

## An efficient technique to solve nonlinear equations using multiplicative calculus

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**Abstract:** In this paper, we develop an efficient technique in the framework of multiplicative calculus and suggest a new class of numerical methods for solving multiplicative nonlinear equations  $g(x) = 1$ . We also develop the convergence criteria of the proposed methods. We solve the population growth model and minimization problem, which demonstrate the implementation and efficiency of the new techniques. We also show that these techniques perform much better as compared to the similar ordinary methods for solving ordinary nonlinear equations  $f(x) = 0$ .

**Key words:** Multiplicative derivative, numerical techniques, multiplicative nonlinear equation, convergence criteria

### 1. Introduction

In ordinary calculus, the commonly used integral and differential operators are the generalization of basic operations subtraction and addition of numbers. The concept of this differential and integral calculus was introduced in the seventeenth century. In the 1970s, Grossman and Katz [13] introduced a different definition of differential and integral operators that is based on multiplication and division rather than addition and subtraction. This idea guides the development of the new multiplicative calculus. In recent years, much interest has been given to developing the theory and applications of multiplicative calculus; see [1–6, 8–13, 16–18, 24–27, 29] and the references therein. Bashirov et al. [3] provided the theoretical surroundings of multiplicative calculus. In [12] the authors initiated the applications of multiplicative calculus in the biomedical analysis of images. Filip and Piatecki [11] connected multiplicative calculus in the developments in economics and finance. Applications of multiplicative calculus in actuarial science, finance, and demography were shown in [4]. Other different applications of multiplicative calculus in the structure of abstract writings and in biology were also considered in [2, 10].

There is a broad range of applications of multiplicative nonlinear equations  $g(x) = 1$ , almost in every field of life, especially in engineering and science. In this paper, we develop a variational iteration technique in the multiplicative calculus framework and suggest a new class of numerical methods for solving multiplicative nonlinear equations. The base of the variational iteration technique is linked back to Inokuti et al. [15], but it was He [14] who identified the significance and importance of the variational iteration method for solving various kinds of nonlinear problems. Noor [19], Noor and Shah [21], and Noor et al. [23] have used the variational

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iteration technique to develop several iterative methods for solving some nonlinear problems. By using the idea of Noor [19], we develop the said technique for multiplicative nonlinear equations. We prove the convergence analysis of the proposed methods. We perform some numerical tests to show the efficiency and performance of the new methods. We solve the population growth model and minimization problem using the new multiplicative techniques and compare the results with the ordinary iterative methods. We remark that using multiplicative methods one can obtain the solution of nonlinear equations in very few iterations instead of using the similar ordinary techniques.

## 2. Development of iterative methods

In this section, first we describe some basic results in the field of multiplicative calculus and then, using those results, construct a new class of iterative methods.

**Multiplicative derivative** [3]. The function  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  is said to be  $*$ differentiable at  $x$  or on  $D$  if it is positive and differentiable, respectively, at  $x$  or on  $D$ , and the multiplicative derivative (denoted by  $*$ derivative or  $f^*$ ) satisfies the following limit:

$$f^*(x) = \frac{d^* f}{dx} = \lim_{h \rightarrow 0} \left( \frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}}. \tag{1}$$

Using (1) one can calculate the  $*$ derivatives [3] of the function  $f(x)$  as:

$$f^*(x) = e^{\frac{f'(x)}{f(x)}} = e^{(\ln \circ f)'(x)}, \tag{2}$$

where  $(\ln \circ f)(x) = \ln f(x)$ .

$$f^{**}(x) = e^{(\ln \circ f^*)'(x)} = e^{(\ln \circ f)''(x)}, \tag{3}$$

and in general

$$f^{*(n)}(x) = e^{(\ln \circ f)^{(n)}(x)}, \quad n = 0, 1, \dots \tag{4}$$

Note that in formula (4) the case  $n = 0$  means no multiplicative derivative and it represents the original function  $f(x) = e^{(\ln \circ f)(x)}$ .

### 2.1. Some rules of $*$ differentiation [3]

Let  $c$  be a positive constant,  $f$ , and  $g$  be  $*$ differentiable, and  $h$  be differentiable; then the following results holds:

$$\left. \begin{aligned} \text{(i)} \quad & (c)^* = 1, \\ \text{(ii)} \quad & (cf)^*(x) = f^*(x), \\ \text{(iii)} \quad & (fg)^*(x) = f^*(x)g^*(x), \\ \text{(iv)} \quad & \left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}, \\ \text{(v)} \quad & (f^h)^*(x) = f^*(x)^{h(x)} \cdot f(x)^{h'(x)}, \\ \text{(vi)} \quad & (f \circ h)^*(x) = f^*(h(x))^{h'(x)}. \end{aligned} \right\} \tag{5}$$

For more detail, see [3, 27]. Now we develop the iterative methods using the above results.

**2.2. Multiplicative nonlinear equation**

Consider  $g : D \subset \mathbb{R} \rightarrow \mathbb{R}^+$  to be a positive nonlinear function. Then, in the framework of multiplicative calculus, the solution of the following nonlinear equation:

$$g(x) = 1, \tag{6}$$

exists and the equation is known as a multiplicative nonlinear equation. We remark that (6) is a nonlinear equation in the framework of multiplicative calculus. The aim of this text is to develop methods for (6). Suppose that  $\alpha$  is the solution of multiplicative nonlinear equation (6).

Consider the following relation in the framework of multiplicative calculus:

$$e^x = G(x), \tag{7}$$

with

$$G(x) = e^x (g(x))^{\lambda\phi(x)}, \tag{8}$$

where  $\phi(x) \neq 0$  is an arbitrary function and  $\lambda \neq 0$  is a parameter. By applying the natural log to both sides of (7) and using (8), we get the following fixed point problem:

$$x = x + \lambda\phi(x) \ln(g(x)). \tag{9}$$

Now, by taking the first \*derivative of  $G(x)$  in (8) by applying the rules (iii) and (v) of (5), we have

$$\frac{d^*G(x)}{dx} = e \cdot \left( (g^*(x))^{\lambda\phi(x)} \cdot (g(x))^{\lambda\phi'(x)} \right). \tag{10}$$

To find the minimum of a function using the ordinary derivative we use  $f'(x) = 0$ , which is the optimality criterion [7] in ordinary calculus. Generalizing this concept to multiplicative calculus, the optimality condition for finding the minimum of function  $G(x)$  is given by

$$\frac{d^*G(x)}{dx} = 1. \tag{11}$$

By using the optimality criterion (11) in (10), we obtain

$$1 = e \cdot \left( (g^*(x))^{\lambda\phi(x)} \cdot (g(x))^{\lambda\phi'(x)} \right). \tag{12}$$

Now, applying the natural log to both sides of (12), we get

$$\lambda = \frac{-1}{[\phi(x) \ln(g^*(x)) + \phi'(x) \ln(g(x))]} \tag{13}$$

By using the value of  $\lambda$  in the fixed point problem (9), we have

$$x = x - \frac{\phi(x) \ln(g(x))}{\phi(x) \ln(g^*(x)) + \phi'(x) \ln(g(x))}. \tag{14}$$

From the fixed point problem (14), we suggest the following iterative scheme for solving the multiplicative nonlinear equation  $g(x) = 1$ :

$$x_{n+1} = x_n - \frac{\phi(x_n) \ln(g(x_n))}{\phi(x_n) \ln(g^*(x_n)) + \phi'(x_n) \ln(g(x_n))}. \tag{15}$$

Iterative scheme (15) is the main recurrence relation for the multiplicative iterative methods. By taking different values of arbitrary function  $\phi(x_n)$  in (15), we infer numerous multiplicative iterative methods for solving multiplicative nonlinear equations (6) and this is the core aim of this article.

Now consider an individual case of the arbitrary functions  $\phi(x_n)$  in (15).

Let  $\phi(x_n) = e^{\beta x_n}$ . Then  $\phi'(x_n) = \beta e^{\beta x_n}$  and, accordingly from (15), we obtain the following iterative method for solving multiplicative nonlinear equation  $g(x) = 1$ :

**Method 2.1** For a given initial value  $x_0$ , approximate solution  $x_{n+1}$  of multiplicative nonlinear equation  $g(x) = 1$  by the following iterative scheme:

$$x_{n+1} = x_n - \frac{\ln(g(x_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}, \text{ for } n = 0, 1, 2, \dots, \tag{16}$$

where  $\beta$  is a parameter and a suitable selection of  $\beta$  makes the method more rapidly convergent. Other different choices of  $\phi(x_n)$  can be taken, for example  $\phi(x_n) = e^{\frac{\beta}{g^*(x_n)}}$ ,  $\phi(x_n) = e^{\frac{\beta g(x_n)}{g^*(x_n)}}$ , etc., which gives different new methods. The convergence and numerical efficiency of these methods may be different from each other.

For  $\beta = 0$  in (16), it reduces to the following iterative scheme:

**Method 2.2** For a given initial guess  $x_0$ , approximate the solution  $x_{n+1}$  of multiplicative nonlinear equation  $g(x) = 1$  by the scheme:

$$x_{n+1} = x_n - \frac{\ln(g(x_n))}{\ln(g^*(x_n))}, \text{ for } n = 0, 1, 2, \dots. \tag{17}$$

Method 2.2 is the multiplicative Newton method for solving the multiplicative nonlinear equation (6). We would like to mention that Method 2.2 was derived in [25].

Now, for  $\beta = -\frac{\ln(g^{**}(x_n))}{2 \ln(g^*(x_n))}$  in (16), it lessens to the following method for  $g(x) = 1$ :

**Method 2.3** For a given initial guess  $x_0$ , approximate the solution  $x_{n+1}$  of multiplicative nonlinear equation  $g(x) = 1$  by the iterative method:

$$x_{n+1} = x_n - \frac{2(\ln g(x_n)) \ln(g^*(x_n))}{\left[2(\ln(g^*(x_n)))^2 - \ln(g(x_n)) \ln(g^{**}(x_n))\right]}, \text{ for } n = 0, 1, 2, \dots. \tag{18}$$

Method 2.3 is the modified Halley method (see [20] for the ordinary Halley method) for solving (6) in the framework of multiplicative calculus.

Similarly, by suitable and apposite selection of  $\beta$  in (16) one can acquire a variety of various iterative methods. Now we propose the following two-step method by taking (16) as a predictor and the identical as a corrector, but with frozen denominator:

**Method 2.4** For a given initial guess  $x_0$ , approximate the solution  $x_{n+1}$  of multiplicative nonlinear equation  $g(x) = 1$  by the following iterative scheme:

$$\begin{aligned} y_n &= x_n - \frac{\ln(g(x_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}, \\ x_{n+1} &= y_n - \frac{\ln(g(y_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}, \text{ for } n = 0, 1, 2, \dots \end{aligned} \tag{19}$$

Method 2.4 is also a new predictor-corrector iterative scheme for solving the multiplicative nonlinear equation (6).

### 3. Convergence analysis

In this section, first we describe the multiplicative Taylor theorem for one variable.

**Multiplicative Taylor theorem** [3]. Let  $D$  be an open interval and let  $f : D \rightarrow \mathbb{R}$  be  $n + 1$  times \*differentiable on  $D$ . Then for any  $x, x + h \in D$ , there exists a number  $\theta \in (0, 1)$  such that

$$f(x + h) = \left( \prod_{m=0}^n \left( f^{*(m)}(x) \right)^{\frac{h^m}{m!}} \right) \left( f^{*(n+1)}(x + \theta h) \right)^{\frac{h^{n+1}}{(n+1)!}}. \tag{20}$$

Now we study the convergence criteria of the methods proposed in the previous section.

**Theorem 3.1** Consider  $g : D \rightarrow \mathbb{R}^+$  to be a positive function. Let  $g$  be a sufficiently \*differentiable and differentiable function for all  $x \in D$ . Assume that there exists a solution  $\alpha \in [a, b] \subset D$  of nonlinear equation (6) such that  $g(\alpha) = 1$ . Then there exists a  $\delta > 0$ , such that the iteration scheme defined by Method 2.4 converges to the solution  $\alpha$  for any initial approximation  $x_0 \in [\alpha - \delta, \alpha + \delta]$  and its rate of convergence is three.

**Proof** Consider the iteration defined by Method 2.4:

$$x_{n+1} = y_n - \frac{\ln(g(y_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}, \tag{21}$$

where

$$y_n = x_n - \frac{\ln(g(x_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}. \tag{22}$$

From (21), by taking  $x_n - \alpha = e_n$ ,  $y_n - \alpha = e_n^*$  and  $x_{n+1} - \alpha = e_{n+1}$ , we have

$$e_{n+1} - e_n^* = - \frac{\ln(g(y_n))}{\ln(g^*(x_n)) + \beta \ln(g(x_n))}. \tag{23}$$

From (23), after some simplification, we get

$$\ln \left[ \left( g^*(x_n) (g(x_n))^\beta \right)^{(e_{n+1}-e_n^*)} (g(y_n)) \right] = 0. \tag{24}$$

By applying natural exponential on both sides of (24), we obtain

$$\left( g^*(x_n) (g(x_n))^\beta \right)^{e_{n+1}} \left( g^*(x_n) (g(x_n))^\beta \right)^{-e_n^*} g(y_n) = 1. \tag{25}$$

From (25), we get

$$\left( g^*(x_n) (g(x_n))^\beta \right)^{e_{n+1}} = \frac{\left( g^*(x_n) (g(x_n))^\beta \right)^{e_n^*}}{g(y_n)}. \tag{26}$$

Similarly, from (22), we obtain

$$\left( g^*(x_n) (g(x_n))^\beta \right)^{e_n^*} = \frac{\left( g^*(x_n) (g(x_n))^\beta \right)^{e_n}}{g(x_n)}. \tag{27}$$

Now, by expanding  $g(x_n)$  using the multiplicative Taylor theorem (20) about  $\alpha$ , up to the first 4 factors, we have

$$g(x_n) \approx g(\alpha) g^*(\alpha)^{(x_n-\alpha)} g^{**}(\alpha)^{\frac{(x_n-\alpha)^2}{2!}} g^{***}(\alpha)^{\frac{(x_n-\alpha)^3}{3!}}. \tag{28}$$

Using  $g(\alpha) = 1$ , we get

$$g(x_n) \approx g^*(\alpha)^{e_n} g^{**}(\alpha)^{\frac{e_n^2}{2!}} g^{***}(\alpha)^{\frac{e_n^3}{3!}}. \tag{29}$$

From (29), we have the result

$$\left( g(x_n) \right)^\beta \approx g^*(\alpha)^{\beta e_n} g^{**}(\alpha)^{\frac{\beta e_n^2}{2!}} g^{***}(\alpha)^{\frac{\beta e_n^3}{3!}}. \tag{30}$$

Similarly, by expanding  $g^*(x_n)$  using (20) about  $\alpha$ , up to the first 4 factors, we have

$$g^*(x_n) \approx g^*(\alpha) g^{**}(\alpha)^{e_n} g^{***}(\alpha)^{\frac{e_n^2}{2!}} g^{*iv}(\alpha)^{\frac{e_n^3}{3!}}. \tag{31}$$

By using (29)–(31), we obtain

$$\frac{\left( g^*(x_n) (g(x_n))^\beta \right)^{e_n}}{g(x_n)} = \left( g^*(\alpha)^\beta g^{**}(\alpha)^{\left(\frac{1}{2!} + \frac{\beta e_n}{2!}\right)} g^{***}(\alpha)^{\left(\frac{2e_n}{3!} + \frac{\beta e_n^2}{3!}\right)} \right)^{e_n^2}, \tag{32}$$

and similarly, using (30) and (31), we get

$$\begin{aligned} & \left( g^*(x_n) (g(x_n))^\beta \right)^{e_n^*} \\ &= \left( g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left(e_n + \frac{\beta e_n^2}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!}\right)} \right)^{e_n^*}. \end{aligned} \tag{33}$$

Thus, using (32) and (33) in (27), we obtain the following result:

$$\begin{aligned} & \left( g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left( e_n + \frac{\beta e_n^2}{2!} \right)} g^{***}(\alpha)^{\left( \frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!} \right)} \right)^{e_n^*} \\ &= \left( g^*(\alpha)^\beta g^{**}(\alpha)^{\left( \frac{1}{2!} + \frac{\beta e_n}{2!} \right)} g^{***}(\alpha)^{\left( \frac{2e_n}{3!} + \frac{\beta e_n^2}{3!} \right)} \right)^{e_n^2}. \end{aligned} \tag{34}$$

By applying the natural log on both sides of (34), we have

$$\begin{aligned} & e_n^* \ln \left( g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left( e_n + \frac{\beta e_n^2}{2!} \right)} g^{***}(\alpha)^{\left( \frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!} \right)} \right) \\ &= e_n^2 \ln \left( g^*(\alpha)^\beta g^{**}(\alpha)^{\left( \frac{1}{2!} + \frac{\beta e_n}{2!} \right)} g^{***}(\alpha)^{\left( \frac{2e_n}{3!} + \frac{\beta e_n^2}{3!} \right)} \right). \end{aligned} \tag{35}$$

From (35), we have the following result:

$$e_n^* = \lambda_1 e_n^2, \tag{36}$$

where  $\lambda_1 > 0$ , is given by

$$\lambda_1 = \frac{\ln \left( g^*(\alpha)^\beta g^{**}(\alpha)^{\left( \frac{1}{2!} + \frac{\beta e_n}{2!} \right)} g^{***}(\alpha)^{\left( \frac{2e_n}{3!} + \frac{\beta e_n^2}{3!} \right)} \right)}{\ln \left( g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left( e_n + \frac{\beta e_n^2}{2!} \right)} g^{***}(\alpha)^{\left( \frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!} \right)} \right)}. \tag{37}$$

Now, by expanding  $g(y_n)$  using the multiplicative Taylor theorem (20) about  $\alpha$ , up to the first 4 factors, we have

$$g(y_n) \approx g(\alpha) g^*(\alpha)^{(y_n-\alpha)} g^{**}(\alpha)^{\frac{(y_n-\alpha)^2}{2!}} g^{***}(\alpha)^{\frac{(y_n-\alpha)^3}{3!}}. \tag{38}$$

Using  $g(\alpha) = 1$ , we get

$$g(y_n) \approx g^*(\alpha)^{e_n^*} g^{**}(\alpha)^{\frac{(e_n^*)^2}{2!}} g^{***}(\alpha)^{\frac{(e_n^*)^3}{3!}}. \tag{39}$$

By using (30), (31), and (39), and also using the value of  $e_n^*$  from (36), we obtain

$$\begin{aligned} & \frac{\left( g^*(x_n) (g(x_n))^\beta \right)^{e_n^*}}{g(y_n)} \\ &= \left[ g^*(\alpha)^\beta g^{**}(\alpha)^{\left( 1 + \frac{\beta e_n}{2!} - \frac{\lambda_1 e_n}{2!} \right)} g^{***}(\alpha)^{\left( \frac{e_n}{2!} + \frac{\beta e_n^2}{3!} - \frac{\lambda_1^2 e_n^3}{3!} \right)} \right]^{\lambda_1 e_n^3}, \end{aligned} \tag{40}$$

and similarly, by using (30) and (31), we get

$$\begin{aligned} & \left( g^*(x_n) (g(x_n))^\beta \right)^{e_{n+1}} \\ &= \left( g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left( e_n + \frac{\beta e_n^2}{2!} \right)} g^{***}(\alpha)^{\left( \frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!} \right)} \right)^{e_{n+1}}. \end{aligned} \tag{41}$$

Thus, by using (40) and (41) in (26), we obtain the following result:

$$\begin{aligned} & \left[ g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left(e_n + \frac{\beta e_n^2}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!}\right)} \right]^{e_{n+1}} \\ &= \left[ g^*(\alpha)^\beta g^{**}(\alpha)^{\left(1 + \frac{\beta e_n}{2!} - \frac{\lambda_1 e_n}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n}{2!} + \frac{\beta e_n^2}{3!} - \frac{\lambda_1^2 e_n^3}{3!}\right)} \right]^{\lambda_1 e_n^3}. \end{aligned} \tag{42}$$

By applying the natural log on both sides of (42), we have

$$\begin{aligned} & e_{n+1} \ln \left[ g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left(e_n + \frac{\beta e_n^2}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!}\right)} \right] \\ &= \lambda_1 e_n^3 \ln \left[ g^*(\alpha)^\beta g^{**}(\alpha)^{\left(1 + \frac{\beta e_n}{2!} - \frac{\lambda_1 e_n}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n}{2!} + \frac{\beta e_n^2}{3!} - \frac{\lambda_1^2 e_n^3}{3!}\right)} \right]. \end{aligned} \tag{43}$$

Finally, we have the following error condition:

$$e_{n+1} = \lambda_2 e_n^3, \tag{44}$$

where  $\lambda_2 > 0$  is given by

$$\lambda_2 = \lambda_1 \frac{\ln \left[ g^*(\alpha)^\beta g^{**}(\alpha)^{\left(1 + \frac{\beta e_n}{2!} - \frac{\lambda_1 e_n}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n}{2!} + \frac{\beta e_n^2}{3!} - \frac{\lambda_1^2 e_n^3}{3!}\right)} \right]}{\ln \left[ g^*(\alpha)^{(1+\beta e_n)} g^{**}(\alpha)^{\left(e_n + \frac{\beta e_n^2}{2!}\right)} g^{***}(\alpha)^{\left(\frac{e_n^2}{2!} + \frac{\beta e_n^3}{3!}\right)} \right]}. \tag{45}$$

It is clear from the error condition (44) that Method 2.4 converges to the solution  $\alpha$  with a convergence rate of the third order. □

Similarly, using the above technique one can easily prove the convergence of Methods 2.1–2.3.

#### 4. Numerical examples

In this section, we consider some numerical examples to demonstrate the efficiency and performance of newly developed methods; see Tables 1–3 and Figures 1–3. We compare our new methods (16) (\*VIM), (17) (\*Halley), and (19) (\*VIM2) with (\*NM) [25], (18), and the following methods:

- (i) Ordinary variational iteration method [19] (VIM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \beta f(x_n)}.$$

- (ii) Ordinary Newton method [7] (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



(iii) Ordinary Halley method [20] (Halley):

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{\left[2(f'(x_n))^2 - f(x_n)f''(x_n)\right]}.$$

(iv) Two-step third-order method [28] (TR):

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}, \quad \text{where } y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note that the main problem of this paper is to solve  $g(x) = 1$ . For the ordinary method we use  $f(x) = g(x) - 1 = 0$ . We use an Intel Core 2 Duo CPU 2 GHz computer (32 bit Windows XP) with 1 GB of RAM. We use Maple for all computational work with 150 digits floating point arithmetic (digits: 150). For the computer programs, we employ the stopping criteria  $|g(x_n) - 1| < \epsilon$  for multiplicative methods and  $|f(x_n) - 0| < \epsilon$  for ordinary methods, where  $\epsilon = 10^{-40}$ . We approximate the computational order of convergence (COC) (see [22]) using the following formula:

$$\text{COC} \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}.$$

**Example 4.1** [20] Consider the following multiplicative nonlinear equation:

$$g(x) = e^{x^2+7x-30} = 1.$$

The solution of this problem is  $x = 3$ . We take  $x_0 = 3.5$  as an initial approximation for the computer programs. We also chose  $\beta = -0.1$  for multiplicative and ordinary VIM methods. The computational results for this problem are shown in Table 1. The results of this example show that the multiplicative methods are very efficient as compared to the ordinary methods. The multiplicative methods converge to the solution in a few iterations with high accuracy as compared to the other methods. This fact is also clear from Figure 1. One can see that the ordinary VIM and NM methods approximate the solution in 13 iterations, whereas \*VIM and \*NM provide the same solution in only 5 iterations. A similar situation can be seen in the comparison of the ordinary Halley and \*Halley methods.

**Table 1.** Computational results for Example 4.1.

Method	IT	$x_{n+1}$	Rel. err.	COC
*VIM	5	3.0000000000000000	4.29e-59	2.00000
*NM	5	3.0000000000000000	2.66e-44	2.00000
*Halley	4	3.0000000000000000	1.94e-114	2.99999
*VIM2	3	3.0000000000000000	2.57e-45	3.00619
VIM [19]	13	3.0000000000000000	1.47e-49	2.00000
NM [7]	13	3.0000000000000000	1.52e-47	2.00000
Halley [20]	7	3.0000000000000000	2.56e-44	3.00032
TR [28]	9	3.0000000000000000	4.04e-45	2.99572

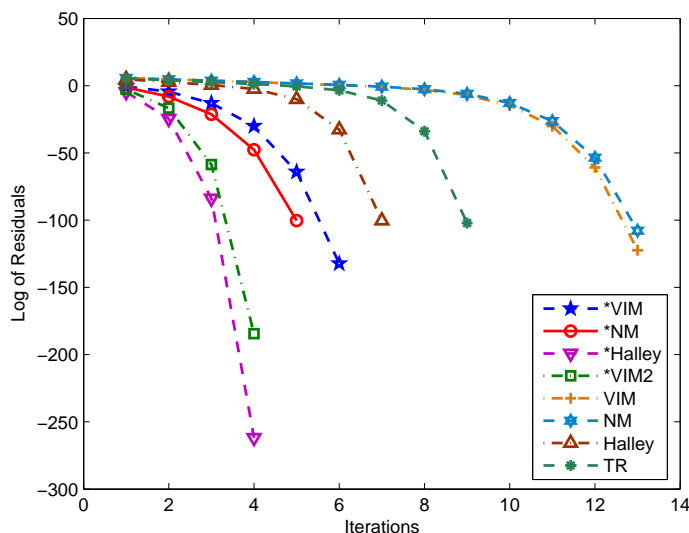


Figure 1. Log of residuals for Example 4.1.

**Example 4.2 (Application to population growth model) [7].** Consider the multiplicative nonlinear equation that arises in mathematical modeling of the growth of population over short periods of time:

$$g(x) = \frac{1000}{1564}e^x + \frac{435}{1,564x}(e^x - 1) = 1,$$

where  $x$  denotes the constant birth rate of the population and whose value needs to be found. We use  $x_0 = 1$  as an initial guess for the computer programs. We also chose  $\beta = -0.01$  for the multiplicative and ordinary VIM methods. The solution of this problem approximated to 15 decimal digits is  $x = 0.100997929685750$ . The numerical results for this problem are given in Table 2. Figure 2 shows the fall of residuals for this example. It is clear from the results of this problem that new multiplicative methods are outstanding in their performance as compared to the ordinary methods. It is clear that number of iterations of \*VIM, \*NM, \*Halley, and \*VIM2 methods is very lower as compared to the ordinary VIM, NM, Halley, and TR methods.

**Example 4.3 (Application to minimization problem) [16, 24].** Consider the following Lenard–Jones

Table 2. Computational results for Example 4.2.

Method	IT	$x_{n+1}$	Rel. err.	COC
*VIM	5	0.100997929685750	2.29e-51	2.00000
*NM	5	0.100997929685750	4.89e-47	2.00000
*Halley	4	0.100997929685750	4.70e-91	2.99989
*VIM2	4	0.100997929685750	1.61e-117	2.99996
VIM [19]	11	0.100997929685750	6.70e-57	2.00000
NM [7]	11	0.100997929685750	2.16e-56	2.00000
Halley [20]	7	0.100997929685750	2.18e-75	2.99974
TR [28]	8	0.100997929685750	3.11e-103	2.99998

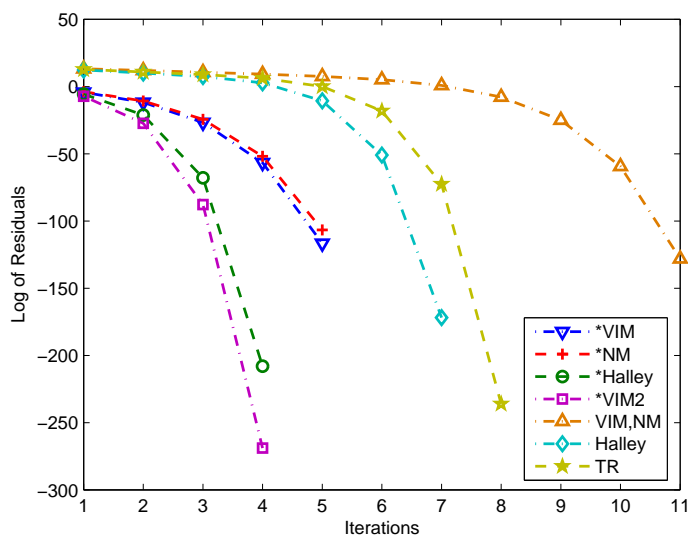


Figure 2. Log of residuals for Example 4.2.

potential [16]:

$$V(x) = V_0 \left[ \left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^6 \right], \tag{46}$$

which is well known in atomic physics and physical chemistry as a model in the interaction between two neutral atoms or molecules, where  $V_0$  denotes the depth of the potential and  $\sigma$  the position where the interparticle interaction becomes zero. We are interested to find the minimum value of function  $V(x)$ . The function  $V(x)$  has minimum value at  $x = 2^{\frac{1}{6}}\sigma$ . We select  $V_0 = 1$ ,  $\sigma = 2$ . Then the actual minimum value of function  $V(x)$  will be  $x = 2^{\frac{7}{6}}$ .

By taking the derivative of function  $V(x)$  in (46), we convert the minimization problem into the problem of finding the solution of the nonlinear equation given by

$$f(x) = V'(x) = V_0 \left[ -\frac{12\sigma^{12}}{x^{13}} + \frac{6\sigma^6}{x^7} \right] = 0. \tag{47}$$

Table 3. Computational results for Example 4.3.

Method	IT	$x_{n+1}$	Rel. err.	COC
*VIM	5	2.244924096618746	3.95e-44	2.00000
*NM	6	2.244924096618746	7.40e-44	2.00000
*Halley	4	2.244924096618746	4.14e-69	2.99885
*VIM2	4	2.244924096618746	3.60e-89	2.99996
VIM [19]	13	2.244924096618746	9.00e-41	2.00000
NM [7]	14	2.244924096618746	1.02e-65	2.00000
Halley [20]	8	2.244924096618746	2.67e-97	2.99998
TR [28]	10	2.244924096618746	1.02e-87	2.99993

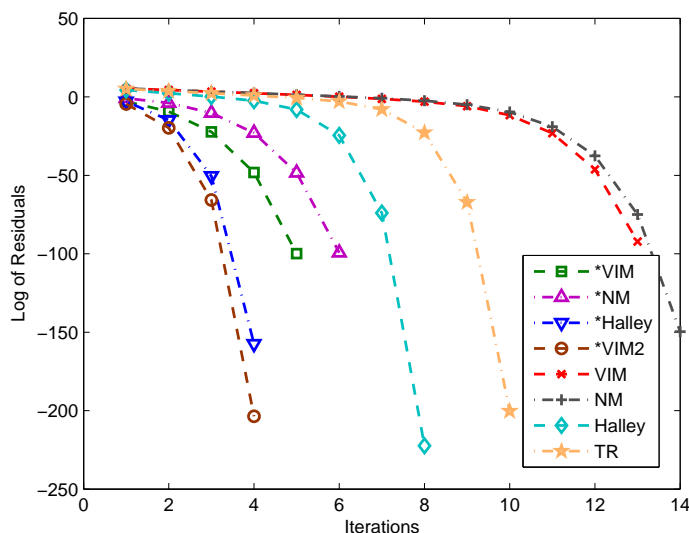


Figure 3. Log of residuals for Example 4.3.

Now we can rewrite nonlinear equation (47) in an equivalent form as multiplicative nonlinear equation:

$$g(x) = \left( \frac{6V_0\sigma^6}{x^7} \right) \left( \frac{x^{13}}{12V_0\sigma^{12}} \right) = 1.$$

We use  $x_0 = 1.4$  as initial values for the computer programs in this example. We also chose  $\beta = 0.3$  for the multiplicative and ordinary VIM methods. The comparison of numerical results for this problem is given in Table 3. Figure 3 shows the fall of residuals for this problem. It is clear from Table 3 and Figure 3 that the effectiveness and presentation of the new methods are much better than those of the other similar ordinary methods. Results of this example also show that the multiplicative methods converge more rapidly to the solution as compared to the ordinary ones.

### 5. Conclusion

In this paper, we have developed some new multiplicative iterative methods. By solving some examples, we showed that the multiplicative methods are more efficient and provide the approximate solution within the given tolerance in a very small number of iterations as compared to the similar ordinary techniques; see Tables 1–3 and Figures 1–3. This technique can also be generalized to nonlinear systems of equations.

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