The equation $dd' + d'd = D^2$ for derivations on $C^*$-algebras

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Abstract: Let $\mathcal{A}$ be an algebra. A linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for each $a, b \in \mathcal{A}$. Given two derivations $d$ and $d'$ on a $C^*$-algebra $\mathcal{A}$, we prove that there exists a derivation $D$ on $\mathcal{A}$ such that $dd' + d'd = D^2$ if and only if $d$ and $d'$ are linearly dependent.

Key words: Derivation; $C^*$-algebra

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation if it satisfies the Leibniz rule $d(ab) = d(a)b + ad(b)$ for each $a, b \in \mathcal{A}$. When $\mathcal{A}$ is a $*$-algebra, $d$ is called a $*$-derivation if $d(a^*) = d(a)^*$ for each $a \in \mathcal{A}$.

As a typical example of a nonzero derivation in a noncommutative algebra, we can consider the inner derivation $\delta_a$ implemented by an element $a \in \mathcal{A}$, which is defined as $\delta_a(x) = xa - ax$ for all $x \in \mathcal{A}$. There are known algebras $\mathcal{A}$ such that each derivation on $\mathcal{A}$ is inner, which is implemented by an element of the algebra $\mathcal{A}$ or an algebra $\mathcal{B}$ containing $\mathcal{A}$. For example, each derivation on a von Neumann algebra $\mathcal{M}$ is inner and is implemented by an element of $\mathcal{M}$. Moreover, each derivation on a $C^*$-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is inner and implemented by an element of the weak closure $\mathcal{M}$ of $\mathcal{A}$ in $B(\mathcal{H})$ (see [4, 10]).

Even for an inner derivation $\delta_a$ on an algebra $\mathcal{A}$, it is very probable that $\delta_a^2$ is not a derivation. In fact, if $d$ is a $*$-derivation on a $C^*$-algebra $\mathcal{A}$, then $d^2$ is a derivation if and only if $d = 0$. To see this, note that $d^2$ is a derivation if and only if

$$d^2(x)y + 2d(x)d(y) + xd^2(y) = d^2(xy) = d^2(x)y + xd^2(y).$$

The latter is equivalent to the fact that $d(x)d(y) = 0$ for each $x, y \in \mathcal{A}$. Thus $d(x)d(x^*) = d(x)d(x^*) = 0$ for each $x \in \mathcal{A}$. Hence $\|d(x)\|^2 = \|d(x)d(x^*)\| = 0$. This shows that $d(x) = 0$ for each $x \in \mathcal{A}$.

These considerations show that the set of derivations on an algebra $\mathcal{A}$ is not in general closed under product. There are various studies seeking some conditions under which the product of two derivations will be again a derivation. Posner [9] was the first to study the product of two derivations on a prime ring. He showed that if the product of two derivations on a prime ring, with characteristic not equal to 2, is a derivation then

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We thus have This shows that $AE \leq 1/2$. Proposition 2.2 Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leq i, k, \ell, j \leq n$ if and only if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$. Proof Let $\{E_{ij}\}_{1 \leq i, j \leq n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. If $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leq i, k, \ell, j \leq n$ then we can write 

$$(E_{ii}AE_{kl})(E_{\ell \ell}BE_{jj}) = a_{ik}b_{\ell j}E_{ij} = b_{ik}a_{\ell j}E_{ij} = (E_{ii}BE_{kl})(E_{\ell \ell}AE_{jj}).$$

We thus have

$$\left(\sum_{i=1}^{n} E_{ii}\right)AE_{kl}B\left(\sum_{j=1}^{n} E_{jj}\right) = \left(\sum_{i=1}^{n} E_{ii}\right)BE_{kl}A\left(\sum_{j=1}^{n} E_{jj}\right).$$

This shows that $AE_{kl}B = BE_{kl}A$ for each $1 \leq k, \ell \leq n$. We can therefore deduce that $AXB = BXA$ for all $X \in M_n(\mathbb{C})$.

On the other hand, if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$, then setting $X = E_{jk}E_{kk}$, we get

$$a_{ij}b_{k\ell}E_{il} = (E_{ii}AE_{jk})(E_{kk}BE_{\ell \ell}) = E_{ii}(AE_{jk}E_{kk}B)E_{\ell \ell} = (E_{ii}BE_{jk})(E_{kk}AE_{\ell \ell}) = b_{ij}a_{k\ell}E_{il}.$$  

\qed

Let $A = [a_{ij}] \in M_n(\mathbb{C})$. We denote the diagonal matrix whose diagonal entries are $a_{ii}$ by $A^D$.

Proposition 2.2 Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then there exists a $C = [c_{ij}] \in M_n(\mathbb{C})$ such that $\delta_A^D + \delta_B^D = \delta_C^D$ if and only if $\alpha A = \beta B + r I$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$. Proof We can assume that $a_{11} = b_{11} = c_{11} = 0$. This is due to the fact that $\delta_{A - a_{11}I} = \delta_A$, $\delta_{B - b_{11}I} = \delta_B$, and $\delta_{C - c_{11}I} = \delta_C$.

Let $\{E_{ij}\}_{1 \leq i, j \leq n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. Then $\delta_A^D + \delta_B^D = \delta_C^D$ if and only if $\delta_A^D(\delta_B(E_{kl})) + \delta_B^D(\delta_A(E_{kl})) = \delta_C^D(\delta_C(E_{kl}))$ for each $1 \leq k, \ell \leq n$ or equivalently

$$E_{kl}(AB + BA) - 2AE_{kl}B - 2BE_{kl}A + (AB + BA)E_{kl} = E_{kl}C^2 - 2CE_{kl}C + C^2E_{kl}.$$
for each $1 \leq k, \ell \leq n$. This is equivalent to the fact that
\[ E_{ii}(E_{kt}(AB + BA) - 2AE_{kt}B - 2BE_{kt}A + (AB + BA)E_{kt})E_{jj} = E_{ii}(E_{kt}C^2 - 2CE_{kt}C + C^2E_{kt})E_{jj}, \]
for each $1 \leq i, j, k, \ell \leq n$. Now for $i \neq k$ and $j \neq \ell$ we have, as in lemma 2.1
\[ (a_{ik}b_{\ell j} + b_{ik}a_{\ell j})E_{ij} = (c_{ik}c_{\ell j})E_{ij}. \]  
(2.1)

Similarly, for $i \neq k$ and $j = \ell$ we have
\[ (-2a_{ik}b_{\ell\ell} - 2b_{ik}a_{\ell\ell} + \sum_{m=1}^{n}(a_{im}b_{mk} + b_{im}a_{mk}))E_{i\ell} = (-2c_{ik}c_{\ell\ell} + \sum_{m=1}^{n}c_{im}c_{mk})E_{i\ell}. \]  
(2.2)

Moreover, for $i = k$ and $j \neq \ell$ we have
\[ \left( \sum_{m=1}^{n}(a_{\ell m}b_{mj} + b_{\ell m}a_{mj}) - 2a_{kk}b_{\ell j} - 2b_{kk}a_{\ell j} \right)E_{kj} = \left( \sum_{m=1}^{n}c_{\ell m}c_{mj} - 2c_{kk}c_{\ell j} \right)E_{kj}. \]  
(2.3)

Finally for $i = k$ and $j = \ell$ we have
\[ \left( \sum_{m=1}^{n}(a_{\ell m}b_{m\ell} + b_{\ell m}a_{m\ell}) - 2a_{kk}b_{\ell\ell} - 2b_{kk}a_{\ell\ell} + \sum_{m=1}^{n}(a_{km}b_{mk} + b_{km}a_{mk}) \right)E_{k\ell} \]
\[ = \left( \sum_{m=1}^{n}c_{\ell m}c_{m\ell} - 2c_{kk}c_{\ell\ell} + \sum_{m=1}^{n}c_{km}c_{mk} \right)E_{k\ell}. \]  
(2.4)

If $k \neq \ell$ then putting $i = \ell$ and $j = k$ in the equation (2.1) we have $c_{\ell k}^2 = 2a_{\ell k}b_{\ell k}$. Thus for $i \neq k$ and $j \neq k$ we have $(a_{ik}b_{\ell j} + b_{ik}a_{\ell j})^2 = c_{ik}^2c_{\ell j}^2 = 4a_{ik}b_{kk}a_{\ell j}b_{\ell j}$. This implies that
\[ a_{ik}b_{\ell j} = b_{ik}a_{\ell j}, \text{ for } i \neq k, j \neq \ell. \]  
(2.5)

Now, if $b_{\ell j} \neq 0$ for some $1 \leq \ell, j \leq n$ with $\ell \neq j$, then the equation
\[ a_{ik} = \frac{a_{ik}}{b_{\ell j}}b_{ik}, \text{ for } i \neq k, \]
implies the existence of some $\alpha$ and $\beta$ with $|\alpha| + |\beta| \neq 0$ such that
\[ \alpha(A - A^D) = \beta(B - B^D). \]  
(2.6)

If $b_{\ell j} = 0$ for all $1 \leq \ell, j \leq n$ with $\ell \neq j$, then $B = B^D$ and so the equation (2.6) holds for $\alpha = 0$ and any nonzero $\beta \in \mathbb{C}$.

Putting $\ell = k$ in (2.4) we get
\[ \sum_{m=1}^{n}(a_{km}b_{mk} + b_{km}a_{mk}) - 2a_{kk}b_{kk} = \sum_{m=1}^{n}c_{km}c_{mk} - c_{kk}c_{kk}. \]
Thus it follows from (2.4) that
\[ 2a_{\ell\ell}b_{\ell\ell} - 2a_{kk}b_{kk} - 2b_{kk}a_{\ell\ell} + 2a_{kk}b_{kk} = c_{\ell\ell}c_{\ell\ell} - 2c_{kk}c_{kk}, \]
or simply
\[ 2(a_{\ell\ell} - a_{kk})(b_{\ell\ell} - b_{kk}) = (c_{\ell\ell} - c_{kk})^2. \]
For \( \ell = 1 \) we have
\[ c_{kk}^2 = 2a_{kk}b_{kk}, \] for each \( 1 \leq k \leq n, \)
and then
\[ a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell} = c_{kk}c_{\ell\ell}. \tag{2.7} \]
Thus for all \( 1 \leq k, \ell \leq n \) we have \((a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell})^2 = c_{kk}^2c_{\ell\ell}^2 = 4a_{kk}b_{kk}a_{\ell\ell}b_{\ell\ell} \). This implies that
\[ a_{kk}b_{\ell\ell} = b_{kk}a_{\ell\ell}, \] for all \( 1 \leq k, \ell \leq n. \tag{2.8} \]
A similar argument as about the equation (2.5) implies the existence of some \( \alpha' \) and \( \beta' \) with \( |\alpha'| + |\beta'| \neq 0 \) such that
\[ \alpha' A^D = \beta' B^D. \]
Returning to the fact that \( a_{im} b_{mk} = b_{im} a_{mk} = \frac{1}{2}c_{im} c_{mk} \) for \( m \neq i, k \), we have
\[ \sum_{m=1}^{n} (a_{im} b_{mk} + b_{im} a_{mk}) = \sum_{m=1}^{n} c_{im} c_{mk}, \] for \( i \neq k \). \tag{2.9}
Thus letting \( \ell = i \) in (2.2) we have
\[ a_{ik}(b_{ii} - b_{kk}) + b_{ik}(a_{ii} - a_{kk}) = c_{ik}(c_{ii} - c_{kk}), \] for \( i \neq k, \) \tag{2.10}
and then
\[ (a_{ik}(b_{ii} - b_{kk}) + b_{ik}(a_{ii} - a_{kk}))^2 = c_{ik}^2(c_{ii} - c_{kk})^2 = 4a_{ik}b_{ik}(b_{ii} - b_{kk})(a_{ii} - a_{kk}), \] for \( i \neq k. \]
This implies that
\[ a_{ik}(b_{ii} - b_{kk}) = b_{ik}(a_{ii} - a_{kk}), \] for \( i \neq k. \tag{2.11} \]
Using (2.11) and (2.8) we have
\[ b_{jj}a_{ik}(b_{ii} - b_{kk}) = b_{ik}b_{jj}(a_{ii} - a_{kk}) = b_{ik}a_{jj}(b_{ii} - b_{kk}). \]
Now let \( B^D \notin CI \). Then \( b_{ii} \neq b_{kk} \) for some \( i \) and \( k \). This shows that \( b_{jj}a_{ik} = a_{jj}b_{ik} \). Hence we have \( \alpha = \alpha' \) and \( \beta = \beta' \). By a similar argument we can say that if \( A^D \notin CI \) then \( \alpha = \alpha' \) and \( \beta = \beta' \). We therefore have
\[ \text{if } A^D \notin CI \text{ or } B^D \notin CI \text{ then } \alpha A = \beta B \text{ for some } \alpha \text{ and } \beta \text{ with } |\alpha| + |\beta| \neq 0. \]
On the other hand, if \( A^D = sI \) and \( B^D = tI \) for some \( s, t \in \mathbb{C} \) then
\[ \alpha' A^D + \alpha(A - A^D) = s(\alpha' - \alpha)I + \alpha A, \]
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\[ \beta' B^D + \beta (B - B^D) = t(\beta' - \beta) I + \beta B. \]

Therefore, \( s(\alpha' - \alpha) I + \alpha A = t(\beta' - \beta) I + \beta B. \)

Summarizing these we can say that \( \delta_A \delta_B + \delta_B \delta_A = \delta C^2 \) implies \( \alpha A = \beta B + rI \) for some \( \alpha, \beta, r \in \mathbb{C} \) with \( |\alpha| + |\beta| \neq 0. \)

Conversely, suppose that \( \alpha A = \beta B + rI \) for some \( \alpha, \beta, r \in \mathbb{C} \) with \( |\alpha| + |\beta| \neq 0. \) If \( \alpha \neq 0 \) put \( C = \sqrt{2\beta \alpha^{-1} B} \), and if \( \beta \neq 0 \) put \( C = \sqrt{2\alpha \beta^{-1} A} \). Then it is easy to see that \( \delta_A \delta_B + \delta_B \delta_A = \delta C^2. \)

\[ \square \]

**Remark 2.3** The condition \( \alpha A = \beta B + rI \) for some \( \alpha, \beta, r \in \mathbb{C} \) with \( |\alpha| + |\beta| \neq 0 \) in \( M_n(\mathbb{C}) \) is equivalent to the fact that \( \delta_A \) and \( \delta_B \) are linearly dependent.

A natural question is the following: Is it true in general that \( dd' + d'd = D^2 \) on an algebra \( A \) is equivalent to \( d \) and \( d' \) being linearly dependent? In this case we of course have \( D = \sqrt{2 \lambda d} = \sqrt{2 \lambda} d' \) for some \( \lambda, \lambda' \in \mathbb{C} \). The following example shows that the answer is not affirmative in general.

**Example 2.4** Let \( A \) be the subalgebra of \( M_4(\mathbb{C}) \) generated by \( E_{11}, E_{12}, E_{34}, \) and \( E_{44} \). If \( d = \delta_{E_{12}} \) and \( d' = \delta_{E_{34}} \), then for each \( X = x E_{11} + y E_{12} + z E_{34} + w E_{44} \in A \) we have

\[ (dd' + d'd)(X) = \delta_{E_{12}}(-w E_{34}) + \delta_{E_{34}}(x E_{12}) = 0. \]

Thus \( dd' + d'd = \delta_0^2 \). However, \( d \) and \( d' \) are linearly independent.

**Lemma 2.5** Let \( A \) be the subalgebra of \( M_4(\mathbb{C}) \) generated by \( E_{11}, E_{12}, E_{34}, \) and \( E_{44} \). Then each derivation on \( A \) is of the form \( d = \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}} \) for some \( \alpha, \beta, \gamma, \theta \in \mathbb{C} \).

**Proof** Let \( d : A \to A \) be a derivation. For each projection \( P \),

\[ d(P) = d(P^2) = d(P)P + Pd(P). \]

If \( d(E_{11}) = \alpha E_{11} + \beta E_{12} + \gamma E_{34} + \theta E_{44} \) for some \( \alpha, \beta, \gamma, \theta \in \mathbb{C} \), we have from (2.12) that \( \alpha = \gamma = \theta = 0 \), so that \( d(E_{11}) = \beta E_{12} \) for some \( \beta \in \mathbb{C} \). Similarly, we get \( d(E_{44}) = \gamma E_{34} \) for some \( \gamma \in \mathbb{C} \). Therefore,

\[ d(E_{12}) = d(E_{11}E_{12}) = d(E_{11})E_{12} + E_{11}d(E_{12}) = E_{11}d(E_{12}), \]

so that \( d(E_{12}) = \lambda E_{11} + \alpha E_{12} \) for some \( \lambda, \alpha \in \mathbb{C} \). Then

\[ 0 = d(E_{12}E_{11}) = d(E_{12})E_{11} + E_{12}d(E_{11}) = d(E_{12})E_{11} = \lambda E_{11}. \]

Thus \( \lambda = 0 \) and so \( d(E_{12}) = \alpha E_{12} \) for some \( \alpha \in \mathbb{C} \). Similarly, we get \( d(E_{34}) = \theta E_{34} \) for some \( \theta \in \mathbb{C} \).

For each \( X \in A \) we have

\[ d(X) = d(x E_{11} + y E_{12} + z E_{34} + w E_{44}) = xd(E_{11}) + yd(E_{12}) + zd(E_{34}) + wd(E_{44}) \]

\[ = (\beta x + \alpha y)E_{12} + (\theta z + \gamma w)E_{34} + \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}}(X). \]

Therefore, \( d = \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}} \) for some \( \alpha, \beta, \gamma, \theta \in \mathbb{C} \).

\[ \square \]
Remark 2.6 Let $A$ be the subalgebra of $M_4(\mathbb{C})$ generated by $E_{11}$, $E_{12}$, $E_{34}$, and $E_{44}$ and let $d, d'$, and $D$ be derivations on $A$. By Lemma 2.5, we can assume that $d = \delta_{\beta}E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}$, $d' = \delta_{\beta'}E_{12} - \alpha' E_{11} - \gamma' E_{34} + \theta' E_{44}$, and $D = \delta_tE_{12} - \alpha_E_{11} - \gamma E_{34} + \theta E_{44}$ for some $\alpha, \beta, \gamma, \theta, \alpha', \beta', \gamma', \theta', r, s, t, u \in \mathbb{C}$. Then $dd' + d'd = D^2$ if and only if $rs = \alpha\beta' + \alpha'\beta$, $tu = \gamma\theta' + \gamma'\theta$, $r^2 = 2\alpha\alpha'$, and $u^2 = 2\beta\beta'$.

3. Derivations on $C^*$-algebras

In this section, let $\mathcal{H}$ be a Hilbert space with the orthonormal basis $\{\xi_i\}_{i \in I}$. For a bounded operator $T \in B(\mathcal{H})$, let $t_{ij} = \langle T\xi_j, \xi_i \rangle$ for $i, j \in I$. We thus have $T\xi_j = \sum_{i \in I} t_{ij}\xi_i$ and we can write $T = [t_{ij}]_{i, j \in I}$. The latter is called the matrix representation of the operator $T \in B(\mathcal{H})$.

For $i, j \in I$, let $E_{ij} \in B(\mathcal{H})$ be the operator defined by $E_{ij}\xi_j = \xi_i$ and $E_{ij}\xi_k = 0$ for $k \neq j$. Then $E_{ij}E_{jk} = E_{ik}$ and $E_{ij}E_{ik} = 0$ for $j \neq k$. Using the fact that $E_{ij}(x) = \langle x, \xi_j \rangle \xi_i$ for all $x \in \mathcal{H}$, we get

$$T = \sum_{p \in I} \sum_{q \in I} t_{pq}E_{qp} \quad (3.1)$$

for every $T \in B(\mathcal{H})$. Moreover, putting $r_{ij} = \langle R\xi_j, \xi_i \rangle$, $s_{ij} = \langle S\xi_j, \xi_i \rangle$ we have

$$RS = \sum_{p \in I} \sum_{q \in I} \sum_{m \in I} r_{qm}s_{mp}E_{qp} \quad (3.2)$$

for every $R, S \in B(\mathcal{H})$. It follows from (3.1) and (3.2) that

$$E_{ij}TE_{kl} = t_{jk}E_{i\ell}, \quad E_{ij}RSE_{kl} = \sum_{p \in I} r_{jp}s_{pk}E_{i\ell} \quad (T, R, S \in B(\mathcal{H})).$$

Using these facts, we are ready to prove the next theorem.

Theorem 3.1 Let $A$ be a $C^*$-algebra and let $d, d'$ be two derivations on $A$. Then there exists a derivation $D$ on $A$ such that $dd' + d'd = D^2$ if and only if $d$ and $d'$ are linearly dependent.

Proof Let $A$ act faithfully on the Hilbert space $\mathcal{H}$ with the orthonormal basis $\{\xi_i\}_{i \in I}$. By the Kadison–Sakai theorem [4, 10], $d = \delta_R, d' = \delta_S$, and $D = \delta_T$ for some $R, S,$ and $T$ in $B(\mathcal{H})$.

Then $dd' + d'd = D^2$ if and only if $(dd' + d'd)(E_{kl}) = D^2(E_{kl})$ for each $k, \ell \in I$, or equivalently

$$E_{kl}(RS + SR) - 2RE_{kl}S - 2SE_{kl}R + (RS + SR)E_{kl} = E_{kl}T^2 - 2TE_{kl}T + T^2E_{kl},$$

for each $k, \ell \in I$. This is equivalent to the fact that

$$E_{ii}(E_{kl}(RS + SR) - 2RE_{kl}S - 2SE_{kl}R + (RS + SR)E_{kl})E_{jj} = E_{ii}(E_{kl}T^2 - 2TE_{kl}T + T^2E_{kl})E_{jj},$$

for each $i, j, k, \ell \in I$. Now for $i \neq k$ and $j \neq \ell$ we have

$$r_{ik}s_{\ell j} + s_{ik}r_{\ell j} = t_{ik}t_{\ell j}.$$

Similarly, for $i \neq k$ and $j = \ell$ we have

$$-2r_{ik}s_{\ell\ell} - 2s_{ik}r_{\ell\ell} + \sum_{p \in I} (r_{ip}s_{pk} + s_{ip}r_{pk}) = -2t_{ik}t_{\ell\ell} + \sum_{p \in I} t_{ip}t_{pk}.$$
Furthermore, for $i = k$ and $j \neq \ell$ we have

$$
\sum_{p \in I} (r_{lp}s_{pj} + s_{tp}r_{pj}) - 2r_{kk}s_{\ell j} - 2s_{kk}r_{\ell j} = \sum_{p \in I} t_{tp}t_{pj} - 2t_{kk}t_{\ell j}.
$$

Finally for $i = k$ and $j = \ell$ we have

$$
\sum_{p \in I} (r_{lp}s_{lp} + s_{tp}r_{tp}) - 2r_{kk}s_{\ell \ell} - 2s_{kk}r_{\ell \ell} + \sum_{p \in I} (r_{kp}s_{pk} + s_{kp}r_{pk})
\quad = \sum_{p \in I} t_{tp}t_{tp} - 2t_{kk}t_{\ell \ell} + \sum_{p \in I} t_{kp}t_{pk}.
$$

Now a similar verification as in Proposition 2.2 implies the result. 

**Remark 3.2** Note that the subalgebra $A$ of $M_4(\mathbb{C})$ generated by $E_{11}$, $E_{12}$, $E_{34}$, and $E_{44}$ is finite-dimensional (being a subalgebra of $M_4(\mathbb{C})$), and therefore it is complete with respect to the norm inherited from $M_4(\mathbb{C})$. Hence $A$ is a Banach algebra. However, $A$ is not a $*$-subalgebra of $M_4(\mathbb{C})$, since $X = E_{12} \in A$, but $X^* = E_{21} \notin A$. Therefore, $A$ is not a $C^*$-algebra and Example 2.4 does not contradict the statement of Theorem 3.1.

**References**


