A result on the maximal length of consecutive 0 digits in $\beta$-expansions

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Abstract: Let $\beta > 1$ be a real number. For any $x \in [0, 1]$, let $r_n(x, \beta)$ be the maximal length of consecutive 0 digits in the first $n$ digits of the $\beta$-expansion of $x$. In this note, it is proved that for any $0 < a < b < +\infty$, the set

$$E_{a, b} = \{x \in [0, 1] : \liminf_{n \to \infty} \frac{r_n(x, \beta)}{\log_n n} = a, \quad \limsup_{n \to \infty} \frac{r_n(x, \beta)}{\log_n n} = b\}$$

has the full Hausdorff dimension.

Key words: $\beta$-Expansion, consecutive zero digits, Hausdorff dimension

1. Introduction

For any real number $\beta > 1$, let

$$T_\beta : [0, 1] \to [0, 1]$$

be the $\beta$-transformation defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ means the largest integer no more than $\xi$. Then for any $x \in [0, 1]$, $T_\beta$ leads to the following series representation of $x$:

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots,$$

where $\varepsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ is said to be the $n$-th digit of $x$ with base $\beta$. The infinite digit sequence

$$\varepsilon_1(x, \beta)\varepsilon_2(x, \beta)\cdots\varepsilon_n(x, \beta)\cdots$$

is said to be the $\beta$-expansion of $x$.

For $n \geq 1$, we denote by $r_n(x, \beta)$ the maximal length of consecutive 0 digits in $\varepsilon_1(x, \beta)\cdots\varepsilon_n(x, \beta)$, i.e.

$$r_n(x, \beta) = \max\{k \geq 1 : \varepsilon_{i+1}(x, \beta) = \cdots = \varepsilon_{i+k}(x, \beta) = 0 \text{ for some } 0 \leq i \leq n - k\}.$$ 

Here we agree to define $r_n(x, \beta) = 0$ if there is no 0 digit in $\varepsilon_1(x, \beta)\cdots\varepsilon_n(x, \beta)$. Of course, $r_n(x, \beta)$ is monotonically nondecreasing with respect to $n$. There are many results about the growth speed of $r_n(x, \beta)$.

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For $\beta = 2$, Erdős and Rényi [3] proved that for Lebesgue almost all $x \in [0, 1],$
\[
\lim_{n \to \infty} \frac{r_n(x, 2)}{\log 2 n} = 1,
\]
(1.1)
in 1970. See also [12] for a proof of (1.1). Furthermore, Ma et al. [9] showed that the set of points that violate (1.1) is of Hausdorff dimension one. Recently, Sun and Xu [13] determined the Hausdorff dimension of the set
\[
D_{a,b} = \left\{ x \in [0, 1] : \liminf_{n \to \infty} \frac{r_n(x, 2)}{\log 2 n} = a, \limsup_{n \to \infty} \frac{r_n(x, 2)}{\log 2 n} = b \right\}
\]
with $0 < a < b < +\infty$. They proved that the exceptional set $D_{a,b}$ has Hausdorff dimension one. In 2016, Li and Wu [8] replaced $\log 2 n$ in (1.1) by a general function $\varphi(n)$, where $\varphi : N \to \mathbb{R}^+$ is a monotonically increasing function with $\lim_{n \to \infty} \varphi(n) = +\infty$. They considered the following set:
\[
D_\varphi = \left\{ x \in [0, 1] : \liminf_{n \to \infty} \frac{r_n(x, 2)}{\varphi(n)} = 0, \limsup_{n \to \infty} \frac{r_n(x, 2)}{\varphi(n)} = +\infty \right\}.
\]
Li and Wu [8] showed that the exceptional set $D_\varphi$ has Hausdorff dimension 1 if $\limsup_{n \to \infty} \frac{n}{\varphi(n)} = +\infty$; otherwise $D_\varphi$ has Hausdorff dimension 0. Naturally, for any $\beta > 1$, what is the growth rate of $r_n(x, \beta)$? It is known that for $\beta = 2$, $T_\beta$ is a finite expanding Markov map. However, when $T_\beta$ without the Markov property, things become more difficult. Recently, Tong et al. [14] gave the answer to this question as follows.

**Theorem 1.1 ([14])** Let $\beta > 1$ be a real number.

(i) For Lebesgue almost all $x \in [0, 1]$, we have
\[
\lim_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} = 1.
\]
(ii) Let $\alpha > 0$ and
\[
E_\alpha = \left\{ x \in [0, 1] : \lim_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} = \alpha \right\}.
\]
Then the set $E_\alpha$ has Hausdorff dimension 1.

In this note, we consider the following kind of exceptional set for $r_n(x, \beta)$. For any $0 < a < b < +\infty$, let
\[
E_{a,b} = \left\{ x \in [0, 1] : \liminf_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} = a, \limsup_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} = b \right\}.
\]
Intuitively, the set $E_{a,b}$ is small because it consists of points that cannot satisfy the above law in Theorem 1.1. However, we prove the following dimensional result of the set $E_{a,b}$.

**Theorem 1.2** For any real numbers $0 < a < b < +\infty$, the Hausdorff dimension of the set $E_{a,b}$ is full.
2. Preliminary

In this section, we list some basic properties of $\beta$-expansions and give some notations. We write $uv$ for the concatenation of words $u$ and $v$. In particular, $u^i$ denotes the $i$ times self-concatenation of $u$ for $i \geq 1$. Denote by $|u|$ the length of the word $u$.

**Definition 2.1** We say that a finite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ or an infinite word $\varepsilon_1\varepsilon_2\cdots$ is $\beta$-admissible, if there exists $x \in [0, 1)$ such that $\varepsilon_i(x, \beta) = \varepsilon_i$ for all $1 \leq i \leq n$ or $i \geq 1$, respectively.

Let us denote by $\Sigma^1_\beta$ the set of admissible words of length $n$ and $\Sigma_\beta$ the set of all admissible words of infinite length. We define the lexicographical order $\lex$ between two infinite words as follows:

$$\varepsilon_1\varepsilon_2\cdots <_\lex \varepsilon'_1\varepsilon'_2\cdots$$

if there exists some integer $k \geq 1$ satisfying $\varepsilon_j = \varepsilon'_j$ for all $1 \leq j < k$ and $\varepsilon_k < \varepsilon'_k$. In fact, we can extend the order $\lex$ to finite words by identifying a finite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ with the infinite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n0^\infty$ where $\xi^\infty$ means the periodic sequence $\xi\cdots$. Now we define an infinite word $\varepsilon^*_1\varepsilon^*_2\cdots$ from the $\beta$-expansions of 1. If there exists an integer $m \geq 1$ such that $\varepsilon_m(1, \beta) \geq 1$ but $\varepsilon_n(1, \beta) = 0$ for all $n > m$, then we write

$$\varepsilon^*_1(1, \beta)\varepsilon^*_2(1, \beta)\cdots = (\varepsilon_1(1, \beta)\cdots(\varepsilon_m(1, \beta) - 1))^{\infty}.$$

Otherwise, we write

$$\varepsilon^*_1(1, \beta)\varepsilon^*_2(1, \beta)\cdots = \varepsilon_1(1, \beta)\varepsilon_2(1, \beta)\cdots.$$

We list some basic properties about admissible words in the following lemma.

**Lemma 2.1** ([10, 11])  

(i) An infinite word $\varepsilon_1\varepsilon_2\cdots \in \Sigma_\beta$ if and only if

$$\forall k \geq 1, \quad \varepsilon_k\varepsilon_{k+1}\cdots <_\lex \varepsilon^*_1(1, \beta)\varepsilon^*_2(1, \beta)\cdots.$$

(ii) For any $x, y \in [0, 1)$, $x < y$ if and only if $\varepsilon_1(x, \beta)\varepsilon_2(x, \beta)\cdots <_\lex \varepsilon_1(y, \beta)\varepsilon_2(y, \beta)\cdots$. Moreover, if $1 < \beta < \beta'$, then

$$\Sigma_\beta \subset \Sigma_{\beta'}.$$

(iii) For any $\beta > 1$,

$$\beta^n - \frac{1}{2} \leq \#\Sigma^n_\beta \leq \beta^{n+1}/(\beta - 1),$$

where $\#$ means the number of elements of a finite set.

(iv) An infinite word $\varepsilon_1\varepsilon_2\cdots$ is the $\beta$-expansion of 1 for some $\beta > 1$ if and only if for all $k \geq 2$,

$$\varepsilon_k\varepsilon_{k+1}\cdots <_\lex \varepsilon_1\varepsilon_2\cdots.$$

For any admissible word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$, define

$$I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \{x \in [0, 1] : \varepsilon_1(x, \beta)\varepsilon_2(x, \beta)\cdots\varepsilon_n(x, \beta) = \varepsilon_1\varepsilon_2\cdots\varepsilon_n \},$$

which is called an $n$-th order cylinder. We define the $n$-th order cylinder containing $x$ for $x \in [0, 1]$, denoted by $I_n(x, \beta)$, which is the set of points $y \in [0, 1]$ with the property that $\varepsilon_i(y, \beta) = \varepsilon_i(x, \beta)$ for all $1 \leq i \leq n$. We write $|I_n(x, \beta)|$ for the length of $I_n(x, \beta)$. The following basic properties of cylinders are proved in [6] and [7].
Lemma 2.2 ([6, 7]) The cylinder \( I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) \) is an interval whose left endpoint is \( \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n} \) and

\[ |I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)| \leq \frac{1}{\beta^n}. \]

Here we consider a kind of cylinders with maximal lengths, which are known as full cylinders.

Definition 2.2 For any \( \varepsilon_1\varepsilon_2\cdots\varepsilon_n \in \Sigma^n_\beta \), we say that the cylinder \( I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) \) is full if its length satisfies

\[ |I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)| = \frac{1}{\beta^n}. \]

The characterization of full cylinders was obtained by Fan and Wang [6] as follows.

Lemma 2.3 ([6]) Let \( \varepsilon_1\varepsilon_2\cdots\varepsilon_n \) be an admissible word. The following conditions are equivalent:

(i) The cylinder \( I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) \) is full;

(ii) \( \mathcal{T}_n(I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)) = [0, 1); \)

(iii) For any \( w_1w_2\cdots w_m \in \Sigma^m_\beta \), the concatenation \( \varepsilon_1\varepsilon_2\cdots\varepsilon_n w_1w_2\cdots w_m \) is still \( \beta \)-admissible.

We shall make use of the following two lemmas from Bugeaud and Wang [1] to construct new full cylinders and estimate the number of full cylinders.

Lemma 2.4 ([1]) If \( I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) \) and \( I_m(w_1w_2\cdots w_m) \) are two full cylinders, then the concatenation \( I_{n+m}(\varepsilon_1\varepsilon_2\cdots\varepsilon_n w_1w_2\cdots w_m) \) is still a full cylinder.

Lemma 2.5 ([1]) For any \( n+1 \) consecutive cylinders of order \( n \), there is at least one full cylinder.

3. Proof of Theorem 1.2

3.1. The construction of a Cantor subset of \( E_{a,b} \)

Let us recall that for any \( 0 < a < b < +\infty, \)

\[ E_{a,b} = \left\{ x \in [0, 1] : \liminf_{n \to \infty} \frac{r_n(x, \beta)}{\log_\beta n} = a, \limsup_{n \to \infty} \frac{r_n(x, \beta)}{\log_\beta n} = b \right\}. \]

The main idea of the proof is to construct a Cantor subset of \( E_{a,b} \) denoted by \( E^*_{a,b} \) with \( \dim_H E^*_{a,b} = 1 \). Here we denote by \( \dim_H \) the Hausdorff dimension. For convenience, we shall make use of full cylinders repeatedly in the construction of \( E^*_{a,b} \). The construction is divided into four steps.

Step 1. Let \( h \) be the smallest integer such that \( I_{h+1}(10^h) \) is full. Take an integer \( N \geq 1 \) large enough with the property that

\[ \lfloor b \log_\beta m \rfloor \geq h + 1 \quad \text{and} \quad \frac{\lfloor m^{b/a} \rfloor - m}{b \log_\beta m} \geq 2, \]
for all \( m \geq N \). Let \( n_1 = N \). For any \( k \geq 2 \), we define \( n_k \) and \( d_{k-1} \) by the following recursive formulae:

\[
n_k = \lfloor n_{k-1}^\frac{1}{2} \rfloor, \quad d_{k-1} = \lfloor b \log_b n_{k-1} \rfloor.
\]

Step 2. For any \( k \geq 1 \), we set

\[
W_k = \{ \varepsilon_1 \cdots \varepsilon_{d_k} \in \Sigma_{d_k}^a : I_{d_k} (\varepsilon_1 \cdots \varepsilon_{d_k}) \text{ is full with } \varepsilon_1 = 1 \}.
\]

Let

\[
t_k = \left\lfloor \frac{n_{k+1} - n_k}{d_k} \right\rfloor - 1.
\]

Set

\[
U_k = \{ w^{(1)}w^{(2)}\cdots w^{(t_k)} : w^{(j)} \in W_k, \text{ for all } 1 \leq j \leq t_k \}.
\]

In other words, \( U_k \) consists of finite words that are possible concatenations of any \( t_k \) words from \( W_k \).

Step 3. For any \( k \geq 1 \), we first define a finite word \( v^{(k)} \) as follows:

\[
v^{(k)} = \begin{cases} 0^{k}, & \text{if } 0 \leq \delta_k \leq h \\ 10^{d_k-1}, & \text{if } h + 1 \leq \delta_k < d_k, \end{cases}
\]

where \( \delta_k = n_{k+1} - n_k - \lfloor \frac{n_{k+1} - n_k}{d_k} \rfloor dk \). Next, we define

\[
D_k = \{ 10^{d_k-1}u^{(k)}v^{(k)} : u^{(k)} \in U_k \}.
\]

Note that for each word in \( D_k \), the maximal length of consecutive 0 digits is at least \( d_k - 1 \) and at most \( d_k - 1 + h \) by our construction.

Step 4. The desired Cantor set is defined as

\[
E_{a,b}^* = \{ x \in [0,1] : \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots = 0^N\sigma^{(1)}\sigma^{(2)}\cdots, \sigma^{(k)} \in D_k, \ k \geq 1 \},
\]

which will be shown to be a subset of \( E_{a,b} \) with Hausdorff dimension 1. By the properties of full cylinders, the construction is well defined (see the Figure).

The following results are obtained by direct calculations. For convenience, we list them as a lemma.

**Lemma 3.1** Let \( \{ n_k \}_{k \geq 1} \) and \( \{ d_k \}_{k \geq 1} \) be defined as above. Then:

1. \( \lim_{k \to \infty} \frac{d_k}{n_k} = 0 \), \( \lim_{k \to \infty} \frac{\log_b n_k}{\log_b n_{k+1}} = \frac{a}{b} \);

2. \( \lim_{k \to \infty} \frac{d_k}{\log_b n_{k+1}} = a \), \( \lim_{k \to \infty} \frac{d_k}{\log_b n_k} = b \).
Proposition 3.1 \( E^*_a,b \) is a Cantor subset of \( E_{a,b} \).

Proof We first show

\[
  a \leq \liminf_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} \leq \limsup_{n \to \infty} \frac{r_n(x, \beta)}{\log \beta n} \leq b. \tag{3.4}
\]

By the definition of \( d_i \), there exists some \( K \geq 1 \) such that \( d_i > N \) for any \( i \geq K \). Then for any \( n > n_{K+1} \), there exists some \( k \geq K + 1 \), such that \( n_k < n \leq n_{k+1} \). We distinguish two cases.

Case 1: If \( n_k < n \leq n_k + d_k \), then \( d_{k-1} - 1 \leq r_n(x, \beta) \leq \max\{d_k - 1, d_{k-1} - 1 + h\} \). Thus, for all \( k \) large enough,

\[
  \frac{d_{k-1} - 1}{\log \beta (n_k + d_k)} \leq \frac{r_n(x, \beta)}{\log \beta n} \leq \frac{d_k - 1}{\log \beta n_k}. \tag{3.5}
\]

Case 2: If \( n_k + d_k < n \leq n_{k+1} \), then we have \( d_k - 1 \leq r_n(x, \beta) \leq d_k - 1 + h \). Thus,

\[
  \frac{d_k - 1}{\log \beta n_{k+1}} \leq \frac{r_n(x, \beta)}{\log \beta n} \leq \frac{d_k - 1 + h}{\log \beta n_k}. \tag{3.6}
\]

By (3.5), (3.6), and Lemma 3.1, (3.4) holds.

It remains to prove that there exist subsequences \( \{m_k\}_{k \geq 1} \) and \( \{m'_k\}_{k \geq 1} \) such that \( \lim_{k \to \infty} \frac{r_{m_k}(x, \beta)}{\log \beta m_k} = a \) and \( \lim_{k \to \infty} \frac{r_{m'_k}(x, \beta)}{\log \beta m'_k} = b \), respectively. Let \( m_k = n_{k+1} + d_k \). Then \( d_k - 1 \leq r_{m_k}(x, \beta) \leq d_k - 1 + h \) for all \( k \) large enough. Thus, by Lemma 3.1, we have

\[
  \lim_{k \to \infty} \frac{r_{m_k}(x, \beta)}{\log \beta m_k} = \lim_{k \to \infty} \frac{d_k}{\log \beta (n_{k+1} + d_k)} = a.
\]

Let \( m'_k = n_k + d_k \). Similarly, we have \( \lim_{k \to \infty} \frac{r_{m'_k}(x, \beta)}{\log \beta m'_k} = b \). \( \square \)

3.2. Hausdorff dimension of \( E^*_a,b \)

Our next goal is to get a lower bound of \( \dim_H E^*_a,b \). For any \( \beta_* < \beta \), we shall prove that \( \dim_H E^*_a,b \geq \log \frac{\beta_*}{\log \beta} \).

We first introduce the following modified mass distribution principle, which is helpful for the estimate of lower bounds of Hausdorff dimensions. The usual mass distribution principle can be found in [5].

Proposition 3.2 ([1]) Let \( E \) be a Borel measurable set in \([0,1]\) and \( \mu \) be a Borel measure with \( \mu(E) > 0 \). For some \( s > 0 \), there exist a constant \( C > 0 \) and an integer \( M \) with the property that for any \( n > M \) and any \( n \)-th order cylinder \( I_n \),

\[
  \mu(I_n) \leq C \cdot |I_n|^s.
\]

Then \( \dim_H E \geq s \).

For any \( k \geq 1 \), let \( W_k \) and \( D_k \) be defined by (3.1) and (3.3), respectively. We set

\[
  p_k := \# W_k
\]
and

\[ q_{k+1} := \# \{0^N \sigma^{(1)} \sigma^{(2)} \cdots \sigma^{(k)}, \quad \sigma^{(j)} \in D_j, \quad 1 \leq j \leq k \}. \]

The following lemma gives lower bounds of \( p_k \) and \( q_k \) for \( k \) large enough.

**Lemma 3.2** For any \( \beta_* < \beta \), there exists an integer \( K(\beta_*) \geq 1 \) such that for any \( k > K(\beta_*) \),

\[ p_k \geq \beta_*^{d_k} \]

and

\[ q_k \geq C(\beta_*) \gamma_*^k, \]

where \( \gamma_k = n_k - n_1 - 2 \sum_{j=1}^{k-1} d_j \) and \( C(\beta_*) \) is a constant depending only on \( \beta_* \).

**Proof** We first estimate \( p_k \). For any \( k \geq 1 \), write

\[ W'_k = \{ \varepsilon_1 \cdots \varepsilon_{d_k} \in \Sigma^\beta_{d_k} : I_{d_k-h-1}(\varepsilon_{h+2} \cdots \varepsilon_{d_k}) \text{ is full and } \varepsilon_1 \cdots \varepsilon_{h+1} = 10^h \} \]

and

\[ p'_k := \# W'_k. \]

Then \( W'_k \subset W_k \) by the properties of full cylinders. Therefore, \( p_k \geq p'_k \). From Lemma 2.1 we obtain that the number of admissible words with length \( d_k - h - 1 \) is at least \( \beta_*^{d_k-h-1} \). According to Lemma 2.5, we have

\[ p'_k \geq \left\lfloor \frac{\beta_*^{d_k-h-1}}{d_k-h} \right\rfloor \]

for any \( k \geq 1 \). It is easy to check that there exists \( K(\beta_*) > 1 \) such that for any \( k > K(\beta_*) \),

\[ \left\lfloor \frac{\beta_*^{d_k-h-1}}{d_k-h} \right\rfloor \geq \beta_*^{d_k}. \]

Thus,

\[ p_k \geq p'_k \geq \beta_*^{d_k}. \tag{3.7} \]

We conclude from (3.7) that

\[ \# U_k = p'_k \geq (\beta_*^{d_k}) \left\lfloor \frac{n_{k+1}-n_k}{d_k} \right\rfloor^{-1} \geq \beta_*^{n_{k+1}-n_k-2d_k}, \]

and hence that

\[ \# D_k = \# U_k \geq \beta_*^{n_{k+1}-n_k-2d_k}, \]

for any \( k > K(\beta_*) \). Thus, there exists a constant \( C(\beta_*) > 0 \) depending only on \( \beta_* \) such that

\[ q_k = \prod_{j=1}^{k-1} \# D_j \geq C(\beta_*) \gamma_*^k, \]

where \( \gamma_k = n_k - n_1 - 2 \sum_{j=1}^{k-1} d_j \). \( \Box \)
Proposition 3.3 The set \( E^*_{a,b} \) has Hausdorff dimension 1.

Proof It suffices to show that \( \dim_H E^*_{a,b} \geq \frac{\log \beta_*}{\log 2} \) for any \( \beta_* < \beta \). We first define a probability measure \( \mu \) on \( E^*_{a,b} \) by induction. Set \( \mu[0,1] = 1 \) and \( \mu(I_i(0^i)) = 1 \), for all \( 1 \leq i \leq N \). For any \( k \geq 1 \) and any \( \sigma^{(j)} \in D_j \), \( 1 \leq j \leq k \), we define

\[
\mu(I_{nk+1}\{0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k)}\}) = \mu(I_{nk}\{0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k-1)}\}) \frac{1}{\#D_k}. \tag{3.8}
\]

Now we define \( \mu(I_n(x,\beta)) \) for any \( n_k \leq n < n_{k+1} \) and any \( x \in E^*_{a,b} \). Let

\[
\mu(I_n(x,\beta)) = \sum \mu(I_{nk+1}(\xi)),
\]

where the sum is taken over all \( \xi = 0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k)} \) with \( I_{nk+1}(\xi) \subseteq I_n(x,\beta) \) and \( \sigma^{(j)} \in D_j \) for \( 1 \leq j \leq k \). Then we can extend \( \mu \) to a Borel probability measure uniquely on \( E^*_{a,b} \) by Kolmogorov’s consistency theorem. By (3.8) and Lemma 3.2, we have

\[
\mu(I_{nk}(x,\beta)) = \frac{1}{q_k} \leq C^{-1}(\beta_*)\beta_*^{-\gamma_k},
\]

for all \( k > K(\beta_*) \), where \( K(\beta_*) \) is the integer defined as in Lemma 3.2. For any \( n_k \leq n < n_{k+1} \) with \( k > K(\beta_*) \), either there exists integer \( l \) such that

\[
n_k + ld_k \leq n < n_k + (l+1)d_k, \quad 0 \leq l \leq t_k
\]

or

\[
n_k + (t_k + 1)d_k \leq n < n_{k+1},
\]

where \( t_k \) is defined in (3.2).

Next we distinguish three cases.

Case 1: \( n_k \leq n < n_k + 2d_k \). Then

\[
\mu(I_n(x,\beta)) \leq \mu(I_{nk}(x,\beta)) \leq C^{-1}(\beta_*)\beta_*^{-\gamma_k}.
\]

By the definition of a full cylinder, we have

\[
|I_n(x,\beta)| \geq |I_{nk+2d_k}(x,\beta)| = \beta^{-(n_k+2d_k)}.
\]

Combing the two inequalities above, we have

\[
\frac{\log \mu(I_n(x,\beta))}{\log |I_n(x,\beta)|} \geq \frac{\log \beta_*^{-\gamma_k} - \log C(\beta_*)}{\log \beta^{-(n_k+2d_k)}}.
\]

It follows that

\[
\lim_{k \to \infty} \frac{\log \beta_*^{-\gamma_k}}{\log \beta^{-(n_k+2d_k)}} = \lim_{k \to \infty} \frac{(n_k - n_1 - 2 \sum_{j=1}^{k-1} d_j) \log \beta_*}{(n_k + 2d_k) \log \beta} = \frac{\log \beta_*}{\log \beta} \tag{3.9}
\]

by Lemma 3.1.
Case 2: \( n_k + l d_k \leq n < n_k + (l + 1) d_k \) for some \( 2 \leq l \leq t_k \). Then

\[
\mu(I_n(x, \beta)) \leq \mu(I_{n_k + l d_k}(x, \beta)) = \mu(I_{n_k}(x, \beta)) \cdot p_k^{-(l-1)} \leq C^{-1}(\beta_*) \beta^{-\gamma_k} \beta^{-d_k(l-1)}
\]

by Lemma 3.2 and (3.8). By the definition of a full cylinder and Lemma 2.4, we have

\[
| I_n(x, \beta) | \geq | I_{n_k + (l+1) d_k}(x, \beta) | = \beta^{- (n_k + (l+1) d_k)}.
\]

Hence,

\[
\frac{\log \mu(I_n(x, \beta))}{\log | I_n(x, \beta) |} \geq \frac{\log \beta^{-\gamma_k - d_k(l-1)} - \log C(\beta_*)}{\log \beta^{- (n_k + (l+1) d_k)}}.
\]

Similarly,

\[
\lim_{k \to \infty} \frac{\log \beta^{-\gamma_k - d_k(l-1)}}{\log \beta^{- (n_k + (l+1) d_k)}} = \frac{\log \beta_*}{\log \beta}.
\]

Case 3: \( n_k + (t_k + 1) d_k \leq n < n_{k+1} \). Note that

\[
\mu(I_n(x, \beta)) = \mu(I_{n_k + 1}(x, \beta)) \leq C^{-1}(\beta_*) \beta^{-\gamma_{k+1}}
\]

and

\[
| I_n(x, \beta) | \geq | I_{n_{k+1}}(x, \beta) | = \beta^{-n_{k+1}}.
\]

Then

\[
\frac{\log \mu(I_n(x, \beta))}{\log | I_n(x, \beta) |} \geq \frac{\log \beta^{-\gamma_{k+1}} - \log C(\beta_*)}{\log \beta^{-n_{k+1}}} \quad \text{and} \quad \lim_{k \to \infty} \frac{\log \beta^{-\gamma_{k+1}}}{\log \beta^{-n_{k+1}}} = \frac{\log \beta_*}{\log \beta}.
\]

By (3.9), (3.10), and (3.11), it follows that for any \( \varepsilon > 0 \), there exists an integer \( K' \) such that for any \( n \geq n_K \) and any \( x \in E^*_{a,b} \)

\[
\mu(I_n(x, \beta)) \leq | I_n(x, \beta) | \log \beta_* / \log \beta - \varepsilon.
\]

Using Proposition 3.2, we conclude that

\[
\dim_H E^*_{a,b} \geq \log \beta_* / \log \beta - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, it follows that \( \dim_H E^*_{a,b} \geq \log \beta_* / \log \beta \).

**Proof** [Proof of Theorem 1.2] With the help of Proposition 3.1 and Proposition 3.3, the conclusion is obtained immediately.

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