^{(2)}$ is equivalent to $h(\mathbb{K}_j^+) = 2$ for some $j \in \{1, 2, 3\}$. On the other hand, the class number formula implies that

$$\begin{aligned} h(\mathbb{K}_1^+) &= \tfrac{1}{4} q(\mathbb{K}_1^+) h(p_1) h(2p_2) h(2p_1p_2) = 2q(\mathbb{K}_1^+), \\ h(\mathbb{K}_2^+) &= \tfrac{1}{4} q(\mathbb{K}_2^+) h(p_2) h(2p_1) h(2p_1p_2) = h(2p_1) q(\mathbb{K}_2^+), \text{ and} \\ h(\mathbb{K}_3^+) &= \tfrac{1}{4} q(\mathbb{K}_3^+) h(2) h(p_1p_2) h(2p_1p_2) = 2q(\mathbb{K}_3^+). \end{aligned}$$

Thus the results. □

Theorem 3.2 *Let p_1 and p_2 be different primes satisfying the conditions (1). Denote by $\mathfrak{z}$ the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ lies above 2 and by $h(2p_1) = 2^n$, $n \geq 1$, the 2-class number of $\mathbb{Q}(\sqrt{2p_1})$. Then the following assertions hold:*

1. $q(\mathbb{K}_2^+) = 1$ *if and only if $\mathfrak{z}$ capitulates in* $\mathbb{K}_2^+$.

2. $\mathbf{C}l_2(\mathbb{K}_2^+)$ *is cyclic generated by the class of $\mathcal{P}_2$ a prime ideal of $\mathbb{K}_2^+$ above p_2. Moreover,* $h(\mathbb{K}_2^+) = 2^n \Longleftrightarrow q(\mathbb{K}_2^+) = 1$, *i.e.* $h(\mathbb{K}_2^+) = 2^{n+1} \Longleftrightarrow q(\mathbb{K}_2^+) = 2$.

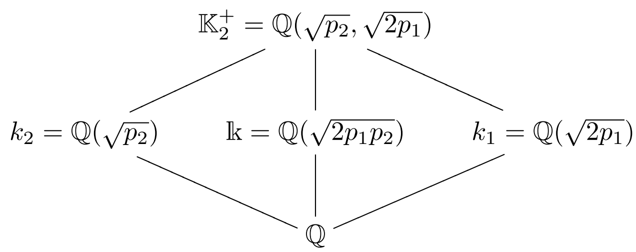

Figure 2. Subfields of $\mathbb{K}_2^+/\mathbb{Q}$.

Proof To prove this theorem, we need Figure 2 below. It is easy to see that $\mathbb{K}_2^+/k_1$ and $\mathbb{K}_2^+/k_2$ are ramified, but $\mathbb{K}_2^+/\Bbbk$ is not, so by class field theory $N_{\mathbb{K}_2^+/k_1}(\mathbf{C}l_2(\mathbb{K}_2^+)) = \mathbf{C}l_2(k_1)$, $N_{\mathbb{K}_2^+/k_2}(\mathbf{C}l_2(\mathbb{K}_2^+)) = \mathbf{C}l_2(k_2)$ (which has an odd class number), and $[\mathbf{C}l_2(\Bbbk) : N_{\mathbb{K}_2^+/\Bbbk}(\mathbf{C}l_2(\mathbb{K}_2^+))] = 2$.

On the other hand, it is easy to see also that $\mathfrak{p}_2$ capitulates and splits in $\mathbb{K}_2^+$. Letting $\mathcal{P}_2$ be a prime ideal of $\mathbb{K}_2^+$ above p_2, then $\mathcal{P}_2$ is not principal, as otherwise we will get $N_{\mathbb{K}_2^+/k_1}(\mathcal{P}_2) \sim P_2 \sim 1$, which is absurd (Lemma 2.3).

We claim that, in $\mathbf{C}l_2(\mathbb{K}_2^+)$, $\mathcal{P}_2^2 \sim P_2$. To this end, let s and t be the elements of order 2 in $\mathrm{Gal}(\mathbb{K}_2^+/\mathbb{Q})$ that fix k_1 and $\Bbbk$, respectively. Using the identity $2 + (1+s+t+st) = (1+s) + (1+t) + (1+st)$ of the group ring $\mathbb{Z}[\mathrm{Gal}(\mathbb{K}_2^+/\mathbb{Q})]$ and observing that $\mathbb{Q}$ and the fixed field of st have odd class numbers, we find:

$$\mathcal{P}_2^2 \sim \mathcal{P}_2^{1+s}\mathcal{P}_2^{1+t}\mathcal{P}_2^{1+st} \sim \mathfrak{p}_2 P_2 \sim P_2,$$

where the last relation (in $\mathbf{C}l_2(\mathbb{K}_2^+)$) comes from the fact that $\mathfrak{p}_2$ capitulates in $\mathbb{K}_2^+$. Thus,

$$\mathcal{P}_2^{2^n} \sim P_2^{2^{n-1}} \sim \mathfrak{2} \quad \text{and} \quad \mathcal{P}_2^{2^{n+1}} \sim P_2^{2^n} \sim 1. \tag{2}$$

Note that for all $i \leq n-1$, $\mathcal{P}_2^{2^i} \not\sim 1$; otherwise, we get $P_2^{2^i} \sim 1$, which is absurd by Lemma 2.3. Hence, the class of $\mathcal{P}_2$ generates a subgroup of $\mathbf{C}l_2(\mathbb{K}_2^+)$ of order 2^n or 2^{n+1} accordingly as $\mathfrak{2}$ capitulates or not in $\mathbb{K}_2^+$.

On one hand, $N_{\mathbb{K}_2^+/k_1}(\langle[\mathcal{P}_2]\rangle)) = \langle[P_2]\rangle$ and $N_{\mathbb{K}_2^+/\Bbbk}(\langle[\mathcal{P}_2]\rangle)) = \langle[\mathfrak{p}_2]\rangle$, which is of index 2 in $\mathbf{C}l_2(\Bbbk)$; on the other hand, in $\mathbf{C}l_2(\mathbb{K}_2^+)$, we have $\mathcal{P}_2^{2^n} \sim P_2^{2^{n-1}} \sim \mathfrak{2} \sim \mathcal{P}_1$, where $\mathcal{P}_1$ is the prime ideal of $\mathbb{K}_2^+$ above p_1. Therefore, $\mathbf{C}l_2(\mathbb{K}_2^+)$ is cyclic generated by the class of $\mathcal{P}_2$, i.e.

$$\mathbf{C}l_2(\mathbb{K}_2^+) = \langle[\mathcal{P}_2]\rangle.$$

Finally, the class number formula implies that $h(\mathbb{K}_2^+) = q(\mathbb{K}_2^+)h(2p_1)$; thus, by the equation (2), $\mathfrak{2}$ capitulates in $\mathbb{K}_2^+$ if and only if $\mathcal{P}_2^{2^n} \sim 1$. Therefore, $\mathfrak{2}$ capitulates in $\mathbb{K}_2^+$ if and only if $q(\mathbb{K}_2^+) = 1$. Thus the results. $\square$

Remark 3.3 *Let p_1 and p_2 be primes as above and keep the previous notations. Then, for $j \in \{1, 3\}$, we have:*

$$q(\mathbb{K}_j^+) = 1 \Leftrightarrow \left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1 \;\Leftrightarrow\; p_1 \neq x^2 + 32y^2, \text{ where } x, y \in \mathbb{N}.$$

Proof Note first that $N(\epsilon_2) = -1$ and, by Lemma 2.2, $N(\epsilon_{2p_1p_2}) = -1$. Moreover, since $\left(\frac{p_1}{p_2}\right) = -1$ and $\left(\frac{2}{p_2}\right) = -1$, then according to [24] $N(\epsilon_{p_1p_2}) = -1$ and $N(\epsilon_{2p_2}) = -1$. Thus, [18] implies that $q(\mathbb{K}_j^+) = 1$ or 2. Hence, the first equivalence is assured by Theorem 3.1, and the second one is assured by [6]. Thus the results derived. □

Corollary 3.4 *Let p_1 and p_2 be different primes satisfying the conditions (1). Let $\mathbb{K}$ be an unramified quadratic extension of $\Bbbk$. Denote by ϵ_j, $1 \leq j \leq 3$, the three fundamental units of the three quadratic subfields of $\mathbb{K}$. Denote by $\mathcal{Z}$ the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ above 2. If $N(\epsilon_1) = N(\epsilon_2) = N(\epsilon_3) = -1$, then the equation $X^2 - \epsilon_1\epsilon_2\epsilon_3 = 0$ has a solution in $\mathbb{K}$ if and only if one of the following statements holds:*

1. $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = 1$, *if $\mathbb{K} = \mathbb{K}_1^+$ or $\mathbb{K}_3^+$.*

2. $\mathcal{Z}$ *does not capitulate in $\mathbb{K} = \mathbb{K}_2^+$.*

Proof Follows immediately from Theorems 3.1 and 3.2 and Lemma 2.1. □

Corollary 3.5 *Let p_1 and p_2 be primes as above and keep the previous notations. Put $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ and denote by $\Bbbk_2^{(1)}$ its first Hilbert 2-class field and by $\Bbbk_2^{(2)}$ its second Hilbert 2-class field. Put $G = \mathbf{G}al(\Bbbk_2^{(2)}/\Bbbk)$ and denote by $\mathcal{Z}$ the prime ideal of $\mathbb{Q}(\sqrt{2p_1})$ lies above 2. Then:*

1. *For $j \in \{1, 2, 3\}$ the following statements are equivalent:*
   a. $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1$,
   b. $\mathbf{C}l_2(\mathbb{K}_j^+) \simeq (2)$,
   c. *All the classes of $\mathbf{C}l_2(\Bbbk)$ capitulate in $\mathbb{K}_j^+$,*
   d. $G \sim (2, 2)$.

2. *For $j \in \{1, 3\}$ the following statements are equivalent:*
   a. $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = 1$,
   b. *Two classes of $\mathbf{C}l_2(\Bbbk)$ capitulate in $\mathbb{K}_j^+$,*
   c. $\mathbf{C}l_2(\mathbb{K}_j^+) \simeq (2, 2)$,
   d. *G is dihedral of order 2^m $(m \geq 8)$ or quaternionic of order 2^m $(m > 3)$, and, moreover, G is dihedral of order 2^m $(m \geq 8)$ if and only if $\mathcal{Z}$ capitulates in $\mathbb{K}_2^+$.*

Proof Let $E_{\mathbb{K}_j^+}$ and $E_\Bbbk$ be the unit groups of $\mathbb{K}_j^+$ and $\Bbbk$, respectively. It is well known from [10] that the number of classes of $E_\Bbbk$ that capitulate in $\mathbb{K}_j^+$ is $2[E_\Bbbk : N_{\mathbb{K}_j^+/\Bbbk}(E_{\mathbb{K}_j^+})]$. On the other hand, as $q(\mathbb{K}_j^+) = 1$ or 2 and, under our conditions, $[E_\Bbbk : N_{\mathbb{K}_j^+/\Bbbk}(E_{\mathbb{K}_j^+})] = 1$ or 2, then we deduce easily that:

$$[E_\Bbbk : N_{\mathbb{K}_j^+/\Bbbk}(E_{\mathbb{K}_j^+})] = 1 \Longleftrightarrow q(\mathbb{K}_j^+) = 2. \tag{3}$$

1. a. is equivalent by Theorem 3.1 to $q(\mathbb{K}_j^+) = 1$, which is equivalent by the equation (3) to c., and a. is also equivalent by [24] to $h(\mathbb{K}_j^+) = 2$. This in turn is equivalent by [15] to d.

2. a. is equivalent by Theorem 3.1 to $q(\mathbb{K}_j^+) = 2$, which is equivalent by the equation (3) to b.

We know from Lemma 2.4 that $\mathbf{C}l_2(\Bbbk) = \langle [\mathfrak{p}_1], [\mathfrak{p}_2] \rangle$, where $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2})$ and $\mathfrak{p}_1$, $\mathfrak{p}_2$ are the prime ideals above p_1 and p_2, respectively. We know also from Theorem 3.2 that $\mathbf{C}l_2(\mathbb{K}_2^+) = \langle [\mathcal{P}_2] \rangle$ with $\mathcal{P}_2$ being a prime ideal of $\mathbb{K}_2^+$ above p_2. Thus, $N_{\mathbb{K}_2^+/\Bbbk}(\mathbf{C}l_2(\mathbb{K}_2^+)) = \langle [\mathfrak{p}_2] \rangle$. As $\mathbb{K}_2^+ = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1})$, it is easy to see $N_{\mathbb{K}_2^+/\Bbbk}(\mathbf{C}l_2(\mathbb{K}_2^+)) \subset \kappa_{\mathbb{K}_2^+}$, where $\kappa_{\mathbb{K}_2^+}$ is the set of ideal classes of $\Bbbk$ that capitulate in $\mathbb{K}_2^+$. Hence, $\mathbb{K}_2^+$ satisfies Taussky's condition A. Therefore, G is never a semidihedral group (see [15]).

Proceeding as in the proof of Theorem 3.2, we determine the generators of $\mathbf{C}l_2(\mathbb{K}_1^+)$ and $\mathbf{C}l_2(\mathbb{K}_3^+)$. From that we deduce that b. is equivalent to c. By calculating $N_{\mathbb{K}_j^+/\Bbbk}(\mathbf{C}l_2(\mathbb{K}_j^+)$, $1 \leq j \leq 3$, we notice (using Taussky's conditions) that we have two cases of capitulation: $4 \quad 2B \quad 2B$ or $2A \quad 2B \quad 2B$.

The first case occurs if and only if G is dihedral of order 2^m $(m \geq 3)$, and the second one occurs if and only if G is quaternionic of order 2^m $(m > 3)$ (for more details, see [15]). Therefore, the equivalence between c. and d. is assured by Theorem 3.2 and [15]. □

4. The 2-rank of some CM-fields

Recall that p_1 and p_2 are different primes satisfying the following conditions:

$$p_1 \equiv p_2 \equiv 1 \pmod 4 \text{ and } \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = -\left(\frac{2}{p_1}\right) = -1.$$

Consider the field $\mathbb{F} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$. The goal of this section is to compute the 2-rank of the 2-class groups of the fields $\mathbb{K}_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, \sqrt{-1})$, $\mathbb{K}_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1}, \sqrt{-1})$, $\mathbb{K}_3 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, \sqrt{-1})$, and $\mathbb{F}^{(*)} = \mathbb{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, i)$. Let us begin by $\mathbb{K}_2$.

Theorem 4.1 *Let p_1 and p_2 be different primes satisfying the conditions (1). Assume $\left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1$, and consider $\mathbb{K}_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1}, i)$. Then $\mathbf{C}l_2(\mathbb{K}_2)$, the 2-class group of $\mathbb{K}_2$, is of type $(2, 2^{\ell+1})$, where $2^\ell = h(-2p_1)$ and $\ell \in \mathbb{N}^*$.*

Proof Setting $F = \mathbb{Q}(\sqrt{p_2}, i)$, then according to [2] the unit group of F is $E_F = \langle i, \epsilon_{p_2} \rangle$. As $h(F) = \frac{1}{2}q(F)h(p_2)h(-p_2) = 1$, so the class number of F is odd. Therefore, the 2-rank of the 2-class group of $\mathbb{K}_2$ is equal to $r = t - e - 1$, where t is the number of finite and infinite primes of F ramified in $\mathbb{K}_2/F$ and $2^e = [E_F : E_F \cap N_{\mathbb{K}_2/F}(\mathbb{K}_2^\times)]$.

Let us compute t. Let p be a prime number of $\mathbb{Q}$ and denote by $\mathfrak{p}_M$ a prime ideal of some extension $M/\mathbb{Q}$, which lies above p, and $e(\mathfrak{p}_M/p)$ its ramification index.

As the extension $\mathbb{K}_2/\mathbb{F}$ is unramified, then $e(\mathfrak{p}_F/p).e(\mathfrak{p}_{\mathbb{K}_2}/\mathfrak{p}_F) = e(\mathfrak{p}_F/p)$. Since 2 is totally ramified in $\mathbb{F}$ and inert in $\mathbb{Q}(\sqrt{p_2})$, then there is only one ideal prime of F above 2 that ramifies in $\mathbb{K}_2$. On the other hand, p_1 is inert in $\mathbb{Q}(\sqrt{p_2})$ and hence $e({\mathfrak{p}_1}_F/p_1) = 1$, and since $e({\mathfrak{p}_1}_F/p_1) = 2$, then $e({\mathfrak{p}_1}_{\mathbb{K}_2}/{\mathfrak{p}_1}_F) = 2$.

Finally, $e(\mathfrak{p}_{2F}/p_2)=2$, and as $e(\mathfrak{p}_{2\mathbb{F}}/p_2)=2$, we deduce that $e(\mathfrak{p}_{2\mathbb{K}_2}/\mathfrak{p}_{2F})=1$. Thus, $t=3$ and $r=2-e$, i.e. the 2-rank of $\mathbb{K}_2$ is $r=2-e$.

To compute e, we have to find units of F that are norms of some elements of $\mathbb{K}_2^\times$. Letting $\mathfrak{p}$ be an ideal of F such that $\mathfrak{p}\neq 2_F$, then we have:

- If $\mathfrak{p}$ is not above p_1, then $v_\mathfrak{p}(\epsilon_{p_2})=v_\mathfrak{p}(2p_1)=v_\mathfrak{p}(i)=0$. Hence, $\left(\frac{2p_1,\epsilon_{p_2}}{\mathfrak{p}}\right)=1$ and $\left(\frac{2p_1,i}{\mathfrak{p}}\right)=1$.
- If $\mathfrak{p}=\mathfrak{p}_{1F}$ is above p_1, then $v_\mathfrak{p}(\epsilon_{p_2})=v_\mathfrak{p}(i)=0$ and $v_\mathfrak{p}(2p_1)=1$. As $\mathfrak{p}$ is not ramified in both of $F(\sqrt{i})$ and $F(\sqrt{\epsilon_{p_2}})$, so

$$\begin{cases}\left(\frac{\epsilon_{p_2},2p_1}{\mathfrak{p}_{1F}}\right)=\left(\frac{\epsilon_{p_2}}{\mathfrak{p}_{1F}}\right)=\left(\frac{\epsilon_{p_2}^2}{\mathfrak{p}_{1\mathbb{Q}(\sqrt{p_2})}}\right)=1,\\ \left(\frac{i,2p_1}{\mathfrak{p}_{1F}}\right)=\left(\frac{i}{\mathfrak{p}_{1F}}\right)=\left(\frac{-1}{\mathfrak{p}_{1\mathbb{Q}(i)}}\right)=1.\end{cases}$$

Therefore, for all prime ideal $\mathfrak{p}$ of F, the product formula for the Hilbert symbol implies that $\left(\frac{\epsilon_{p_2},2p_1}{\mathfrak{p}}\right)=\left(\frac{i,2p_1}{\mathfrak{p}}\right)=1$.

From this, we deduce that $e=0$ and $r=2$.

We prove now that the 4-rank of $\mathbf{C}l_2(\mathbb{K}_2)$ is 1. For this, put $k=\mathbb{Q}(\sqrt{-2p_1p_2})$ and denote by $k^{(*)}=\mathbb{Q}(\sqrt{-2},\sqrt{p_1},\sqrt{p_2})$ its genus field (see Figure 3). Note that $q(L_1)=q(L_2)=q(L_3)=1$, so the 2-class

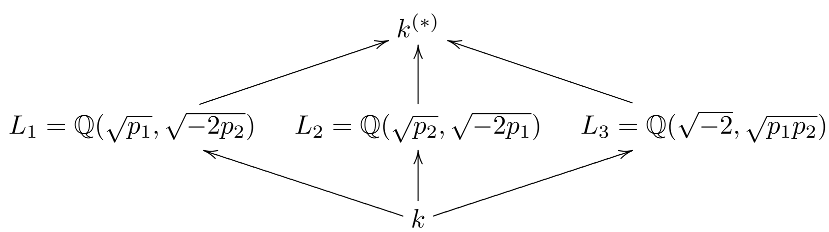

Figure 3. Subfields of $k^{(*)}/k$.

group of k is of type $(2,2)$ (see [14]). The class number formula implies that $h(L_1)=4$, $h(L_2)=2h(-2p_1)$, and $h(L_3)=4$. On the other hand, according to [3], the 2-class group of L_3 is of type $(2,2)$. Thus, by [7] we have $\left(\frac{2}{p_1}\right)_4\left(\frac{p_1}{2}\right)_4=-1$ if and only if $k^{(*)}=k_2^{(1)}=k_2^{(2)}$. Hence, by [15] and [25] we get that the 2-class group of L_1 is of type $(2,2)$ and that of L_2 is cyclic of order $2h(-2p_1)$. To this end, consider the application:

$$\begin{array}{rccl}\varphi: & \mathbf{C}l_2(\mathbb{K}_2) & \longrightarrow & \mathbf{C}l_2(L_2)\\ & c & \longmapsto & N_{\mathbb{K}_2/L_2}(c).\end{array}$$

As $h(\mathbb{K}_2)=4h(-2p_1)$ and $h(L_2)=2h(-2p_1)$, so $|ker\varphi|=2$. Since also the 2-rank of $\mathbf{C}l_2(\mathbb{K}_2)$ is 2 and that of $\mathbf{C}l_2(L_2)$ is 1, then the 4-rank of $\mathbf{C}l_2(\mathbb{K}_2)$ is 1. Hence, $\mathbf{C}l_2(\mathbb{K}_2)$ is of type $(2,2h(-2p_1))=(2,2^{\ell+1})$, where $2^\ell=h(-2p_1)$. □

Theorem 4.2 *Let p_1 and p_2 be different primes satisfying the conditions (1). Assume $\left(\frac{2}{p_1}\right)_4\left(\frac{p_1}{2}\right)_4 = -1$, and consider the field $\mathbb{K}_3 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$; then $\mathbf{C}l_2(\mathbb{K}_3)$, the 2-class group of $\mathbb{K}_3$, is of type $(2, 2, 2)$.*

Proof Putting $F = \mathbb{Q}(\sqrt{2}, i)$ and letting ϵ_2 be the fundamental unit of $\mathbb{Q}(\sqrt{2})$, from [2] we get that the unit group of F is $E_F = \langle i, \epsilon_2\rangle$. It is well known that the class number of F is odd. Thus, the 2-rank of the 2-class group of $\mathbb{K}_3$ is $r = t - e - 1$, where t is the number of finite and infinite primes of F ramified in $\mathbb{K}_3/F$ and $2^e = [E_F : E_F \cap N_{\mathbb{K}_3/F}(\mathbb{K}_3^\times)]$. Proceeding as in Theorem 4.1 we prove that $r = 3$. On the other hand, the 2-class number of $\mathbb{K}_3$ is $h(\mathbb{K}_3) = 4q(\mathbb{K}_3) = 8$. Hence the result. □

Theorem 4.3 *Let p_1 and p_2 be different primes satisfying the conditions (1). Assume $\left(\frac{2}{p_1}\right)_4\left(\frac{p_1}{2}\right)_4 = -1$, and consider the field $\mathbb{K}_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$; then $\mathbf{C}l_2(\mathbb{K}_1)$, the 2-class group of $\mathbb{K}_1$, is of type $(2, 2, 4)$.*

Proof Putting $F = \mathbb{Q}(\sqrt{p_1}, i)$ and letting ϵ_{p_1} be the fundamental unit of $\mathbb{Q}(\sqrt{p_1})$, then by [2] the unit group of F is $E_F = \langle i, \epsilon_{p_1}\rangle$. As $p_1 \equiv 1 \pmod 8$, then the class number of F is even and hence the 2-rank of the class group of $\mathbb{K}_1$ satisfies $r \geq t - e - 1$, where t is the number of finite and infinite primes of F ramified in $\mathbb{K}_1/F$, and $2^e = [E_F : E_F \cap N_{\mathbb{K}_1/F}(\mathbb{K}_1^\times)]$. Proceeding as in Theorem 4.1 we prove that $t = 4$. Thus, $r \geq 3 - e$. Let us calculate e by computing the units of F that are norms of some elements of $\mathbb{K}_1^\times$.

Keep the notation that $\mathfrak{p}_M$ denotes a prime ideal of some extension $M/\mathbb{Q}$ lying above a prime number p of $\mathbb{Q}$, and let $e(\mathfrak{p}_M/p)$ be its ramification index.

Since $\mathfrak{p}_{2F}$ is unramified in both of $F(\sqrt{i})$ and $F(\sqrt{\epsilon_{p_1}})$, so

$$\begin{cases} \left(\dfrac{\epsilon_{p_1}, 2p_2}{\mathfrak{p}_{2F}}\right) = \left(\dfrac{\epsilon_{p_1}}{\mathfrak{p}_{2F}}\right) = \left(\dfrac{-1}{\mathfrak{p}_{2\mathbb{Q}(\sqrt{i})}}\right) = 1 \\ \left(\dfrac{i, 2p_2}{\mathfrak{p}_{2F}}\right) = \left(\dfrac{i}{\mathfrak{p}_{2F}}\right) = \left(\dfrac{-1}{\mathfrak{p}_{2\mathbb{Q}(\sqrt{i})}}\right) = 1. \end{cases}$$

Similarly, as 2_F is unramified in $F(\sqrt{p_2})$, so

$$\left(\frac{i, 2p_2}{2_F}\right) = \left(\frac{i, 2}{2_F}\right)\left(\frac{i, p_2}{2_F}\right) = \left(\frac{i, 2}{2_F}\right) = \left(\frac{i, i^{-1}}{2_F}\right)\left(\frac{i, 2i}{2_F}\right) = 1.$$

Finally, since $N(\epsilon_{p_1}) = -1$, then $2\pi_1\epsilon_{p_1}$ is a square in F (where π_1, $\pi_2 \in \mathbb{Z}[i]$ and $p_1 = \pi_1\pi_2$), and hence $\left(\dfrac{\epsilon_{p_1}, 2}{\mathfrak{p}_{2F}}\right) = \left(\dfrac{2\pi_1, 2}{\mathfrak{p}_{2F}}\right)$, so

$$\left(\frac{\epsilon_{p_1}, 2p_2}{2_F}\right) = \left(\frac{\epsilon_{p_1}, 2}{2_F}\right)\left(\frac{\epsilon_{p_1}, p_2}{2_F}\right) = \left(\frac{\epsilon_{p_1}, 2}{2_F}\right) = \left(\frac{2\pi_1, 2}{2_F}\right) = \left(\frac{\pi_1}{2_F}\right)^{v_{2_F}(2)} = 1.$$

Consequently, $e = 0$, and thus $r \geq 3$.

Setting $k_0 = \mathbb{Q}(\sqrt{-p_1}, \sqrt{2p_2})$, we will compute the 2-rank of the class group of k_0. For this, we use the notations of [22]. Putting $k_1 = \mathbb{Q}(\sqrt{-p_1})$, $k_2 = \sqrt{2p_2}$, $k_3 = \mathbb{Q}(\sqrt{-2p_1p_2})$, $l = p_1$, $q = 2$, and $r = 2$, then

$t_1 = 2$, $t_2 = 2$, and $t_3 = 3$. Thus, $t = 4$, $r_a = 4$, $w = 1$, $x = 0$, and $y = 1$ and consequently the 2-rank of the class group of k_0 is $r_2 = 4 - 1 - 0 - 1 = 2$.

On the other hand, since $q(k_0) = 1$ and $q(\mathbb{K}_1) = 2$, then the class number formula implies that $h(k_0) = h(\mathbb{K}_1) = 16$. Hence, $\mathbf{C}l_2(\mathbb{K}_1)$ is of type $(2,2,2,2)$ or $(2,2,4)$.

To this end, $\mathbb{K}_1$ is an unramified quadratic extension of k_0, and then

$$\mathbf{C}l_2(\mathbb{K}_1)/\mathbf{C}l_2(\mathbb{K}_1)^{1-\sigma} \simeq N_{\mathbb{K}_1/k_0}(\mathbf{C}l_2(\mathbb{K}_1)),$$

where $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{K}_1/k_0)$. If we suppose that $\mathbf{C}l_2(\mathbb{K}_1)$ is of type $(2,2,2,2)$, we will get that $N_{\mathbb{K}_1/k_0}(\mathbf{C}l_2(\mathbb{K}_1))$ is of type $(2,2,2)$ since $\mathbf{C}l_2(\mathbb{K}_1)^{1-\sigma}$ is of index 2. However, this contradicts the fact that $N_{\mathbb{K}_1/k_0}(\mathbf{C}l_2(\mathbb{K}_1))$ is a subgroup of $\mathbf{C}l_2(\mathbb{k})$ that is of 2-rank equal to 2. Therefore, $\mathbf{C}l_2(\mathbb{K}_1)$ is of type $(2,2,4)$. □

Theorem 4.4 *Let p_1 and p_2 be different primes satisfying the conditions (1). Put $\mathbb{F} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$ and denote by $\mathbb{F}^{(*)} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{2}, i)$ its genus field. Then the rank of $\mathbf{C}l_2(\mathbb{F}^{(*)})$, the 2-class group of $\mathbb{F}^{(*)}$, is 2 and $h(\mathbb{F}^{(*)}) = 4h(-2p_1)$.*

Proof Put $K = \mathbb{Q}(\sqrt{-p_1}, \sqrt{p_2}, \sqrt{2})$, $F = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{2})$, and $L = \mathbb{Q}(\sqrt{p_2}, \sqrt{2}, i)$. It is easy to see that $\mathbb{F}^{(*)}/L^+$ is a V_4-extension of CM-type fields, The following diagram (Figure 4) clarifies this. According to [21]

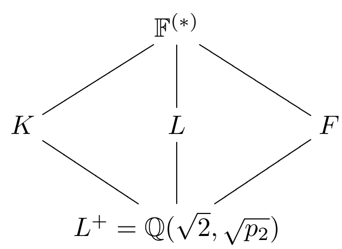

Figure 4. Subfields of $\mathbb{F}^{(*)}/L^+$.

we have:

$$h(\mathbb{F}^{(*)}) = \frac{Q_{\mathbb{F}^{(*)}}}{Q_K Q_L} \cdot \frac{\omega_{\mathbb{F}^{(*)}}}{\omega_K \omega_L} \cdot \frac{h(K)h(L)h(F)}{h(L^+)^2}. \tag{4}$$

To this end, note that $\omega_{\mathbb{F}^{(*)}} = \omega_L = 4\omega_K = 8$, $W_{\mathbb{F}^{(*)}} = W_L$, and $W_K = \{\pm 1\}$. On the other hand, by [12] we get $Q_{\mathbb{F}^{(*)}} = 1$; thus, $Q_L = 1$, since by [21], we have $Q_L | Q_{\mathbb{F}^{(*)}}[W_{\mathbb{F}^{(*)}} : W_L]$.

As $q(L^+) = 2$, i.e. $\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}$ is a square in L^+, then according to [2] $\{\epsilon_2, \epsilon_{p_2}, \sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}\}$ is not a fundamental system of units of K if and only if there exist α, β, and γ in $\{0,1\}$, not all zero, such that $p_1\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}^{\alpha}\epsilon_2^{\beta}\epsilon_{p_2}^{\gamma}$ is a square in L^+. Supposing that $p_1\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}^{\alpha}\epsilon_2^{\beta}\epsilon_{p_2}^{\gamma} = X^2$, where $X \in L^+$, then $N_{L^+/\mathbb{Q}(\sqrt{2})}(X^2) = p_1^2\epsilon_2^{\alpha}\epsilon_2^{2\beta}(-1)^{\gamma}$, and thus $\gamma = 0$ and $\alpha = 0$ since ϵ_2 is not a square in $\mathbb{Q}(\sqrt{2})$. Consequently, $X^2 = p_1\epsilon_2^{\beta}$, and this implies that $\beta = 1$. Hence, $N_{L^+/\mathbb{Q}(\sqrt{p_2})}(X^2) = -p_1^2$, which is false. Therefore, $\{\epsilon_2, \epsilon_{p_2}, \sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}\}$ is a fundamental system of units of K. We conclude that $q(K) = 2$ and $Q_K = 1$. Similarly, we prove that $\{\epsilon_2, \epsilon_{p_2}, \sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}\}$ is a fundamental system of units of L and $q(L) = 4$.

Finally, by Theorem 4.1, $h(F)=1$. The class number formula yields that $h(L)=1$ and $h(K)=2h(-p_1)h(-2p_1)$. By replacement in formula (4) we get: $h(\mathbb{F}^{(*)})=h(-p_1)h(-2p_1)$. As also $\left(\frac{2}{p_1}\right)_4\left(\frac{p_1}{2}\right)_4=-1$, so $h(-p_1)=4$, and hence $h(\mathbb{F}^{(*)})=4h(-2p_1)$.

We know that the class number of $L=\mathbb{Q}(\sqrt{2},\sqrt{p_2},i)$ is odd, so then the 2-rank of the class group of $\mathbb{F}^{(*)}$ is given by the formula $r=t-e-1$, where t is the number of finite and infinite primes of L ramified in $\mathbb{F}^{(*)}/L$ and $2^e=[E_L:E_L\cap N_{\mathbb{F}^{(*)}/L}(\mathbb{F}^{(*)\times})]$. We compute t by using Figure 5.

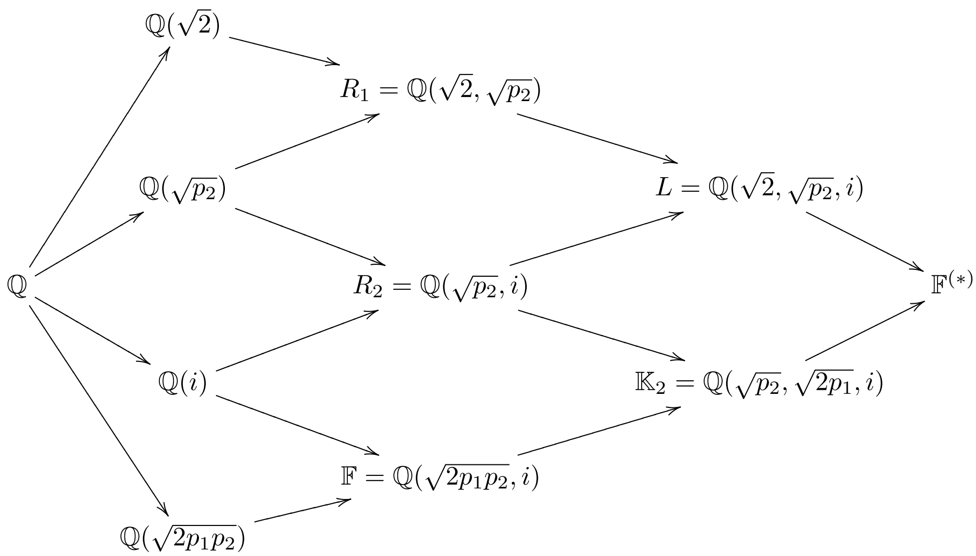

Figure 5. Subfields of $\mathbb{F}^{(*)}/\mathbb{Q}$.

Since $\mathbb{F}^{(*)}$ is an unramified extension of $\mathbb{K}_2$, and $\mathbb{K}_2$ is also an unramified extension of $\Bbbk$, then it is easy to see that there are 4 prime ideals of L that ramify in $\mathbb{F}^{(*)}$ and they all lie above p_1. Thus, $t=4$, and $r=3-e$.

Let us now compute e. For this we will use the Hilbert symbol. We know that $E_L=\langle\sqrt{i},\epsilon_2,\epsilon_{p_2},\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}\rangle$; denote by $\mathfrak{p}_{jL}$, $j\in\{1,2,3,4\}$, the prime ideals of L above p_1; and denote also by $\mathfrak{p}_{1M}$ an ideal prime of some extension $M/\mathbb{Q}$ that is above p_1.

Since $\mathfrak{p}_{jL}$ is unramified in $L(\sqrt{\sqrt{i}})$ and $v_{\mathfrak{p}_{jL}}(p_1)=1$, then

$$\left(\frac{\sqrt{i},p_1}{\mathfrak{p}_{jL}}\right)=\left(\frac{\sqrt{i}}{\mathfrak{p}_{jL}}\right)=\left(\frac{\sqrt{2}(1+i)}{\mathfrak{p}_{jL}}\right)=\left(\frac{\sqrt{2}}{\mathfrak{p}_{jL}}\right)\left(\frac{1+i}{\mathfrak{p}_{jL}}\right)=\left(\frac{1+i}{\mathfrak{p}_{1R_2}}\right)\left(\frac{\sqrt{2}}{\mathfrak{p}_{1R_1}}\right)=$$
$$\left(\frac{(1+i)^2}{\mathfrak{p}_{1\mathbb{Q}(i)}}\right)\left(\frac{2}{\mathfrak{p}_{1\mathbb{Q}(\sqrt{2})}}\right)=\left(\frac{2}{p_1}\right)=1.$$

We have also that $\mathfrak{p}_{jL}$ is unramified in both of $L(\sqrt{\epsilon_2})$ and $L(\sqrt{\epsilon_{p_2}})$, and $v_{\mathfrak{p}_{jL}}(p_1)=1$, so then

$$\begin{cases} \left(\dfrac{\epsilon_2, p_1}{\mathfrak{p}_{1L}}\right) = \left(\dfrac{\epsilon_2}{\mathfrak{p}_{1R_1}}\right) = \left(\dfrac{\epsilon_2^2}{\mathfrak{p}_{1\mathbb{Q}(\sqrt{2})}}\right) = 1 \\ \left(\dfrac{\epsilon_{p_2}, p_1}{\mathfrak{p}_{1L}}\right) = \left(\dfrac{\epsilon_{p_2}}{\mathfrak{p}_{1R_1}}\right) = \left(\dfrac{\epsilon_{p_2}}{\mathfrak{p}_{1\mathbb{Q}(\sqrt{p_2})}}\right) = \left(\dfrac{-1}{p_1}\right) = 1. \end{cases}$$

Similarly, we get

$$\left(\frac{\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}, p_1}{\mathfrak{p}_{1L}}\right) = \left(\frac{\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}}{\mathfrak{p}_{1R_1}}\right) = \left(\frac{\epsilon_2}{\mathfrak{p}_{1\mathbb{Q}(\sqrt{2})}}\right) = \left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4 = -1.$$

Consequently, $\sqrt{\epsilon_2\epsilon_{p_2}\epsilon_{2p_2}}$ is not a norm of some element from $\mathbb{F}^{(*)}$. Thus, $e = 1$, and the 2-rank of $\mathbb{F}^{(*)}$ is $r = 2$. □

5. Numerical examples

In this section and in the following Table, we give examples that illustrate our results. The first column gives the number $d = 2p_1p_2$, the second (resp. third, fourth, fifth, sixth) gives the class group of the field $\mathbb{F} = \mathbb{Q}(\sqrt{d}, i)$ (resp. $\mathbb{K}_1$, $\mathbb{K}_2$, $\mathbb{K}_3$, $\mathbb{F}^{(*)}$), and the seventh (resp. eighth) column gives the biquadratic residue symbol $a = \left(\frac{2}{p_1}\right)_4 \left(\frac{p_1}{2}\right)_4$ (resp. the unit index $b = q(\mathbb{K}_3^+)$). The computations are made using PARI/GP [23].

Table. Numerical examples.

$2.p_1.p_2$	$\mathbf{C}l(\mathbb{k})$	$\mathbf{C}l(\mathbb{K}_1)$	$\mathbf{C}l(\mathbb{K}_2)$	$\mathbf{C}l(\mathbb{K}_3)$	$\mathbf{C}l(\mathbb{F}^{(*)})$	a	b
2.17.5	[6, 2, 2]	[6, 2, 2]	[12, 2, 2]	[24, 2]	[12, 4]	−1	1
2.73.5	[6, 6, 2]	[30, 6, 2]	[12, 6, 2]	[96, 6]	[240, 12]	−1	1
2.97.5	[6, 2, 2]	[30, 2, 2]	[12, 2, 2]	[120, 2]	[60, 20]	−1	1
2.17.29	[22, 2, 2]	[66, 2, 2]	[44, 2, 2]	[264, 2]	[132, 12]	−1	1
2.41.13	[10, 2, 2]	[30, 2, 2, 2]	[120, 2, 2]	[40, 4]	[120, 12, 2]	1	2
2.113.5	[22, 2, 2]	[66, 2, 2, 2]	[88, 2, 2]	[176, 8]	[264, 8, 8]	1	2
2.17.37	[6, 2, 2]	[18, 6, 2]	[60, 2, 2]	[24, 2]	[180, 12]	−1	1
2.137.5	[22, 2, 2]	[66, 2, 2, 2]	[88, 2, 2]	[264, 4]	[264, 12, 2]	1	2
2.73.13	[14, 2, 2]	[42, 2, 2]	[84, 2, 2]	[224, 2]	[336, 12]	−1	1
2.193.5	[10, 2, 2]	[110, 2, 2]	[20, 2, 2]	[40, 10]	[220, 20]	−1	1
2.17.61	[10, 2, 2]	[10, 10, 2]	[20, 10, 2]	[120, 2]	[60, 20, 5]	−1	1
2.89.13	[14, 2, 2]	[14, 14, 2]	[84, 6, 2]	[112, 2]	[168, 84]	−1	1
2.233.5	[30, 2, 2]	[30, 10, 2]	[60, 6, 2]	[240, 2]	[120, 60]	−1	1
2.41.29	[30, 2, 2]	[30, 10, 2, 2]	[120, 2, 2]	[120, 12]	[240, 60, 2]	1	2
2.97.13	[18, 2, 2]	[90, 2, 2]	[36, 6, 2]	[360, 2]	[180, 60]	−1	1
2.257.5	[22, 2, 2]	[66, 2, 2, 2]	[528, 2, 2]	[352, 4]	[528, 48, 2]	1	2
2.313.5	[14, 2, 2]	[14, 14, 2, 2]	[56, 2, 2]	[504, 4]	[1008, 28, 2]	1	2
2.337.5	[18, 2, 2]	[234, 2, 2, 2]	[72, 2, 2]	[144, 12]	[936, 24, 2]	1	2
2.353.5	[26, 2, 2]	[390, 2, 2, 2]	[208, 2, 2]	[624, 4]	[3120, 24, 2]	1	2

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