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Small covers over products of a simple polytope with a simplex

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Abstract: This paper proves that the number of small covers over products of a simple polytope with a n-simplex, up to D-J equivalence, is a polynomial in the variable 2^n. A similar result holds for orientable small covers. We also provide a new way of computation, namely computing the finite number of representatives and interpolating polynomially. The ratio between the number of orientable small covers and the number of small covers is given. As an application, by interpolation, we determine the polynomials related to small covers and orientable small covers over products of a prism with a simplex up to D-J equivalence. A formula for calculating the number of equivariant homeomorphism classes of small covers over the product is also provided.

Key words: Small cover, equivalence class, equivariant homeomorphism, polytope, polynomial

1. Introduction

The small cover, introduced by Davis and Januszkiewicz [9], has recently become one of the most interesting objects in toric topology. A closed manifold M^m is called a small cover if M^m admits a locally standard $(\mathbb{Z}_2)^m$-action such that its orbit space is homeomorphic to a simple convex m-polytope P^m. A typical example of a small cover is the Stong manifold [8] defined by means of a real projective bundle, which can be used as generators in the Thom unoriented cobordism ring. More generally, a generalized real Bott manifold was introduced as a small cover in [7] by Choi et al., which appears in a sequence of projective bundles starting with a point

$$\mathbb{R}B_k \xrightarrow{\pi_k} \mathbb{R}B_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} \mathbb{R}B_1 \xrightarrow{\pi_1} \mathbb{R}B_0 = \{\text{a point}\},$$

where $\mathbb{R}B_i$ is the projectivization of the Whitney sum of n_i+1 real line bundles over $\mathbb{R}B_{i-1}$. Each $\mathbb{R}B_i$ provides a small cover over $\prod_{j=1}^{i}\Delta^{n_j}$ where Δ^{n_j} is the n_j-simplex. $\mathbb{R}B_i$ is called a generalized real Bott manifold of height i starting with a point. If each $n_i = 1$, then it is called a real Bott manifold; see [15].

In the above sequence, if we replace "a point" with a closed manifold M^m, which is a small cover over P^m, then we get a new sequence starting with M^m:

$$\mathbb{R}B_k \xrightarrow{\pi_k} \mathbb{R}B_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} \mathbb{R}B_1 \xrightarrow{\pi_1} \mathbb{R}B_0 = M^m.$$

We call $\mathbb{R}B_i$ a generalized real Bott manifold of height i starting with M^m. In fact, each $\mathbb{R}B_i$ is a small cover over $P^m \times \prod_{j=1}^{i}\Delta^{n_j}$; see [12].

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For $\mathbb{R}B_0 = \{\text{a point}\}$, Choi et al. [7] proved that every small cover over a product of simplices is D-J equivalent (see definition 4) to a generalized real Bott manifold, and so the number of D-J equivalence classes of small covers over a product of simplices is equal to the number of equivalence classes of generalized real Bott manifolds. We could ask a similar question for $\mathbb{R}B_0 = M^m$. Since the general question seems quite hard to solve, it is reasonable to restrict P^m, the height i, or n_i for enumerating small covers up to various equivalences. Generally speaking, if the height i is fixed, it is more difficult to enumerate small covers for large m, $l-m$, and n_i than for small m, $l-m$, or n_i. For $m \leq 3$ and some n_i, several papers have studied the equivalence classes of small covers over the space.

For $m = 1$, Choi studied small covers over cubes [5], which are obtained as the projectivization of a Whitney sum of two real line bundles. In particular he associated an acyclic digraph with labeled n vertices to a small cover over an n-cube and proved that the correspondence is bijective. Using the number of acyclic digraphs with n labeled vertices, he counted the number of equivariant homeomorphism classes of small covers over an n-cube. In [6] orientable small covers over cubes were counted. Choi also obtained estimates for O_n/R_n, where O_n stands for the number of orientable small covers and R_n denotes the number of all small covers over an n-cube up to D-J equivalence.

For $m = 2$, using combinatorial tools, Cai et al. gave a counting formula for the number of small covers of 3-dimensional prisms [3]. Later, as a generalization of this result, Lü and Masuda investigated a closed smooth manifold of dimension n with a nonfree effective action of $(\mathbb{Z}_2)^n$ [13]. The orbit space Q is a nice manifold with corners. When Q is a simple polytope, the manifold is a small cover. They dealt with the classification of all 2-torus manifolds with a given orbit space up to equivariant homeomorphism and showed that the equivariant homeomorphism classes can be identified with the coset space $(H^1(Q;(\mathbb{Z}_2)^n) \times \Lambda(Q))/Aut(Q)$. On this basis, using the fact that the equivariant homeomorphism classes of small covers correspond to the orbits of the action of the automorphism group of the polytope on the set of colorings, Wang and Chen [17] obtained a formula for calculating the number of equivariant homeomorphism classes of small covers and orientable small covers over products of a polygon with a simplex.

In a similar manner, for $m = 3$, Chen et al. [4] calculated the number of equivariant homeomorphism classes of (orientable) small covers over products of the polar of the cyclic polytope $C^3(6)$ with a simplex.

On the other hand, for $\mathbb{R}B_0 = \{\text{a point}\}$, Kamishima and Masuda [11] studied the cohomological rigidity problem for real toric manifolds and asserted that two real Bott manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}_2$-coefficients are isomorphic as graded rings. Furthermore, in another paper [15], using the structure of stable KO groups of real projective spaces, Masuda gave a necessary and sufficient condition for cohomological rigidity of generalized real Bott manifolds of height two and found counterexamples of cohomological rigidity for real toric manifolds. He also showed that if two generalized real Bott manifolds of height two are homotopy equivalent then they are diffeomorphic.

In [16], Nakayama and Nishimura investigated the topology of small covers using the coloring theory in combinatorics and gave an orientability condition for a small cover. They showed the existence of an orientable small cover over every simple convex 3-polytope and the existence of a nonorientable small cover over every simple convex 3-polytope, except for the 3-simplex.

From a geometric point of view, Garrison and Scott [10] discussed small covers over the right-angled dodecahedron and the right-angled 120-cell. Small covers over right-angled hyperbolic polytopes provide examples of closed hyperbolic manifolds. They used an algorithm to find all the small covers over the dodecahedron (there are 25), and to find small covers over the 120-cell of minimal complexity. In [14], Lü and Yu showed that every 3-dimensional small cover can be obtained from standard actions on $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ through the application of six explicitly described geometric operations.

In this work, suppose that M^m is a small cover over P^m, which is a fixed simple polytope with l facets. Then we generate a series of polytopes $P^m \times \Delta^n (n = 1, 2, \cdots)$. We prove that the number of all small covers (or orientable small covers) over $P^m \times \Delta^n$, up to D-J equivalence, is a polynomial in the variable 2^n (or 2^{n-1}); see Theorem 12, which extends several previous results [3, 5–7, 12, 17]. We thus provide a new way of computation by computing the finite number of representatives in the series and interpolating polynomially. Moreover, we find that the ratio between the number of orientable small covers and the number of small covers over $P^m \times \Delta^n$ approaches $1/2^{l-m}$ as odd n increases, and the ratio approaches 0 as even n increases. Next, as an application of Theorem 12, we intend to specify a generalized real Bott manifold as a starting small cover over $P^m (m \geq 4)$, and, by interpolation, determine the polynomial. For this purpose, we take $P^m = P_6^3 \times I$ as an example, where P_6^3 (see the Figure) is the dual of the cyclic polytope $C^3(6)$ as a convex hull of 6 distinct points on a curve $\gamma(t) = \{(t, t^2, t^3) : t \in \mathbb{R}\}$ in $\mathbb{R}^3$; see [2]. In this case, it turns out that, up to D-J equivalence, the number of small covers is greater than the number of generalized real Bott manifolds. In addition, a formula for calculating the number of equivariant homeomorphism classes of small covers over the product is also provided; see Theorem 18 and Theorem 21.

This paper is organized as follows. In Section 2, we review the basic theory of small covers. In Section 3, we prove polynomial theorems concerning D-J equivalence classes of (orientable) small covers over $P^m \times \Delta^n$ and get the ratio. In Section 4, we take $P^m = P_6^3 \times I$ as an example, by interpolation, to determine the polynomial related to small covers up to D-J equivalence. We also determine the number of equivariant homeomorphism classes of small covers. In Section 5, we deal with the orientable small covers over $P^m \times \Delta^n$ for $P^m = P_6^3 \times I$.

2. Preliminaries

Definition 1 *(see [18]) An m-dimensional convex polytope P^m is simple if precisely m facets (i.e. codimension-one faces) meet at each vertex.*

Definition 2 *(see [9]) An m-dimensional closed manifold M^m is a small cover if it admits a $(\mathbb{Z}_2)^m$-action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}^m$ and the orbit space $M^m/(\mathbb{Z}_2)^m$ is homeomorphic to a simple convex polytope P^m.*

Let P^m be a simple convex polytope of dimension m and $\mathcal{F}(P^m) = \{F_1, \cdots, F_l\}$ be the set of facets of P^m. Suppose that $\pi : M^m \to P^m$ is a small cover over P^m. For any $x \in \pi^{-1}(\text{int}(F_i))$, the isotropy group at x is independent of the choice of x; denote it by $\mathbb{Z}_2(F_i)$, which is a rank-one subgroup. $\mathbb{Z}_2(F_i)$ actually agrees with an element ν_i in $(\mathbb{Z}_2)^m$ as a vector space. In this way, we obtain a characteristic function

$$\lambda : \mathcal{F}(P^m) \longrightarrow (\mathbb{Z}_2)^m$$

defined by $\lambda(F_i) = \nu_i$ such that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is nonempty, $\lambda(F_{i_1}), \cdots,$ $\lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^m$; see [9]. If we regard each nonzero vector of $(\mathbb{Z}_2)^m$ as being a color, then the characteristic function λ means that each facet is colored by a color. Here we also call λ a $(\mathbb{Z}_2)^m$-coloring on P^m.

In fact, using a $(\mathbb{Z}_2)^m$-coloring $\lambda : \mathcal{F}(P^m) \longrightarrow (\mathbb{Z}_2)^m$, Davis and Januszkiewicz gave a reconstruction process for a small cover. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^m$ generated by $\lambda(F_i)$. Given a point $p \in P^m$, we denote by $F(p)$ the minimal face containing p in its relative interior. Assume $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^{u} \mathbb{Z}_2(F_{i_j})$. Let $M(\lambda)$ denote $P^m \times (\mathbb{Z}_2)^m / \sim$, where $(p,g) \sim (q,h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^m$ on $P^m \times (\mathbb{Z}_2)^m$ descends to an action on $M(\lambda)$ with quotient P^m. Thus, $M(\lambda)$ is a small cover over P^m.

Definition 3 *(see [9])Two small covers M_1 and M_2 over P^m are said to be weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^m \to (\mathbb{Z}_2)^m$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^m$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic.*

Definition 4 *(see [9])Two small covers M_1 and M_2 over P^m are said to be Davis–Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^m.*

Let $\Lambda(P^m)$ be the set of all $(\mathbb{Z}_2)^m$-colorings on P^m. The following theorem has been proved in [9].

Theorem 5 *(Davis–Januszkiewicz)All small covers over P^m are given by $\{M(\lambda) | \lambda \in \Lambda(P^m)\}$, i.e. for each small cover M^m over P^m, there is a $(\mathbb{Z}_2)^m$-coloring λ with an equivariant homeomorphism $M(\lambda) \longrightarrow M^m$ covering the identity on P^m.*

In order to determine the D-J equivalence classes of small covers, noting that the characteristic functions differing by the linear automorphism of $(\mathbb{Z}_2)^m$ produce the same small cover up to D-J equivalence, we know that the answer should be the number of characteristic functions divided by $|GL(m, \mathbb{Z}_2)|$. More precisely, a free action of $GL(m, \mathbb{Z}_2)$ on $\Lambda(P^m)$ can be defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$ [17]. Then two small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^m are D-J equivalent if and only if there is a $\sigma \in GL(m, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$.

What is the orbit space of the action? Let $e_1, \cdots, e_m$ be the standard basis of $(\mathbb{Z}_2)^m$ and $F_1, \cdots, F_m$ of $\mathcal{F}(P^m)$ meet at one vertex of P^m. Write $A(P^m) = \{\lambda \in \Lambda(P^m) | \lambda(F_i) = e_i, i = 1, \cdots, m\}$. Then $A(P^m)$ is the orbit space of $\Lambda(P^m)$ under the action of $GL(m, \mathbb{Z}_2)$, and so ***the number of D-J equivalence classes of small covers over P^m is*** $|A(P^m)|$. We have:

Lemma 6 *(see [17])* $|\Lambda(P^m)| = |A(P^m)| \times |GL(m, \mathbb{Z}_2)|$.

For equivariant homeomorphism classes of small covers, from Theorem 5, symmetries of a polytope P^m should be taken into account. In a similar manner, consider that the automorphism group of P^m acts on characteristic functions, but since these symmetries do not act freely, we need the Burnside lemma in computation. Precisely, all faces of a simple convex polytope P^m form a poset (i.e. a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^m)$ is a bijection from $\mathcal{F}(P^m)$ to itself that preserves the poset structure of all faces of P^m. The group of automorphisms of $\mathcal{F}(P^m)$ is denoted by $Aut(\mathcal{F}(P^m))$. One defines a right action of $Aut(\mathcal{F}(P^m))$ on $\Lambda(P^m)$ by $\lambda \times h \longmapsto \lambda \circ h$, where $\lambda \in \Lambda(P^m)$ and $h \in Aut(\mathcal{F}(P^m))$. The following theorem is known.

Theorem 7 *(see [13]) Two small covers over an m-dimensional simple convex polytope P^m are equivariantly homeomorphic if and only if there is an $h \in Aut(\mathcal{F}(P^m))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are the corresponding $(\mathbb{Z}_2)^m$-colorings on P^m.*

Consequently, ***the number of orbits of*** $\Lambda(P^m)$ ***under the action of*** $Aut(\mathcal{F}(P^m))$ ***is just the number of equivariant homeomorphism classes of small covers over*** P^m. The enumeration of the number of orbits can be accomplished by using the Burnside lemma.

Lemma 8 *(Burnside lemma) Let G be a finite group acting on a set X. Then the number of orbits X under the action of G equals* $\frac{1}{|G|}\sum_{g\in G}|X_g|$, *where* $X_g = \{x \in X | gx = x\}$.

For an orientable small cover $M(\lambda)$ over a simple convex polytope P^m, Nakayama and Nishimura proved the following:

Theorem 9 *(see [16]) For a basis* $\{e_1, \cdots, e_m\}$ *of* $(\mathbb{Z}_2)^m$, *a homomorphism* $\varepsilon : (\mathbb{Z}_2)^m \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ *is defined by* $\varepsilon(e_i) = 1 (i = 1, \cdots, m)$. *A small cover* $M(\lambda)$ *over a simple convex polytope* P^m *is orientable if and only if there exists a basis* $\{e_1, \cdots, e_m\}$ *of* $(\mathbb{Z}_2)^m$ *such that the image of* $\varepsilon\lambda$ *is* $\{1\}$.

Definition 10 *A* $(\mathbb{Z}_2)^m$*-coloring is orientable if it satisfies the orientability condition in Theorem 9.*

Similarly, let $O(P^m)$ be the set of all orientable colorings on P^m. There is a free action of $GL(m, \mathbb{Z}_2)$ on $O(P^m)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$. Let $e_1, \cdots, e_m$ be the standard basis of $(\mathbb{Z}_2)^m$ and $F_1, \cdots, F_m$ of $\mathcal{F}(P^m)$ meet at a vertex of P^m. Write $B(P^m) = \{\lambda \in O(P^m) | \lambda(F_i) = e_i, i = 1, \cdots, m\}$. Then $B(P^m)$ is the orbit space of $O(P^m)$ under the action of $GL(m, \mathbb{Z}_2)$. In fact, we have $B(P^m) = \{\lambda \in O(P^m) | \lambda(F_i) = e_i, i = 1, \cdots, m$ and for $m + 1 \le j \le l, \lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}, 1 \le j_1 < j_2 < \cdots < j_{2h_j+1} \le m\}$.

Lemma 11 *(see [17])* $|O(P^m)| = |B(P^m)| \times |GL(m, \mathbb{Z}_2)|$.

Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^m are D-J equivalent if and only if there is $\sigma \in GL(m, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. *Thus,* ***the number of D-J equivalence classes of orientable small covers over*** P^m ***is*** $|B(P^m)|$**.**

One can also define a right action of $Aut(\mathcal{F}(P^m))$ on $O(P^m)$ by $\lambda \times h \longmapsto \lambda \circ h$, where $\lambda \in O(P^m)$ and $h \in Aut(\mathcal{F}(P^m))$. Theorem 7 holds for orientable small covers; that is, two orientable small covers over P^m are equivariantly homeomorphic if and only if there is $h \in Aut(\mathcal{F}(P^m))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are their corresponding orientable colorings on P^m. Therefore, the number of orbits of $O(P^m)$ under the action of $Aut(\mathcal{F}(P^m))$ is just the number of equivariant homeomorphism classes of orientable small covers over P^m.

3. D-J equivalence classes of small covers and polynomials

Suppose that P^m is a fixed simple polytope with l facets, and we generate a series of polytopes $P^m \times \Delta^n (n = 1, 2, \cdots)$, where Δ^n is an n-simplex. Let $DJ(P^m \times \Delta^n)$ be the number of D-J equivalence classes of small covers over $P^m \times \Delta^n$ and $DJ_o(P^m \times \Delta^n)$ be the number of D-J equivalence classes of orientable small covers over $P^m \times \Delta^n$. Then we have:

Theorem 12 *Let* P^m *be a fixed simple polytope with* l *facets and* $A(P^m) \neq \emptyset$. *We have:*

(1) $DJ(P^m \times \Delta^n)$ *is a polynomial of degree* $l - m$ *in the variable* 2^n;

(2) $DJ_o(P^m \times \Delta^n)$ is a polynomial of degree $l-m$ in the variable 2^{n-1} for odd n, and the coefficient of the term of degree $l-m$ is the same as the coefficient of the term of degree $l-m$ in $DJ(P^m \times \Delta^n)$;

(3) $DJ_o(P^m \times \Delta^n)$ is a polynomial of degree p in the variable 2^{n-1} for even n, where $p \le l-m-1$.

Proof By $F'_1, F'_2, \cdots, F'_l$ we denote all facets of P^m and by $F'_{l+1}, F'_{l+2}, \cdots, F'_{l+n+1}$ the facets of the n-simplex Δ^n. Set $\mathcal{F}' = \{F_i = F'_i \times \Delta^n | 1 \le i \le l\}$, $\mathcal{F}'' = \{F_i = P^m \times F'_i | l+1 \le i \le l+n+1\}$. Then the set of facets of $P^m \times \Delta^n$ is $\mathcal{F}(P^m \times \Delta^n) = \mathcal{F}' \cup \mathcal{F}''$.

(1) Let e_1, e_2, $\cdots$, e_{m+n} be the standard basis of $(\mathbb{Z}_2)^{m+n}$. Then $A(P^m \times \Delta^n) = \{\lambda \in \Lambda(P^m \times \Delta^n) | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \cdots, \lambda(F_m) = e_m, \lambda(F_{l+1}) = e_{m+1}, \lambda(F_{l+2}) = e_{m+2}, \cdots, \lambda(F_{l+n}) = e_{m+n}\}$ and $DJ(P^m \times \Delta^n) = |A(P^m \times \Delta^n)|$.

Assume $\lambda(F_i) = \sum_{j=1}^{m+n} f_{ij} e_j$, where $i = m+1, \cdots, l, l+n+1$, $f_{ij} = 0$ or 1. For $i = l+n+1$, by the linear independence of λ, $\lambda(F_i) = \sum_{j=1}^{m} f_{ij} e_j + e_{m+1} + \cdots + e_{m+n}$.

Consider $(\mathbb{Z}_2)^m$ and $(\mathbb{Z}_2)^n \subset (\mathbb{Z}_2)^{m+n} = (\mathbb{Z}_2)^m \oplus (\mathbb{Z}_2)^n$ as a direct summand, and write $k_1 : (\mathbb{Z}_2)^m \hookrightarrow (\mathbb{Z}_2)^{m+n}$ and $k_2 : (\mathbb{Z}_2)^n \hookrightarrow (\mathbb{Z}_2)^{m+n}$ for inclusions; $p_1 : (\mathbb{Z}_2)^{m+n} \to (\mathbb{Z}_2)^m$ and $p_2 : (\mathbb{Z}_2)^{m+n} \to (\mathbb{Z}_2)^n$ for projections.

If $\lambda \in A(P^m \times \Delta^n)$, then we construct a characteristic function $\eta \in A(P^m)$, defined by $\eta(F'_i) = p_1 \circ \lambda(F_i)(1 \le i \le l)$. Let $F'_{i_1}, F'_{i_2}, \cdots, F'_{i_m}$ meet at a vertex of P^m, where $1 \le i_1 < i_2 < \cdots < i_m \le l$. Then $\eta(F'_{i_1}), \eta(F'_{i_2}), \cdots, \eta(F'_{i_m})$ are linearly independent. It is possible that $p_1 \circ \lambda_1(F_i) = p_1 \circ \lambda_2(F_i)(i = 1, 2, \cdots, l+n+1)$ even though $\lambda_1 \neq \lambda_2$. Next, fixing a collection of vectors $p_1 \circ \lambda_1(F_i)(i = 1, 2, \cdots, l+n+1)$, we consider such λ that $p_1 \circ \lambda_1(F_i) = p_1 \circ \lambda(F_i)$ and investigate relations among $p_2 \circ \lambda(F_i) = (f_{i(m+1)}, f_{i(m+2)}, \cdots, f_{i(m+n)})$ $(i = m+1, \cdots, l)$. Using the linear independence of $\{\lambda(F_{i_1}), \cdots, \lambda(F_{i_m}), \lambda(F_{l+1}), \cdots, \lambda(\hat{F}_d), \cdots, \lambda(F_{l+n}), \lambda(F_{l+n+1})\}$, where $\hat{F}_d$ with the 'hat' symbol indicates that this facet is deleted from a sequence $F_{l+1}, \cdots, F_d, \cdots, F_{l+n}(l+1 \le d \le l+n)$, by calculating determinants we know that there exist $r_1, r_2, \cdots, r_m \in \mathbb{Z}_2$ such that

$$\begin{cases} r_1 f_{i_1(m+1)} + r_2 f_{i_2(m+1)} + \cdots + r_m f_{i_m(m+1)} = 0 \\ r_1 f_{i_1(m+2)} + r_2 f_{i_2(m+2)} + \cdots + r_m f_{i_m(m+2)} = 0 \\ \qquad \vdots \\ r_1 f_{i_1(m+n)} + r_2 f_{i_2(m+n)} + \cdots + r_m f_{i_m(m+n)} = 0. \end{cases}$$

From these equations, vectors $p_2 \circ \lambda(F_i) = (f_{i(m+1)}, f_{i(m+2)}, \cdots, f_{i(m+n)})$ $(i = m+1, \cdots, l)$ can be divided into three groups, G_1, G_2, and G_3. $p_2 \circ \lambda(F_i) \in G_1$ if and only if $\exists\ m+1 \le j_1 < j_2 < \cdots < j_k < i$ such that $p_2 \circ \lambda(F_i) = \sum_{j \in \{j_1, j_2, \cdots, j_k\}} p_2 \circ \lambda(F_j)$; $p_2 \circ \lambda(F_i) \in G_2$ if and only if $p_2 \circ \lambda(F_i) \notin G_1$; and whenever the components $f_{i(m+1)}, f_{i(m+2)}, \cdots, f_{i(m+n)}$ freely vary in $\mathbb{Z}_2$ ($p_1 \circ \lambda(F_i) = p_1 \circ \lambda_1(F_i)$ is fixed), the corresponding λ is in $A(P^m \times \Delta^n)$; $p_2 \circ \lambda(F_i) \in G_3$ if and only if $p_2 \circ \lambda(F_i) = (0, 0, \cdots, 0) \notin G_1 \cup G_2$.

Set $T_k = \{\lambda \in A(P^m \times \Delta^n) |$ exactly $\exists\ k$ facets F_i such that $p_2 \circ \lambda(F_i) \in G_2\}, 0 \le k \le l-m$. Then T_k contributes $(2^n)^k$ to $DJ(P^m \times \Delta^n)$. Thus, $DJ(P^m \times \Delta^n)$ is a polynomial in the variable 2^n.

In order to show that the degree of the polynomial is equal to $l-m$, fixing a characteristic function $\eta \in A(P^m)$ and taking a collection of vectors $\mathbf{f}_j = (f_{j(m+1)}, f_{j(m+2)}, \cdots, f_{j(m+n)}) \in (\mathbb{Z}_2)^n, j = m+1, \cdots, l$, we construct a collection of characteristic functions $\lambda \in A(P^m \times \Delta^n)$, defined by $\lambda(F_i) = e_i (1 \le i \le m)$, $\lambda(F_i) = k_1 \circ \eta(F'_i) + k_2(\mathbf{f}_i)(m+1 \le i \le l)$, $\lambda(F_i) = e_{m+i-l}(l+1 \le i \le l+n)$, $\lambda(F_{l+n+1}) = e_{m+1} + \cdots + e_{m+n}$. These characteristic functions belong to T_{l-m} and contribute $(2^n)^{l-m}$ to $DJ(P^m \times \Delta^n)$. Hence, the degree is equal to $l-m$.

(2) In the proof of the first conclusion (1), replacing $\Lambda(P^m)$ and $A(P^m)$ with $O(P^m)$ and $B(P^m)$, respectively, we consider the orientability condition of a characteristic function to obtain that $p_2 \circ \lambda(F_i) \in G_2$ if and only if $p_2 \circ \lambda(F_i) \notin G_1$, and whenever $n-1$ components in $(f_{i(m+1)}, f_{i(m+2)}, \cdots, f_{i(m+n)})$ freely vary in $\mathbb{Z}_2$ ($p_1 \circ \lambda(F_i) = p_1 \circ \lambda_1(F_i)$ is fixed), the corresponding λ is in $B(P^m \times \Delta^n)$. Therefore, T_k contributes $(2^{n-1})^k$ to $DJ_o(P^m \times \Delta^n), 0 \le k \le l-m$ and $T_{l-m} \neq \emptyset$. Thus, $DJ_o(P^m \times \Delta^n)$ is a polynomial of degree $l-m$ in the variable 2^{n-1}. Using the fact that each edge of a polytope is incident with two vertices, we know that the coefficient of the term of degree $l-m$ is the same as the coefficient of the term of degree $l-m$ in $DJ(P^m \times \Delta^n)$.

(3) Similar to the proof of the second part, but $T_{l-m} = \emptyset$. Hence, the degree of the polynomial is less than $l-m$.

□

Corollary 13 *For odd n, $\frac{DJ_o(P^m \times \Delta^n)}{DJ(P^m \times \Delta^n)} \to \frac{1}{2^{l-m}} (n \to \infty)$; for even n, $\frac{DJ_o(P^m \times \Delta^n)}{DJ(P^m \times \Delta^n)} \to 0 (n \to \infty)$.*

Remark 14 If M^m is a small cover over P^m and $\lambda \in A(P^m \times \Delta^n)$ is a characteristic function that corresponds to a generalized real Bott manifold starting with M^m, by [12] we must have $\lambda(F_{l+n+1}) = e_{m+1} + \cdots + e_{m+n}$. For $P^m = P_6^3 \times I$, we know that there exists a $\lambda \in A(P^m \times \Delta^n)$ such that $\lambda(F_{l+n+1}) \neq e_{m+1} + \cdots + e_{m+n}$. The corresponding small cover must not be equivalent to any generalized real Bott manifold starting with M^m.

4. Equivalence classes of small covers over $P_6^3 \times I \times \Delta^n$

As an application of Theorem 12, we intend to specify a generalized real Bott manifold as a starting small cover over $P^m (m \ge 4)$, and by interpolation determine the polynomial for calculating the number of D-J equivalence classes. For this purpose, we take $P^m = P_6^3 \times I$ as an example, where P_6^3 is the dual of the cyclic polytope $C^3(6)$. From the above remark, we know that, up to D-J equivalence, the number of small covers over $P_6^3 \times I \times \Delta^n$ is greater than the number of generalized real Bott manifolds. Furthermore, we provide a formula for calculating the number of equivariant homeomorphism classes of small covers over $P_6^3 \times I \times \Delta^n$.

Theorem 15 $DJ(P_6^3 \times I \times \Delta^n) = 165 \cdot 2^{4n} + 72 \cdot 2^{3n} + 348 \cdot 2^{2n} + 900 \cdot 2^n + 1155.$

Proof By Theorem 12, $DJ(P_6^3 \times I \times \Delta^n)$ is a polynomial of degree 4 in the variable 2^n. For $n = 1, 2, \cdots, 5$, we have data points $(x_0, y_0) = (2, 7563)$, $(x_1, y_1) = (2^2, 57171)$, $(x_2, y_2) = (2^3, 743331)$, $(x_3, y_3) = (2^4, 11212995)$, $(x_4, y_4) = (2^5, 175760643)$. We write down the polynomial immediately in terms of Lagrange polynomials: $DJ(P_6^3 \times I \times \Delta^n) = \sum_{i=0}^{4} (\prod_{0 \le j \le 4, j \neq i} \frac{x - x_j}{x_i - x_j}) y_i = 165 \cdot 2^{4n} + 72 \cdot 2^{3n} + 348 \cdot 2^{2n} + 900 \cdot 2^n + 1155.$ □

Theorem 16 *The number of* $(\mathbb{Z}_2)^{n+4}$*-colorings on* $P_6^3 \times I \times \Delta^n$ *is equal to*
$\prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)(165\cdot 2^{4n}+72\cdot 2^{3n}+348\cdot 2^{2n}+900\cdot 2^n+1155).$

Proof From [1], we know that $|GL(n+4,\mathbb{Z}_2)| = \prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)$. By Lemma 6 and Theorem 15, we have that

$$|\Lambda(P_6^3 \times I \times \Delta^n)| = \prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)(165\cdot 2^{4n}+72\cdot 2^{3n}+348\cdot 2^{2n}+900\cdot 2^n+1155). \qquad \square$$

In order to calculate the number of equivariant homeomorphism classes of small covers over $P_6^3 \times I \times \Delta^n$, we need to understand the structure of $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))$. P_6^3 has two pentagons, two quadrilaterals, and two triangles (see the Figure).

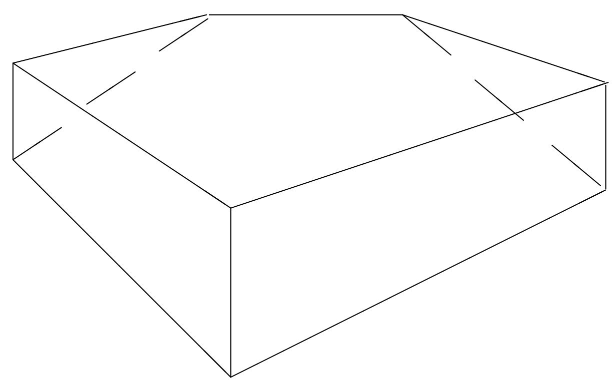

Figure. Polytope P_6^3.

For convenience, by F_1' we denote the top pentagon of P_6^3; by F_2', the left quadrilateral; F_3', the right quadrilateral; F_4', the bottom pentagon; F_5', the right triangle; and by F_6', the left triangle. Similarly, by F_7' and F_8' we denote the two endpoints of I, and by $F_9', F_{10}', \cdots, F_{n+9}'$ all the facets of the n-simplex Δ^n. Set $\mathcal{F}' = \{F_i = F_i' \times I \times \Delta^n | 1 \le i \le 6\}$, $\mathcal{F}'' = \{F_i = P_6^3 \times F_i' \times \Delta^n | i = 7, 8\}, \mathcal{F}''' = \{F_i = P_6^3 \times I \times F_i' | i = 9, \cdots, n+9\}$. Then $\mathcal{F}(P_6^3 \times I \times \Delta^n) = \mathcal{F}' \bigcup \mathcal{F}'' \bigcup \mathcal{F}'''$.

Lemma 17 *The automorphism group* $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))$ *is isomorphic to the direct product* $K_4 \times \mathbb{Z}_2 \times S_{n+1}(n \ge 2)$, *where* $K_4 = \langle x, y | x^2 = y^2 = (xy)^2 = 1\rangle$ *is the Klein four-group and* S_{n+1} *the symmetric group with order* $(n+1)!$.

Proof The automorphism group $Aut(\mathcal{F}(P_6^3))$ is isomorphic to the Klein four-group $K_4 = \langle x, y | x^2 = y^2 = (xy)^2 = 1\rangle$, where one of the generators gives an interchange between top and bottom pentagons and the other gives an interchange between left and right triangles, between left and right quadrilaterals. The automorphism group $Aut(\mathcal{F}(I))$ is isomorphic to the group $\mathbb{Z}_2 = \{-1, 1\}$. The automorphism group $Aut(\mathcal{F}(\Delta^n))$ is isomorphic to the $(n+1)$-symmetric group S_{n+1}. Since an automorphism of $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))(n \ge 2)$ maps $\mathcal{F}'$ to $\mathcal{F}'$, $\mathcal{F}''$ to $\mathcal{F}''$ and maps $\mathcal{F}'''$ to $\mathcal{F}'''$, we have that $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))$ is isomorphic to $K_4 \times \mathbb{Z}_2 \times S_{n+1}$. $\square$

Theorem 18 *Let* $E(P_6^3 \times I \times \Delta^n)$ *denote the number of equivariant homeomorphism classes of small covers* *over* $P_6^3 \times I \times \Delta^n (n \geq 2)$. *Then we have*

$$E(P_6^3 \times I \times \Delta^n) = \frac{\prod_{k=1}^{n+4} \left(2^{n+4} - 2^{k-1}\right)}{8(n+1)!} (165 \cdot 2^{4n} + 27 \cdot 2^{3n+3} + 87 \cdot 2^{2n+2} + 345 \cdot 2^{n+2} + 1683).$$

Proof By Lemma 8, we have

$E(P_6^3 \times I \times \Delta^n) = \frac{1}{8(n+1)!} \sum_{g \in Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))} |\Lambda_g|$, where $\Lambda_g = \{\lambda \in \Lambda(P_6^3 \times I \times \Delta^n) | \lambda = \lambda \circ g\}$.

The automorphism group $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))$ is isomorphic to $K_4 \times \mathbb{Z}_2 \times S_{n+1} (n \geq 2)$. Each automorphism g of $Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))$ can be written as $(x^u y^v, (-1)^\mu, s)$, and the argument is divided into the following cases.

Case 1. $g = (1, -1, 1)$.

If $\lambda \in \Lambda_g$, then $\lambda(F_7) = \lambda(F_8)$. A computation gives $|\Lambda_g| = (9 \cdot 2^{3n+4} + 15 \cdot 2^{n+5} + 528)|GL(n+4, \mathbb{Z}_2)|$ (see supplementary material).

Case 2. $g = (1, 1, 1)$.

g is the identity automorphism, by Theorem 16, $|\Lambda_g| = (165 \cdot 2^{4n} + 72 \cdot 2^{3n} + 348 \cdot 2^{2n} + 900 \cdot 2^n + 1155)|GL(n+4, \mathbb{Z}_2)|$.

Case 3. $g \in Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n)) \backslash \{(1, -1, 1), (1, 1, 1)\}$.

In this case, by linear independence, we have $\Lambda_g = \emptyset$, so

$$E(P_6^3 \times I \times \Delta^n) = \frac{\prod_{k=1}^{n+4} \left(2^{n+4} - 2^{k-1}\right)}{8(n+1)!} (165 \cdot 2^{4n} + 27 \cdot 2^{3n+3} + 87 \cdot 2^{2n+2} + 345 \cdot 2^{n+2} + 1683). \qquad \square$$

5. Orientable small covers over $P_6^3 \times I \times \Delta^n$

Theorem 19 *(1)* $DJ_o(P_6^3 \times I \times \Delta^n) = 165 \cdot 2^{4(n-1)} + 84 \cdot 2^{n-1} + 42$ *if* n *is odd;*

(2) $DJ_o(P_6^3 \times I \times \Delta^n) = 72 \cdot 2^{3(n-1)} + 40 \cdot 2^{2(n-1)} + 68 \cdot 2^{n-1} + 43$ *if* n *is even.*

Proof

(1) By Theorem 12, $DJ_o(P_6^3 \times I \times \Delta^n)$ is a polynomial of degree 4 in the variable 2^{n-1}. From data points $(x_0, y_0) = (2, 291)$, $(x_1, y_1) = (2^3, 42618)$, $(x_2, y_2) = (2^5, 66 \times 163861)$, $(x_3, y_3) = (2^7, 6 \times 461374343)$, $(x_4, y_4) = (2^9, 174 \times 4072813939)$, we have a polynomial $DJ_o(P_6^3 \times I \times \Delta^n) = \sum_{i=0}^{4} (\prod_{0 \leq j \leq 4, j \neq i} \frac{x - x_j}{x_i - x_j}) y_i = 165 \cdot 2^{4(n-1)} + 84 \cdot 2^{n-1} + 42$.

(2) Similarly, $DJ_o(P_6^3 \times I \times \Delta^n)$ is a polynomial of degree ≤ 3 in the variable 2^{n-1}. From data points $(x_0, y_0) = (2^2, 915)$, $(x_1, y_1) = (2^4, 40011)$, $(x_2, y_2) = (2^6, 75 \times 32033)$, $(x_3, y_3) = (2^8, 3 \times 50553017)$, we get a polynomial $DJ_o(P_6^3 \times I \times \Delta^n) = 72 \cdot 2^{3(n-1)} + 40 \cdot 2^{2(n-1)} + 68 \cdot 2^{n-1} + 43$.

$\square$

Theorem 20 *The number of orientable* $(\mathbb{Z}_2)^{n+4}$*-colorings on* $P_6^3 \times I \times \Delta^n$ *is equal to:*

(1) $(165 \cdot 2^{4(n-1)} + 84 \cdot 2^{n-1} + 42) \prod_{k=1}^{n+4} \left(2^{n+4} - 2^{k-1}\right)$ *if* n *is odd;*

(2) $(72 \cdot 2^{3(n-1)} + 40 \cdot 2^{2(n-1)} + 68 \cdot 2^{n-1} + 43) \prod_{k=1}^{n+4} \left(2^{n+4} - 2^{k-1}\right)$ *if* n *is even.*

Proof By Lemma 11 and Theorem 19, we have Theorem 20. □

Theorem 21 *Let* $E_o(P_6^3 \times I \times \Delta^n)$ *denote the number of equivariant homeomorphism classes of all orientable small covers over* $P_6^3 \times I \times \Delta^n (n \geq 2)$*. Then we have:*

$$E_o(P_6^3 \times I \times \Delta^n) = \begin{cases} \frac{\prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)}{8(n+1)!}(165 \cdot 2^{4n-4} + 9 \cdot 2^{3n} + 9 \cdot 2^{n+3} + 75) & if\ n\ is\ odd, \\ \frac{\prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)}{8(n+1)!}(9 \cdot 2^{3n+1} + 5 \cdot 2^{2n+1} + 2^{n+6} + 76) & if\ n\ is\ even. \end{cases}$$

Proof By Lemma 8, we have that

$E_o(P_6^3 \times I \times \Delta^n) = \frac{1}{8(n+1)!} \sum_{g \in Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n))} |\Lambda_g|$, where $\Lambda_g = \{\lambda \in O(P_6^3 \times I \times \Delta^n) | \lambda = \lambda \circ g\}$. □

The argument is divided into the following cases.

Case 1. $g = (1, -1, 1)$.

If $\lambda \in \Lambda_g$, then $\lambda(F_7) = \lambda(F_8)$. A computation gives $|\Lambda_g| = (9 \cdot 2^{3n} + 15 \cdot 2^{n+1} + 33)|GL(n+4, \mathbb{Z}_2)|$ (see supplementary material).

Case 2. $g = (1, 1, 1)$.

g is the identity automorphism, by Theorem 20 $|\Lambda_g| = |O(P_6^3 \times I \times \Delta^n)|$.

Case 3. $g \in Aut(\mathcal{F}(P_6^3 \times I \times \Delta^n)) \backslash \{(1, -1, 1), (1, 1, 1)\}$.

In this case, by linear independence, we have $\Lambda_g = \emptyset$, so $E_o(P_6^3 \times I \times \Delta^n)$

$$= \begin{cases} \frac{\prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)}{8(n+1)!}(165 \cdot 2^{4n-4} + 9 \cdot 2^{3n} + 9 \cdot 2^{n+3} + 75) & if\ n\ is\ odd, \\ \frac{\prod_{k=1}^{n+4}\left(2^{n+4}-2^{k-1}\right)}{8(n+1)!}(9 \cdot 2^{3n+1} + 5 \cdot 2^{2n+1} + 2^{n+6} + 76) & if\ n\ is\ even. \end{cases}$$

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Supplementary material of small covers over products of a simple polytope with a simplex

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1. Computation of $|\Lambda_g|$ for $g=(1,-1,1)$ in determining the number of equivariant homeomorphism classes of small covers over $P_6^3\times I\times\Delta^n$

Let $e_1,\ e_2,\ \cdots,\ e_{n+4}$ be the standard basis of $(\mathbb{Z}_2)^{n+4}$. It is easy to see that $F_1,\ F_2,\ F_3,\ F_7,\ F_9,\ \cdots,\ F_{n+8}$ meet at one vertex of $P_6^3\times I\times\Delta^n$. Set $C(P_6^3\times I\times\Delta^n)=\{\lambda\in\Lambda(P_6^3\times I\times\Delta^n)|\lambda(F_1)=e_1,\lambda(F_2)=e_2,\lambda(F_3)=e_3,\lambda(F_7)=\lambda(F_8)=e_4,\lambda(F_9)=e_5,\cdots,\lambda(F_{n+8})=e_{n+4}\}$.

Assume $\lambda(F_4)=\sum_{\alpha=1}^{n+4}a_\alpha e_\alpha,\ \lambda(F_5)=\sum_{\beta=1}^{n+4}b_\beta e_\beta,\ \lambda(F_6)=\sum_{\gamma=1}^{n+4}c_\gamma e_\gamma,\ \lambda(F_{n+9})=\sum_{\nu=1}^{n+4}g_\nu e_\nu$, where $a_\alpha, b_\beta, c_\gamma$, and $g_\nu=0$ or 1.

In order to calculate $|\Lambda_g|$, we need to determine all possible values of the coefficients a_α, b_β, c_γ, and g_ν, and this can be accomplished by using the linear independence of $\lambda(F_{i_1})$, $\lambda(F_{i_2}),\ \cdots,\ \lambda(F_{i_{n+4}})$ whenever $F_{i_1},\ F_{i_2},\ \cdots,\ F_{i_{n+4}}$ meet at one vertex of $P_6^3\times I\times\Delta^n$.

By the linear independence of $\{\lambda(F_1),\ \lambda(F_4),\ \lambda(F_7),\ \lambda(F_9),\ \cdots,\ \lambda(F_{n+8})\}$ we have $a_2+a_3=1$ or $a_2=a_3=1$. Similarly $\{\lambda(F_2),\ \lambda(F_3),\ \lambda(F_4),\ \lambda(F_7),\ \lambda(F_9),\ \cdots,\ \lambda(F_{n+8})\}$ are linearly independent if and only if $a_1=1$. Consequently, $\lambda(F_4)=e_1+e_2+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha$, $e_1+e_3+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha$, or $e_1+e_2+e_3+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha$. Write

$C_1(P_6^3\times I\times\Delta^n)=\{\lambda\in C(P_6^3\times I\times\Delta^n)|\lambda(F_4)=e_1+e_2+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha\}$;

$C_2(P_6^3\times I\times\Delta^n)=\{\lambda\in C(P_6^3\times I\times\Delta^n)|\lambda(F_4)=e_1+e_3+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha\}$;

$C_3(P_6^3\times I\times\Delta^n)=\{\lambda\in C(P_6^3\times I\times\Delta^n)|\lambda(F_4)=e_1+e_2+e_3+\sum_{\alpha=4}^{n+4}a_\alpha e_\alpha\}$.

By considering the other vertices, we get all $\lambda\in C_1(P_6^3\times I\times\Delta^n)$, which are listed in Table 1. If we exchange e_2 and e_3 in Table 1, we obtain all $\lambda\in C_2(P_6^3\times I\times\Delta^n)$. Similarly, all λ in $C_3(P_6^3\times I\times\Delta^n)$ are listed in Table 2.

So $|\Lambda_g|=|C(P_6^3\times I\times\Delta^n)||GL(n+4,\mathbb{Z}_2)|$=$\sum_{i=1}^{3}|C_i(P_6^3\times I\times\Delta^n)|$=$(9\cdot 2^{3n+4}+15\cdot 2^{n+5}+528)|GL(n+4,\mathbb{Z}_2)|$.

2. Computation of $|\Lambda_g|$ for $g=(1,-1,1)$ in determining the number of equivariant homeomorphism classes of orientable small covers over $P_6^3\times I\times\Delta^n$

Let $e_1,\ e_2,\ \cdots,\ e_{n+4}$ be the standard basis of $(\mathbb{Z}_2)^{n+4}$. Set

$D(P_6^3 \times I \times \Delta^n) = \{\lambda \in \Lambda(P_6^3 \times I \times \Delta^n) | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_3) = e_3, \lambda(F_7) = \lambda(F_8) = e_4, \lambda(F_9) = e_5, \cdots, \lambda(F_{n+8}) = e_{n+4}\}$.

A similar argument shows that $\lambda(F_4) = e_1 + e_2 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha$, $e_1 + e_3 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha$, or $e_1 + e_2 + e_3 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha$. Write

$D_1(P_6^3 \times I \times \Delta^n) = \{\lambda \in D(P_6^3 \times I \times \Delta^n) | \lambda(F_4) = e_1 + e_2 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha, \sum_{\alpha=4}^{n+4} a_\alpha \equiv 1 (mod\ 2)\}$;

$D_2(P_6^3 \times I \times \Delta^n) = \{\lambda \in D(P_6^3 \times I \times \Delta^n) | \lambda(F_4) = e_1 + e_3 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha, \sum_{\alpha=4}^{n+4} a_\alpha \equiv 1 (mod\ 2)\}$;

$D_3(P_6^3 \times I \times \Delta^n) = \{\lambda \in D(P_6^3 \times I \times \Delta^n) | \lambda(F_4) = e_1 + e_2 + e_3 + \sum_{\alpha=4}^{n+4} a_\alpha e_\alpha, \sum_{\alpha=4}^{n+4} a_\alpha \equiv 0 (mod\ 2)\}$.

The determination of $D_i(P_6^3 \times I \times \Delta^n)$ is divided into two cases.

For n is odd, by considering the linear independence of λ on the vertices of $P_6^3 \times I \times \Delta^n$ and the orientability condition in Theorem 9, from Table 1 we obtain all orientable λ in $D_1(P_6^3 \times I \times \Delta^n)$, which are listed in Table 3 where $\sum_{\alpha=4}^{n+4} a_\alpha \equiv 1 (mod\ 2)$ for cases 10-12; $b_4 + \sum_{\alpha=5}^{n+4} a_\alpha \equiv 1 (mod\ 2)$ for case 11; $\sum_{\beta=4}^{n+4} b_\beta \equiv 1 (mod\ 2)$ for case 12; $c_1 + c_2 = 1$ for case 4; $c_1 + c_2 + c_4 \equiv 0 (mod\ 2)$ for cases 5 and 6; $\sum_{\gamma=1, \gamma \neq 3}^{n+4} c_\gamma \equiv 0 (mod\ 2)$ for cases 7-9 and 12. If we exchange e_2 and e_3 in Table 3, then we obtain all λ in $D_2(P_6^3 \times I \times \Delta^n)$. Similarly, from Table 2 we get all orientable λ in $D_3(P_6^3 \times I \times \Delta^n)$, which are listed in Table 4 where $\sum_{\alpha=4}^{n+4} a_\alpha \equiv 0 (mod\ 2)$ for cases 4-6; $b_4 + \sum_{\alpha=5}^{n+4} a_\alpha \equiv 0 (mod\ 2)$ for case 5; $\sum_{\beta=4}^{n+4} b_\beta \equiv 0 (mod\ 2)$ for case 6; $c_4 + \sum_{\alpha=5}^{n+4} a_\alpha \equiv 0 (mod\ 2)$ for case 4; $\sum_{\gamma=4}^{n+4} c_\gamma \equiv 0 (mod\ 2)$ for case 6; $g_2 + g_3 = 1$ for case 3.

For n is even, similarly from Table 1 we obtain all orientable λ in $D_1(P_6^3 \times I \times \Delta^n)$, which are listed in Table 5 where $\sum_{\alpha=4}^{n+4} a_\alpha \equiv 1 (mod\ 2)$ for cases 11-13; $\sum_{\beta=4}^{n+4} b_\beta \equiv 1 (mod\ 2)$ for case 13; $c_1 + c_2 + c_4 \equiv 0 (mod\ 2)$ for cases 6 and 7; $\sum_{\gamma=1, \gamma \neq 3}^{n+4} c_\gamma \equiv 0 (mod\ 2)$ for cases 8-10 and 13. If we exchange e_2 and e_3 in Table 5, we obtain all $\lambda \in D_2(P_6^3 \times I \times \Delta^n)$. Similarly, from Table 2 we get all orientable λ in $D_3(P_6^3 \times I \times \Delta^n)$, which are listed in Table 6 where $\sum_{\alpha=4}^{n+4} a_\alpha \equiv 0 (mod\ 2)$ for cases 4-6; $\sum_{\beta=4}^{n+4} b_\beta \equiv 0 (mod\ 2)$ and $\sum_{\gamma=4}^{n+4} c_\gamma \equiv 0 (mod\ 2)$ for case 6; $g_2 + g_3 = 1$ for case 3.

So $|\Lambda_g| = |D(P_6^3 \times I \times \Delta^n)||GL(n+4, \mathbb{Z}_2)| = \sum_{i=1}^{3} |D_i(P_6^3 \times I \times \Delta^n)| = (9 \cdot 2^{3n} + 15 \cdot 2^{n+1} + 33)|GL(n+4, \mathbb{Z}_2)|$.

Table 1 All λ in $C_1(P^3_6 \times I \times \Delta^n)$, where a_i, b_i, c_i, g_i freely vary in $\mathbb{Z}_2$

	$\lambda(F_4)$						$\lambda(F_5)$						$\lambda(F_6)$						$\lambda(F_{n+9})$					
	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$
1	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	1	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
2	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	1	1	1	0	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
3	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	0	0	$\cdots 0$	0	$1-c_2$	1	0	1	$\cdots 1$
4	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	0	0	$\cdots 0$	1	g_2	1	0	1	$\cdots 1$
5	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	1	0	$\cdots 0$	1	1	1	0	1	$\cdots 1$
6	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	1	0	0	1	$\cdots 1$
7	1	1	0	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	g_2	0	0	1	$\cdots 1$
8	1	1	0	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	1	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
9	1	1	0	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	1	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
10	1	1	0	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	0	0	$\cdots 0$	1	0	1	0	1	$\cdots 1$
11	1	1	0	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	1	1	1	0	1	$\cdots 1$
12	1	1	0	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	g_2	0	0	1	$\cdots 1$
13	1	1	0	0	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	1	0	$\cdots 0$	1	0	1	0	1	$\cdots 1$
14	1	1	0	0	a_5	$\cdots a_{n+4}$	0	1	1	1	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
15	1	1	0	1	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	1	c_2	1	0	0	$\cdots 0$	0	c_2	1	0	1	$\cdots 1$
16	1	1	0	1	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	0	0	$\cdots 0$	0	$1-c_2$	1	0	1	$\cdots 1$
17	1	1	0	1	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	1	0	0	1	$\cdots 1$
18	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	1	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
19	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	1	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
20	1	1	0	1	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	1	g_2	1	0	1	$\cdots 1$
21	1	1	0	1	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	g_2	0	0	1	$\cdots 1$
22	1	1	0	1	a_5	$\cdots a_{n+4}$	0	1	1	0	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
23	1	1	0	a_4	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	1	1	1	1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
24	1	1	0	a_4	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	c_1	0	1	1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
25	1	1	0	a_4	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	1	1	1	c_4	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
26	1	1	0	a_4	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	0	1	c_4	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
27	1	1	0	a_4	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	1	0	0	1	$\cdots 1$
28	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	1	0	1	1	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
29	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	1	1	1	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
30	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	1	c_2	1	c_4	0	$\cdots 0$	0	c_2	1	1	1	$\cdots 1$
31	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	0	$1-c_2$	1	1	1	$\cdots 1$
32	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	1	g_2	1	1	1	$\cdots 1$
33	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	1	0	1	1	$\cdots 1$
34	1	1	0	a_4	0	$\cdots 0$	0	1	1	b_4	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	g_2	0	1	1	$\cdots 1$
35	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	0	0	$\cdots 0$	0	1	1	0	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
36	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	0	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
37	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	1	0	$\cdots 0$	0	1	1	c_4	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
38	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	a_4	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
39	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	0	$\cdots 0$	0	1	1	c_4	0	$\cdots 0$	0	1	1	1	1	$\cdots 1$
40	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	a_5	$\cdots a_{n+4}$	0	0	1	1	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
41	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	a_5	$\cdots a_{n+4}$	0	0	1	c_4	0	$\cdots 0$	0	0	1	1	1	$\cdots 1$
42	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	b_5	$\cdots b_{n+4}$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	0	0	1	1	$\cdots 1$
43	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	b_5	$\cdots b_{n+4}$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	0	0	0	1	$\cdots 1$

Table 2 All λ in $C_3(P_6^3 \times I \times \Delta^n)$, where a_i, b_i, c_i, g_i freely vary in $\mathbb{Z}_2$

	$\lambda(F_4)$						$\lambda(F_5)$						$\lambda(F_6)$						$\lambda(F_{n+9})$					
	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$
1	1	1	1	0	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	0	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
2	1	1	1	0	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	0	0	$\cdots 0$	1	g_2	g_3	0	1	$\cdots 1$
3	1	1	1	0	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	1	0	$\cdots 0$	1	g_2	0	0	1	$\cdots 1$
4	1	1	1	0	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	1	0	$\cdots 0$	1	1	1	0	1	$\cdots 1$
5	1	1	1	0	0	$\cdots 0$	0	1	0	1	0	$\cdots 0$	0	0	1	0	0	$\cdots 0$	1	0	1	0	1	$\cdots 1$
6	1	1	1	0	0	$\cdots 0$	0	1	0	1	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	1	g_2	0	0	1	$\cdots 1$
7	1	1	1	0	0	$\cdots 0$	0	1	0	1	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	1	1	1	0	1	$\cdots 1$
8	1	1	1	0	0	$\cdots 0$	0	1	0	b_4	0	$\cdots 0$	0	0	1	1	0	$\cdots 0$	1	0	1	0	1	$\cdots 1$
9	1	1	1	0	a_5	$\cdots a_{n+4}$	0	1	0	0	0	$\cdots 0$	0	0	1	1	a_5	$\cdots a_{n+4}$	0	1	0	0	1	$\cdots 1$
10	1	1	1	0	a_5	$\cdots a_{n+4}$	0	1	0	1	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
11	1	1	1	1	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	0	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
12	1	1	1	1	0	$\cdots 0$	0	1	0	b_4	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	1	g_2	g_3	0	1	$\cdots 1$
13	1	1	1	1	a_5	$\cdots a_{n+4}$	0	1	0	0	0	$\cdots 0$	0	0	1	0	a_5	$\cdots a_{n+4}$	0	1	0	0	1	$\cdots 1$
14	1	1	1	1	a_5	$\cdots a_{n+4}$	0	1	0	0	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
15	1	1	1	a_4	0	$\cdots 0$	0	1	0	0	0	$\cdots 0$	0	0	1	1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
16	1	1	1	a_4	0	$\cdots 0$	0	1	0	1	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
17	1	1	1	a_4	0	$\cdots 0$	0	1	0	b_4	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	0	1	1	1	1	$\cdots 1$
18	1	1	1	a_4	0	$\cdots 0$	0	1	0	b_4	0	$\cdots 0$	0	0	1	c_4	0	$\cdots 0$	1	g_2	g_3	1	1	$\cdots 1$
19	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	0	0	$\cdots 0$	0	0	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	0	1	$\cdots 1$
20	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	1	0	$\cdots 0$	0	0	1	c_4	a_5	$\cdots a_{n+4}$	0	1	0	0	1	$\cdots 1$
21	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	a_4	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
22	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	b_4	0	$\cdots 0$	0	0	1	c_4	a_5	$\cdots a_{n+4}$	0	1	0	1	1	$\cdots 1$
23	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	b_4	a_5	$\cdots a_{n+4}$	0	0	1	1	0	$\cdots 0$	0	0	1	0	1	$\cdots 1$
24	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	b_4	a_5	$\cdots a_{n+4}$	0	0	1	c_4	0	$\cdots 0$	0	0	1	1	1	$\cdots 1$
25	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	b_4	b_5	$\cdots b_{n+4}$	0	0	1	c_4	c_5	$\cdots c_{n+4}$	0	0	0	1	1	$\cdots 1$
26	1	1	1	a_4	a_5	$\cdots a_{n+4}$	0	1	0	b_4	b_5	$\cdots b_{n+4}$	0	0	1	c_4	c_5	$\cdots c_{n+4}$	0	0	0	0	1	$\cdots 1$

Table 3 All orientable λ in $D_1(P_6^3 \times I \times \Delta^n)$ for odd n

	$\lambda(F_4)$						$\lambda(F_5)$						$\lambda(F_6)$						$\lambda(F_{n+9})$					
	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$	e_1	e_2	e_3	e_4	e_5	$\cdots e_{n+4}$
1	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	1	0	1	1	0	$\cdots 0$	0	0	1	1	1	$\cdots 1$
2	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	1	1	1	0	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
3	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	0	1	c_1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
4	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	1	1	c_4	0	$\cdots 0$	0	0	1	1	1	$\cdots 1$
5	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	1	0	1	0	1	$\cdots 1$
6	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	0	$\cdots 0$	1	1	1	1	1	$\cdots 1$
7	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	1	0	0	1	$\cdots 1$
8	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	1	0	1	1	$\cdots 1$
9	1	1	0	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	1	0	0	1	1	$\cdots 1$
10	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	1	0	$\cdots 0$	0	1	1	1	0	$\cdots 0$	0	1	1	0	1	$\cdots 1$
11	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	a_5	$\cdots a_{n+4}$	0	0	1	0	0	$\cdots 0$	0	0	1	1	1	$\cdots 1$
12	1	1	0	a_4	a_5	$\cdots a_{n+4}$	0	1	1	b_4	b_5	$\cdots b_{n+4}$	c_1	c_2	1	c_4	c_5	$\cdots c_{n+4}$	0	0	0	0	1	$\cdots 1$

Table 4 All orientable λ in $D_3(P_6^3 \times I \times \Delta^n)$ for odd n

	$\lambda(F_4)$	$\lambda(F_5)$	$\lambda(F_6)$	$\lambda(F_{n+9})$
	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$
1	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$1\ g_2\ g_2\ 1\ 1\ \cdots 1$
2	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 1\ 0\ 1\ \cdots 1$
3	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$1\ g_2\ g_3\ 0\ 1\ \cdots 1$
4	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ c_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ 1\ 1\ \cdots 1$
5	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ b_4\ a_5\ \cdots a_{n+4}$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 1\ 1\ \cdots 1$
6	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ b_4\ b_5\ \cdots b_{n+4}$	$0\ 0\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$0\ 0\ 0\ 0\ 1\ \cdots 1$

Table 5 All orientable λ in $D_1(P_6^3 \times I \times \Delta^n)$ for even n

	$\lambda(F_4)$	$\lambda(F_5)$	$\lambda(F_6)$	$\lambda(F_{n+9})$
	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$
1	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 1\ \cdots 1$
2	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$1\ 0\ 1\ 1\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 1\ \cdots 1$
3	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 1\ \cdots 1$
4	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 1\ \cdots 1$
5	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ 0\ 1\ c_1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 1\ \cdots 1$
6	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ c_2\ 1\ c_4\ 0\ \cdots 0$	$1\ 1\ 1\ 0\ 1\ \cdots 1$
7	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ c_2\ 1\ c_4\ 0\ \cdots 0$	$1\ 0\ 1\ 1\ 1\ \cdots 1$
8	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ c_2\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$1\ 0\ 0\ 0\ 1\ \cdots 1$
9	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ c_2\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$0\ 1\ 0\ 0\ 1\ \cdots 1$
10	$1\ 1\ 0\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$c_1\ c_2\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$1\ 1\ 0\ 1\ 1\ \cdots 1$
11	$1\ 1\ 0\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 1\ \cdots 1$
12	$1\ 1\ 0\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 1\ \cdots 1$
13	$1\ 1\ 0\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 1\ b_4\ b_5\ \cdots b_{n+4}$	$c_1\ c_2\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$0\ 0\ 0\ 1\ 1\ \cdots 1$

Table 6 All orientable λ in $D_3(P_6^3 \times I \times \Delta^n)$ for even n

	$\lambda(F_4)$	$\lambda(F_5)$	$\lambda(F_6)$	$\lambda(F_{n+9})$
	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$	$e_1\ e_2\ e_3\ e_4\ e_5\ \cdots\ e_{n+4}$
1	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$1\ g_2\ g_2\ 0\ 1\ \cdots 1$
2	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 1\ 1\ 1\ \cdots 1$
3	$1\ 1\ 1\ 0\ 0\ \cdots 0$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$1\ g_2\ g_3\ 1\ 1\ \cdots 1$
4	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ 0\ 1\ \cdots 1$
5	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 0\ 1\ 0\ 0\ \cdots 0$	$0\ 0\ 1\ 0\ 1\ \cdots 1$
6	$1\ 1\ 1\ a_4\ a_5\ \cdots a_{n+4}$	$0\ 1\ 0\ b_4\ b_5\ \cdots b_{n+4}$	$0\ 0\ 1\ c_4\ c_5\ \cdots c_{n+4}$	$0\ 0\ 0\ 1\ 1\ \cdots 1$