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Braided regular crossed modules bifibered over regular groupoids

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Abstract: We show that the forgetful functor from the category of braided regular crossed modules to the category of regular *(or whiskered)* groupoids is a fibration and also a cofibration.

Key words: Crossed modules, groupoids, fibration, cofibration of categories

1. Introduction

Crossed modules of groups were introduced by Whitehead in [14] to study relative homotopy groups as models for homotopy (connected) 2-types. If $f : G \to H$ is a homomorphism of groups, then there is a pullback or reindexing functor $f^* : \mathfrak{CM}/_H \to \mathfrak{CM}/_G$, where $\mathfrak{CM}/_G$ is the category of crossed G-modules. The left adjoint f_* to this functor f^* was constructed in [5]. Thus, Whitehead's crossed modules fibered and cofibered over groups. Analogous constructions in the crossed modules category in commutative algebras and Lie algebras were given in [13] and [8], respectively. For further accounts of fibered and cofibered categories and an introduction to their literature, see [10, 11] and the references therein.

Brown and Sivera [6] showed that the forgetful functor $\Phi_1 : \mathfrak{XMod} \to \mathfrak{Gpd}$ from the category of crossed modules of groupoids to the category of groupoids that sends a crossed module $M \to P$ to its base groupoid P is a fibration and a cofibration of categories. That is, crossed modules of groupoids bifibered over groupoids. If we consider the category of braided regular crossed modules (cf. [4]) as $\mathfrak{BRCM}$ instead of $\mathfrak{XMod}$ and the same functor Φ_1 sending a braided regular crossed module $M \to P$ to its base groupoid P, we need to give some extra properties over the groupoid $P := (P_1, P_0)$. The required properties are:

(i) P_0 is a group;

(ii) there are actions of P_0 on P_1 on the left and right sides satisfying *whiskering axioms* given in [3].

Considering these properties, we can say that this groupoid is a *regular groupoid* as defined in [9]. Therefore, we can extend the result of Brown and Sivera to the forgetful functor $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ from the category of braided regular crossed modules to that of regular groupoids. Thus, the purpose of this paper is to prove that braided regular crossed modules fibered and cofibered or bifibered over regular groupoids.

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2. Preliminaries

We recall some basic definitions from [4]. A *groupoid* $\mathfrak{C}$ is a small category in which every morphism is an isomorphism. We write a groupoid as $\mathfrak{C} := (C_1, C_0)$, where C_0 is the set of objects and C_1 is the set of morphisms. The set of morphisms $p \to q$ for $p, q \in C_0$ is written $C_1(p,q)$, and p, q are the source and target of such a morphism. The source and target maps are written $s, t : C_1 \to C_0$. If $a \in C_1(p,q)$ and $b \in C_1(q,r)$, their composition is written by $ab \in C_1(p,r)$ and shown diagrammatically as

$$\begin{tikzcd} p \arrow[r, "a"] \arrow[rr, bend right, "ab"'] & q \arrow[r, "b"] & r. \end{tikzcd}$$

We use ab for the composition of the morphisms a, b from the left to the right side if $t(a) = s(b)$. Then $s(ab) = s(a)$ and $t(ab) = t(b)$. We write $C_1(p,p)$ as $C_1(p)$. For any morphism a there exists a (necessarily unique) morphism a^{-1} such that $aa^{-1} = e_{s(a)}$ and $(a)^{-1}a = e_{t(a)}$ where $e : C_0 \to C_1$ gives the identity morphism at an object. For any groupoid $\mathfrak{C}$, if $C_1(p,q)$ is empty whenever p, q are distinct (that is, if $s = t$), then $\mathfrak{C}$ is called *totally disconnected*. In any groupoid $\mathfrak{C} := (C_1, C_0)$, for an element $p \in C_0$, the groupoid $C_1(p)$ becomes a group. A groupoid $\mathfrak{C} := (C_1, C_0)$ is often written diagrammatically as

$$\mathfrak{C} := \begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0. \arrow[l, bend right, "e"'] \end{tikzcd}$$

For a survey of applications of groupoids and an introduction to their literature, see [1] and [2].

Brown, in [3], defined the notion of 'whiskering' for a (small) category $\mathfrak{C}$ and gave the notions of left and right multiplications of two morphisms in a whiskered category $\mathfrak{C}$. In fact, the definition of the whiskered category is the same as the definition of the semiregular category given by Gilbert in [9].

Recall from [3] that a whiskering $\mathbf{m}$ on a groupoid $\mathfrak{C} := (C_1, C_0)$ consists of operations

$$m_{ij} : C_i \times C_j \to C_{i+j}, (a,b) \mapsto a \otimes b, (\ i,j = 0,1,\ i+j \leqslant 1)$$

satisfying the following axioms:

Whisk 1.) m_{00} gives a monoid structure on C_0;

Whisk 2.) m_{01}, m_{10} give respectively left and right actions of the monoid C_0 on the groupoid $\mathfrak{C}$ in the sense that:

Whisk 3.) If $x \in C_0$ and $a : u \longrightarrow v$ in C_1, then $x \otimes a : x \otimes u \to x \otimes v$ in $\mathfrak{C}$ so that:

$$(x \otimes y) \otimes a = x \otimes (y \otimes a),\ x \otimes (ab) = (x \otimes a)(x \otimes b),\ x \otimes e_y = e_{xy}.$$

Whisk 4.) Analogous rules hold for the right action;

Whisk 5.)

$$x \otimes (a \otimes y) = (x \otimes a) \otimes y,$$

for all $x, y, u, v \in C_0$, $a, b \in C_1$.

A groupoid $\mathfrak{C}$ together with a whiskering $\mathbf{m}$ is called a whiskered groupoid [3], or a semiregular groupoid in the sense of [9].

Let $\mathfrak{C} := (C_1, C_0)$ be a semiregular groupoid. If m_{00} gives a group structure on C_0, then $\mathfrak{C}$ is called a *regular groupoid.*

Let $\mathfrak{C} := (C_1, C_0)$ be a regular groupoid. If $a : x \to y$ and $b : u \to v$ are two elements in C_1, then the right and left multiplications of a and b are given by (cf. [3])

$$r(a,b) = (a \otimes u)(y \otimes b) : x \otimes u \to y \otimes v$$

and

$$l(a,b) = (x \otimes b)(a \otimes v) : x \otimes u \to y \otimes v$$

as shown in the following diagram:

$$\begin{tikzcd}
x \otimes u \arrow[r, "a\otimes u"] \arrow[d, "x\otimes b"'] \arrow[dr, shift left, "r(a,b)"] \arrow[dr, shift right, "l(a,b)"'] & y \otimes u \arrow[d, "y\otimes b"] \\
x \otimes v \arrow[r, "a\otimes v"'] & y \otimes v.
\end{tikzcd}$$

The following proposition was proven in [3].

Proposition 2.1 *If $l(a,b) = r(a,b)$ for all $a, b \in C_1$, then $(a,b) \mapsto a \otimes b$ given by this common value makes $\mathfrak{C}$ into a strict monoidal groupoid.*

The commutator of two morphisms $a : x \to y$ and $b : u \to v$ in C_1 is defined by

$$[a,b] = l(a,b)^{-1} r(a,b) = (a^{-1} \otimes v)(x \otimes b^{-1})(a \otimes u)(y \otimes b) : y \otimes v \to y \otimes v$$

as shown in the following diagram:

$$\begin{tikzcd}
x \otimes u \arrow[r, "a\otimes u"] \arrow[dr, shift left, "r(a,b)"] & y \otimes u \arrow[d, "y\otimes b"] \\
x \otimes v \arrow[u, "x\otimes b^{-1}"] & y \otimes v. \arrow[l, "a^{-1}\otimes v"] \arrow[ul, shift left, "l(a,b)^{-1}"]
\end{tikzcd}$$

Thus, we have

$$\begin{tikzcd}
y \otimes v \arrow[r, "l(a,b)^{-1}"] \arrow[rr, bend right, "{[a,b]}"'] & x \otimes u \arrow[r, "r(a,b)"] & y \otimes v.
\end{tikzcd}$$

Note that $[a,b]^{-1}$ is not equal to $[b,a]$ because we have

$$[a,b]^{-1} = (y \otimes b^{-1})(a^{-1} \otimes u)(x \otimes b)(a \otimes v) : y \otimes v \to y \otimes v;$$

on the other hand, we have

$$[b,a] = (b^{-1} \otimes y)(u \otimes a^{-1})(b \otimes x)(v \otimes a) : v \otimes y \to v \otimes y,$$

where the left and right whiskering actions are not equal.

If $l(a,b) = r(a,b)$ for all $a, b \in C_1$, we easily have $[a,b] = id$ and then $\mathfrak{C}$ is called a commutative regular groupoid. A morphism of regular groupoids is a morphism of groupoids preserving the actions. Let $\mathfrak{Rgrpd}$ be the category of regular groupoids.

2.1. (Regular) Crossed modules

Let C_1 and C_1' be regular groupoids over the same group C_0 with C_1' totally disconnected. An action of C_1 on C_1' can be given by a partially defined function

$$C_1 \times C_1' \to C_1'$$

written $(a,x) \mapsto x^a$, which satisfies:

1. x^a is defined if and only if $t(x) = s(a)$, and then $t(x^a) = t(a)$;
2. $(xy)^a = (x^a)(y^a)$, $e_p^a = e_q$;
3. $x^{ab} = (x^a)^b$ and $x^{e_p} = x$;
4. $u \otimes (x^a) = (u \otimes x)^{u \otimes a} \in C_1'(u \otimes q)$;
5. $(x^a) \otimes u = (x \otimes u)^{a \otimes u} \in C_1'(q \otimes u)$,

for all $x, y \in C_1'(p)$, $a \in C_1(p,q)$, $b \in C_1(q,r)$, and $u, p, q, r \in C_0$.

A regular crossed module (C_1', C_1, δ) consists of a pair of regular groupoids C_1 and C_1' over the same group C_0 with an action of C_1 on C_1' and a C_0-equivariant morphism $\delta : C_1' \to C_1$ satisfies:

1. $\delta(x^a) = (a)^{-1}\delta(x)a$,
2. $x^{\delta y} = y^{-1}xy$,

for all $x, y \in C_{1'}(p), a \in C_1(p,q)$. A regular crossed module is often written diagrammatically as

$$\begin{tikzcd} C_1' \arrow[r, "\delta"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0. \arrow[l, bend right=60, "e"'] \end{tikzcd}$$

Definition 2.2 ([4]) A braided regular crossed module

$$\begin{tikzcd} C_2 \arrow[r, "\delta"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend right=60, "e"'] \end{tikzcd}$$

is a regular crossed module with the map $\{-,-\} : C_1 \times C_1 \to C_2$ called a braiding map satisfying the following axioms:

B1 : $\{a,b\} \in C_2((ta) \otimes (tb)), \{1_e, b\} = 1_{tb}, \{a, 1_e\} = 1_{ta}$ where $1_e \in C_1(e)$ is the identity morphism and e is the identity element of the group C_0;

B2 : $\{a, bb'\} = \{a,b\}^{ta \cdot b'}\{a,b'\}$;

B3 : $\{aa', b\} = \{a', b\}\{a,b\}^{a' \cdot tb}$;

B4 : $\delta\{a,b\} = (ta \otimes b)^{-1}(a \otimes sb)^{-1}(sa \otimes b)(a \otimes tb)$;

B5 : $\{a, \delta y\} = (t(a) \otimes y)^{-1}(s(a) \otimes y)^{a \otimes q}$ if $y \in C_2(q)$;

B6 : $\{\delta x, b\} = (x \otimes s(b)^{-1})^{p \otimes b}(x \otimes t(b))$ if $x \in C_2(p)$;

B7 : $p \cdot \{a,b\} = \{p \otimes a, b\}, \{a,b\} \otimes p = \{a, b \otimes p\}, \{a \otimes p, b\} = \{a, p \otimes b\}$,

for all $a, a', b, b' \in C_1, x, y \in C_2$, and $p, q \in C_0$.

We will denote the category of braided regular crossed modules by $\mathfrak{BRCM}$.

3. Fibration of categories

We recall the definition of fibration of categories (cf. [6]). For further information about fibered and cofibered categories, see [10] and [11].

Definition 3.1 *Let $\Phi : \mathfrak{X} \to \mathfrak{B}$ be a functor. A morphism $\varphi : Y \to X$ in $\mathfrak{X}$ over $u := \Phi(\varphi)$ is called Cartesian if and only if for all $v : K \to J$ in $\mathfrak{B}$ and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\psi : Z \to Y$ with $\Phi(\psi) = v$ and $\theta = \varphi\psi$.*

This is illustrated by the following diagram:

$$\begin{tikzcd} Z \arrow[r, dotted, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, "\varphi"'] & X & \\ & & & {} \arrow[d, "\Phi"] \\ K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I & {} \end{tikzcd}$$

A morphism $\alpha : Z \to Y$ is called vertical if and only if $\Phi(\alpha)$ is an identity morphism in $\mathfrak{B}$. In particular, for $I \in \mathfrak{B}$ we write $\mathfrak{X}/_I$ consisting of those morphisms α with $\Phi(\alpha) = id_I$.

Definition 3.2 *The functor $\Phi : \mathfrak{X} \to \mathfrak{B}$ is a fibration or fibered over $\mathfrak{B}$ if and only if for all $u : J \to I$ in $\mathfrak{B}$ and $X \in \mathfrak{X}/_I$ there is Cartesian morphism over u: such a Φ is called a Cartesian lifting of X along u.*

3.1. Regular groupoids fibered over groups

Consider a regular groupoid $\mathfrak{C} := (C_1, C_0)$ together with the whiskering operations m_{00}, m_{01}, and m_{10}. Since the set of objects C_0 is a group with the operation m_{00}, we can define a forgetful functor from the category of regular groupoids to the category of groups:

$$\Phi : \mathfrak{Rgrpd} \to \mathfrak{Gpd},$$

which sends the regular $\mathfrak{C} := (C_1, C_0)$ to its objects group C_0. In this section we show that this functor is a fibration. This is a well-known construction for groupoids; see [6] and [12].

In $\mathfrak{Rgrpd}/_{C_0}$, for a regular groupoid $\mathfrak{C} := (C_1, C_0)$ and for a homomorphism $u : C_0' \to C_0$ in $\mathfrak{Gpd}$, we can define the Cartesian lifting

$$\varphi : \mathfrak{C}' = (C_1', C_0') \to (C_1, C_0) = \mathfrak{C}$$

in $\mathfrak{Rgrpd}$ as follows:

We will define the whiskering operation on it. For $x, y \in C_0'$, we take

$$C_1'(x, y) = \{(x, c, y) : s(c) = u(x), t(c) = u(y)\}$$

with composition

$$(x_1, c_1, y_1)(y_1, c_2, y_2) = (x_1, c_1c_2, y_2)$$

and the morphism $\varphi : C_1' \to C_1$ is given by $\varphi(x,c,y) = c$. From [6], we obtain a groupoid over C_0'. The operations $m'_{ij} : C_i' \times C_j' \to C'_{i+j}$ for $i,j = 0,1,\ i+j \leqslant 1$ are given by

$$m'_{01}(t,(x,c,y)) = (t \otimes x, u(t) \otimes c, t \otimes y)$$

and

$$m'_{10}((x,c,y),t) = (x \otimes t, c \otimes u(t), y \otimes t)$$

for $x,y,t \in C_0'$, and $c \in C_1$. We obtain the following result.

Corollary 3.3 $\mathfrak{C}' = (C_1', C_0')$ *is a regular groupoid.*

The universal property is easily verified. The regular groupoid $\mathfrak{C}' = (C_1', C_0')$ is usually called the *pullback* of the regular groupoid $\mathfrak{C} = (C_1, C_0)$ by u.

For every morphism $u : C_0' \to C_0$ in $\mathfrak{Gpd}$ and a regular groupoid $\mathfrak{C} = (C_1, C_0) \in \mathfrak{Rgrpd}$, we may select a Cartesian lifting of $\mathfrak{C}$ along u,

$$u^{\mathfrak{C}} : u^*(\mathfrak{C}) \to \mathfrak{C}.$$

We call the splitting of Φ such a choice of Cartesian lifting. If we fix the homomorphism $u : C_0' \to C_0$ in the category of groups, the splitting gives a reindexing functor

$$u^* : \mathfrak{Rgrpd}/_{C_0} \to \mathfrak{Rgrpd}/_{C_0'}$$

defined on objects by $\mathfrak{C} \mapsto u^*(\mathfrak{C})$ and the image of a morphism $\alpha : \mathfrak{C} \to \mathfrak{H}$ in $\mathfrak{Rgrpd}/_{C_0}$ is $u^*(\alpha)$, the unique vertical arrow commuting the following diagram:

$$\begin{tikzcd} u^*(\mathfrak{C}) \arrow[r, "u^{\mathfrak{C}}"] \arrow[d, "u^*(\alpha)"'] & \mathfrak{C} \arrow[d, "u"] \\ u^*(\mathfrak{H}) \arrow[r, "u^{\mathfrak{H}}"'] & \mathfrak{H}. \end{tikzcd}$$

Using Proposition 2.5 of [6], we can give the following result.

Corollary 3.4 *Let* $u : C_0' \to C_0$ *be a homomorphism of groups and let*

$$u^* : \mathfrak{Rgrpd}/_{C_0} \to \mathfrak{Rgrpd}/_{C_0'}$$

be the reindexing functor. Then there is a bijection

$$\mathfrak{Rgrpd}/_{C_0'}((H_1, C_0'), u^*(C_1, C_0)) \cong \mathfrak{Rgrpd}/_u((H_1, C_0'), (C_1, C_0))$$

natural in $\mathfrak{H} = (H_1, C_0') \in \mathfrak{Rgrpd}/_{C_0'}$, $\mathfrak{C} = (C_1, C_0) \in \mathfrak{Rgrpd}/_{C_0}$ *where* $\mathfrak{Rgrpd}/_u$ *consists of the morphism*

$$\begin{tikzcd} H_1 \arrow[r, "\alpha"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ C_0' \arrow[r, "u"'] & C_0 \end{tikzcd}$$

with $\Phi(\alpha, u) = u$.

4. Braided regular crossed modules fibered over regular groupoids

We have a forgetful functor $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ from the category of braided regular crossed modules to the category of regular groupoids, which sends a braided regular crossed module

$$\begin{tikzcd} C_2 \arrow[r, "\delta"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend right=60, "e"'] \end{tikzcd}$$

to the regular groupoid (C_1, C_0).

Note that Brown and Sivera in Proposition 7.1 of [6] proved that the forgetful functor $\Phi_1 : \mathfrak{Xmod} \to \mathfrak{Gpd}$ from the category of crossed modules of groupoids to the category of groupoids is a fibration and has a left adjoint. By extending this result to regular groupoids, we obtain the following result.

Proposition 4.1 *The forgetful functor $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ is a fibration.*

Proof We give the pullback construction to prove that Φ_1 is a fibration. We suppose that (f, u) is a morphism of regular groupoids as illustrated in the following diagram:

$$\begin{tikzcd} H_1 \arrow[r, "f"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ H_0 \arrow[r, "u"'] & C_0 \end{tikzcd}$$

where $u : H_0 \to C_0$ is a homomorphism of groups. Let

$$\begin{tikzcd} C_2 \arrow[r, "\delta"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \end{tikzcd}$$

be a braided regular crossed module. We define

$$\begin{tikzcd} f^*(C_2) \arrow[r, "\delta^*"] & H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \end{tikzcd}$$

as follows. For $x \in H_0$,

$$f^*(C_2)(x) = \{(h_1, c_2) : f(h_1) = \delta(c_2), h_1 \in H_1(x), c_2 \in C_2(u(x))\}.$$

The action of $h_1 \in H_1(x, y)$ on $(h, c_2) \in f^*(C_2)(x)$ can be given by

$$(h, c_2)^{h_1} = ((h_1)^{-1} h h_1, c_2^{f(h_1)}) \in f^*(C_2)(y).$$

The morphism δ^* from $f^*(C_2)$ to H_1 is defined by $\delta^*(h, c_2) = h$. We get the following diagram:

$$\begin{tikzcd} f^*(C_2) \arrow[r, "\theta"] \arrow[d, "\delta^*"'] & C_2 \arrow[d, "\delta"] \\ H_1 \arrow[r, "f"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & C_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ H_0 \arrow[r, "u"'] & C_0 \end{tikzcd}$$

in which θ is given by $(h, c_2) \mapsto c_2$. When δ is a crossed module over groupoids, Brown and Sivera in [6] proved that δ^* is a crossed module and the pullback of δ by (f, u). That is, the morphism θ is Cartesian in the category of crossed modules over groupoids. If δ is a braided regular crossed module, we shall show that δ^* is a braided regular crossed module and the morphism (θ, f, u) is Cartesian morphism in $\mathfrak{BRCM}$ over $\Phi_1(\theta, f, u) = (f, u) \in \mathfrak{Rgrpd}$.

First we define the left and right actions of H_0 on $f^*(C_2)$. The left action of $p \in H_0$ on $(h_1, c_2) \in f^*(C_2)(x)$ is defined by

$$p \otimes (h_1, c_2) = (p \otimes h_1, u(p) \otimes c_2).$$

Since $s(p \otimes h_1) = p \otimes x$ and $t(u(p) \otimes c_2) = u(p) \otimes t(c_2) = u(p) \otimes x$, we have $(p \otimes h_1, u(p) \otimes c_2) \in f^*(C_2)(p \otimes x)$. Similarly, the right action is defined by

$$(h_1, c_2) \otimes p = (h_1 \otimes p, c_2 \otimes u(p)) \in f^*(C_2)(x \otimes p).$$

Thus, we can say that $f^*(C_2)$ is a regular totally disconnected groupoid over H_0. Since

$$\delta^*(p \otimes (h, c_2)) = p \otimes h = p \otimes \delta^*(h, c_2),$$

the crossed module δ^* is H_0-equivariant relative to the left and right actions. That is, δ^* becomes a regular crossed module.

For this regular crossed module, the braiding map

$$\{-,-\}' : H_1 \times H_1 \longrightarrow f^*(C_2)$$

is given for any morphisms $h_1 \in H_1(x, y)$ and $h_2 \in H_1(a, b)$ by

$$\{h_1, h_2\}' = \big([h_1, h_2]^{-1}, \{f(h_1), f(h_2)\}\big)$$

where the first coordinate is equal to

$$\begin{aligned}[h_1, h_2]^{-1} &= \big((h_1^{-1} \otimes th_2)(sh_1 \otimes h_2^{-1})(h_1 \otimes sh_2)(th_1 \otimes h_2)\big)^{-1} \\ &= (th_1 \otimes h_2^{-1})(h_1^{-1} \otimes sh_2)(sh_1 \otimes h_2)(h_1 \otimes th_2)\end{aligned}$$

as given in Section 2.

Now we prove the braiding axioms as follows.

B1 : We must show that $\{h_1, h_2\}' \in f^*(C_2)(t(h_1) \otimes t(h_2))$. For $h_1 : x \to y$ and $h_2 : a \to b$ in H_1, we obtain $[h_1, h_2]^{-1} : y \otimes b \to y \otimes b \in H_1(y \otimes b)$. On the other hand, since (f, u) is a morphism of regular groupoids, we have $f(h_1) : u(x) \to u(y)$ and $f(h_2) : u(a) \to u(b)$. From axiom **B1** of δ, we obtain $\{f(h_1), f(h_2)\} \in C_2(t(fh_1) \otimes t(fh_2)) = C_2(u(y) \otimes u(b)) = C_2(u(y \otimes b))$. Moreover, from axiom **B4** of δ, since

$$\delta(\{f(h_1), f(h_2)\}) = [f(h_1), f(h_2)]^{-1} = f([h_1, h_2]^{-1}),$$

we get $\{h_1, h_2\}' \in f^*(C_2)(y \otimes b) = f^*(C_2)(t(h_1) \otimes t(h_2))$.

B2 : For morphisms $h_1 : x \to y, h_2 : a \to b$ and $h_3 : b \to c$ in H_1, we obtain

$$\begin{aligned}
\{h_1, h_2h_3\}' &= \big([h_1, h_2h_3]^{-1}, \{f(h_1), f(h_2)f(h_3)\}\big) \\
&= ((t(h_1) \otimes (h_2h_3)^{-1})(h_1^{-1} \otimes s(h_2))(sh_1 \otimes (h_2h_3)) \\
&\qquad (h_1 \otimes th_3), \{fh_1, fh_2\}^{t(fh_1)\otimes(fh_3)}\{fh_1, fh_3\}) \\
&= (th_1 \otimes h_3^{-1})(th_1 \otimes h_2^{-1})(h_1^{-1} \otimes sh_2)(sh_1 \otimes h_2)(sh_1 \otimes h_3)(h_1 \otimes th_3), \\
&\qquad \{fh_1, fh_2\}^{ut(h_1)\otimes(fh_3)}\{fh_1, fh_3\}) \\
&= (th_1 \otimes h_3^{-1})(th_1 \otimes h_2^{-1})(h_1^{-1} \otimes sh_2)(sh_1 \otimes h_2) \\
&\qquad (h_1 \otimes sh_3)(th_1 \otimes h_3)(th_1 \otimes h_3^{-1})(h_1^{-1} \otimes sh_3)(sh_1 \otimes h_3)(h_1 \otimes th_3), \\
&\qquad \{fh_1, fh_2\}^{f(t(h_1)\otimes h_3)}\{fh_1, fh_3\}) \\
&= (th_1 \otimes h_3^{-1})(th_1 \otimes h_2^{-1})(h_1^{-1} \otimes sh_2)(sh_1 \otimes h_2)(h_1 \otimes sh_3) \\
&\qquad (th_1 \otimes h_3), \{fh_1, fh_2\}^{f(t(h_1)\otimes h_3)}) \\
&\qquad (th_1 \otimes h_3^{-1})(h_1^{-1} \otimes sh_3)(sh_1 \otimes h_3)(h_1 \otimes th_3), \{fh_1, fh_3\}) \\
&= (th_1 \otimes h_3^{-1})[h_1, h_2]^{-1}(th_1 \otimes h_3), \{fh_1, fh_2\}^{f(t(h_1)\otimes h_3)}) \\
&\qquad ([h_1, h_3]^{-1}, \{fh_1, fh_3\}) \text{ (since } s(h_3) = t(h_2)) \\
&= ([h_1, h_2]^{-1}, \{fh_1, fh_2\})^{t(h_1)\otimes h_3}([h_1, h_3]^{-1}, \{fh_1, fh_3\}) \\
&= (\{h_1, h_2\}')^{t(h_1)\otimes h_3}\{h_1, h_3\}'.
\end{aligned}$$

B3 : This axiom can be proved similarly to **B2**.

B4 : For morphisms $h_1 : x \to y, h_2 : a \to b$ in H_1, we obtain

$$\begin{aligned}
\delta^*\{h_1, h_2\}' &= \delta^*([h_1, h_2]^{-1}, \{f(h_1), f(h_2)\}) \\
&= [h_1, h_2]^{-1} \\
&= (th_1 \otimes h_2^{-1})(h_1^{-1} \otimes sh_2)(sh_1 \otimes h_2)(h_1 \otimes th_2).
\end{aligned}$$

B5 : For $h \in H_1(x)$, $c_2 \in C_2(u(x))$ and $h_1 : a \to b \in H_1(a, b)$, let $\mathbf{x} = (h, c_2) \in f^*(C_2)(x)$. Then we obtain

$$
\begin{aligned}
\{h_1, \delta^*(\mathbf{x})\}' &= \{h_1, \delta^*(h, c_2)\}' \\
&= \{h_1, h\}' \\
&= ([h_1, h]^{-1}, \{f(h_1), f(h)\}) \\
&= ([h_1, h]^{-1}, \{f(h_1), \delta(c_2)\}) \text{ (since } f(h) = \delta(c_2)) \\
&= (th_1 \otimes h^{-1})(h_1^{-1} \otimes sh)(sh_1 \otimes h)(h_1 \otimes th), \{f(h_1), \delta(c_2)\}) \\
&\overset{\mathbf{B5} \text{ of } \delta}{=} \Big(th_1 \otimes h^{-1})(h_1^{-1} \otimes x)(sh_1 \otimes h)(h_1 \otimes x), (t(f(h_1)) \otimes c_2^{-1})(sf(h_1) \otimes c_2)^{f(h_1) \otimes u(x)}\Big) \\
&= \Big((th_1 \otimes h^{-1})(h_1^{-1} \otimes x)(sh_1 \otimes h)(h_1 \otimes x), (uth_1 \otimes c_2^{-1})(ush_1 \otimes c_2)^{f(h_1 \otimes x)}\Big) \\
&= ((th_1 \otimes h^{-1}), u(t(h_1)) \otimes c_2^{-1})(h_1^{-1} \otimes x)(sh_1 \otimes h)(h_1 \otimes x), (u(sh_1) \otimes c_2)^{f(h_1 \otimes x)}) \\
&= (th_1 \otimes (h, c_2)^{-1})(sh_1 \otimes (h, c_2))^{h_1 \otimes x} \\
&= (th_1 \otimes \mathbf{x}^{-1})(sh_1 \otimes \mathbf{x})^{h_1 \otimes x}
\end{aligned}
$$

where $\mathbf{x} = (h, c_2) \in f^*(C_2)(x)$ for $x \in H_0$.

B6 : For $\mathbf{x} = (h, c_2) \in f^*(C_2)(x)$ with $c_2 \in C_2(u(x))$ and $h_2 : a \to b \in H_1(a, b)$ we obtain

$$
\begin{aligned}
\{\delta^*(\mathbf{x}), h_2\}' &= \{\delta^*(h, c_2), h_2\}' \\
&= \{h, h_2\}' \\
&= ([h, h_2]^{-1}, \{f(h), f(h_2)\}) \\
&= ([h, h_2]^{-1}, \{\delta(c_2), f(h_2)\}) \text{ (since } f(h) = \delta(c_2)) \\
&\overset{\mathbf{B6} \text{ of } \delta}{=} \Big((x \otimes h_2^{-1})(h^{-1} \otimes sh_2)(x \otimes h_2)(h \otimes th_2), (c_2^{-1} \otimes s(fh_2))^{u(x) \otimes f(h_2)}(c_2 \otimes t(fh_2))\Big) \\
&= ((x \otimes h_2^{-1})(h^{-1} \otimes sh_2)(x \otimes h_2), (c_2^{-1} \otimes u(s(h_2)))^{f(x \otimes h_2)})(h \otimes t(h_2), c_2 \otimes u(t(h_2))) \\
&= (h \otimes sh_2), c_2^{-1} \otimes u(s(h_2)))^{x \otimes h_2}((h, c_2) \otimes th_2) \\
&= ((h, c_2)^{-1} \otimes sh_2)^{x \otimes h_2}((h, c_2) \otimes th_2) \\
&= (\mathbf{x}^{-1} \otimes sh_2)^{x \otimes h_2}(\mathbf{x} \otimes th_2).
\end{aligned}
$$

B7 : For any object $p \in H_0$ and morphisms $h_1 : x \to y, h_2 : a \to b$ in H_1, we obtain

$$
\begin{aligned}
p \otimes \{h_1, h_2\}' &= p \otimes ([h_1, h_2]^{-1}, \{f(h_1), f(h_2)\}) \\
&= (t(p \otimes h_1^{-1}) \otimes h_2)((p \otimes h_1^{-1}) \otimes sh_2)(s(p \otimes h_1) \otimes h_2)((p \otimes h_1) \otimes t(h_2), \\
&\qquad \{u(p) \otimes f(h_1), f(h_2)\}) \\
&= ([p \otimes h_1, h_2]^{-1}, \{f(p \otimes h_1), f(h_2)\}) \\
&= \{p \otimes h_1, h_2\}'.
\end{aligned}
$$

Similarly, it is easily verified that $\{h_1, h_2\}' \otimes p = \{h_1, h_2 \otimes p\}'$. Therefore, we get a braided regular crossed module

$$f^*(C_2) \xrightarrow{\delta^*} H_1 \underset{t}{\overset{s}{\rightrightarrows}} H_0.$$

Now we will show that the morphism (θ, f, u) is a Cartesian morphism in $\mathfrak{BRCM}$ over the morphism $\Phi(\theta, f, u) = (f, u)$. Suppose that there is a morphism (f_2, v)

$$\begin{tikzcd}
H_1' \arrow[r, "f_2"] \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] & H_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
H_0' \arrow[r, "v"'] & H_0
\end{tikzcd}$$

in $\mathfrak{Rgrpd}$. Let

$$Z \xrightarrow{\delta'} H_1' \underset{t'}{\overset{s'}{\rightrightarrows}} H_0'$$

be a braided regular crossed module and let (θ', f_2', v') be a morphism in $\mathfrak{BRCM}$ as illustrated in the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\theta'"] \arrow[d, "\delta'"'] & C_2 \arrow[d, "\delta"] \\
H_1' \arrow[r, "f_2'"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & C_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
H_0' \arrow[r, "v'"'] & C_0
\end{tikzcd}$$

with the properties $f_2' = f f_2$, $v' = uv$ and $\delta\theta' = f_2'\delta'$. Then there exists a unique morphism (θ^*, f_2, v):

$$\begin{tikzcd}
Z \arrow[r, "\theta^*"] \arrow[d, "\delta'"'] & f^*(C_2) \arrow[d, "\delta"] \\
H_1' \arrow[r, "f_2"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & H_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
H_0' \arrow[r, "v"'] & H_0
\end{tikzcd}$$

such that $\Phi(\theta^*, f_2, v) = (f_2, v)$. We get the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "\theta'"] \arrow[rd, dotted, "\theta^*"] \arrow[d, "\delta'"'] & & \\
H_1' \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \arrow[rd, "f_2"'] \arrow[rrd, bend left=10, "f_2'"] & f^*(C_2) \arrow[r, "\theta"] \arrow[d, "\delta^*"'] & C_2 \arrow[d, "\delta"] \\
H_0' \arrow[rd, "v"'] \arrow[rrd, bend left=10, "v'"] & H_1 \arrow[r, "f"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & C_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
& H_0 \arrow[r, "u"'] & C_0.
\end{tikzcd}$$

For any $y \in H_0'$ and $z \in Z(y)$, then $\delta'(z) \in H_1'(y)$ and $f_2\delta'(z) \in H_1(v(y))$. Further, $\theta'(z) \in C_2(v'(y)) = C_2(uv(y))$ since $v' = uv$.

Thus, the necessary unique morphism $\theta^* : Z \to f^*(C_2)$ can be defined by

$$\theta^*(z) = (f_2\delta'(z), \theta'(z))$$

for $z \in Z(y), y \in H_0'$. Since $\theta'(z) \in C_2(u(v(y)))$ and $f_2\delta'(z) \in H_1(v(y))$ and since

$$f(f_2\delta'(z)) = f_2'\delta'(z) = \delta(\theta'(z)),$$

we have $(f_2\delta'(z), \theta'(z)) \in f^*(C_2)(v(y))$.

Therefore, for a morphism $(f, u) : (H_1, H_0) \to (C_1, C_0)$ in $\mathfrak{Rgrpd}$ and an object

$$C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

in $\mathfrak{BRCM}/_{(C_1,C_0)}$, the category of braided regular crossed modules over the same regular groupoid (C_1, C_0), there is Cartesian morphism (θ, f, u) over (f, u). Consequently, the forgetful functor $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ is fibration. □

For a fixed morphism $(f, u) : (H_1, H_0) \to (C_1, C_0)$ in $\mathfrak{Rgrpd}$, the splitting of Φ_1 gives a reindexing functor

$$(f, u)^* : \mathfrak{BRCM}/_{(C_1,C_0)} \longrightarrow \mathfrak{BRCM}/_{(H_1,H_0)}$$

defined on objects by

$$\left(\mathfrak{C} := \ C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0\right) \longmapsto \left((f,u)^*(\mathfrak{C}) := \ f^*(C_2) \xrightarrow{\delta^*} H_1 \underset{t}{\overset{s}{\rightrightarrows}} H_0\right).$$

Suppose that a reindexing functor

$$(f, u)^* : \mathfrak{BRCM}/_{(C_1,C_0)} \longrightarrow \mathfrak{BRCM}/_{(H_1,H_0)}$$

is chosen. Then there is a bijection

$$\mathfrak{BRCM}/_{(H_1,H_0)}(\mathfrak{H}, (f,u)^*(\mathfrak{C})) \cong \mathfrak{BRCM}/_{(f,u)}(\mathfrak{H}, \mathfrak{C})$$

natural in

$$\mathfrak{H} := \left(H_2 \xrightarrow{\delta} H_1 \overset{s}{\underset{t}{\rightrightarrows}} H_0 \right) \in \mathfrak{BRCM}/_{(H_1,H_0)}$$

and

$$\mathfrak{C} := \left(C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0 \right) \in \mathfrak{BRCM}/_{(C_1,C_0)}$$

and where $\mathfrak{BRCM}/_{(f,u)}(\mathfrak{H},\mathfrak{C})$ consists of those morphism (α, f, u) from $\mathfrak{H}$ to $\mathfrak{C}$ with $\Phi(\alpha, f, u) = (f, u)$.

5. Cofibration of categories

We recall the following basic definitions from [6].

Definition 5.1 *Let $\Phi : \mathfrak{X} \to \mathfrak{B}$ be a functor. A morphism $\psi : Z \to Y$ in $\mathfrak{X}$ over $v := \Phi(\psi)$ is called co-Cartesian if and only if for all $u : J \to I$ in $\mathfrak{B}$ and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\varphi : Y \to X$ with $\Phi(\varphi) = u$ and $\theta = \varphi\psi$. This is illustrated by the following diagram:*

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\varphi"'] & X \arrow[d, phantom, ""] \\
K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I
\end{tikzcd}
\qquad
\Big\downarrow \Phi$$

Definition 5.2 *The functor $\Phi : \mathfrak{X} \to \mathfrak{B}$ is a cofibration or category cofibered over $\mathfrak{B}$ if and only if for all $v : K \to J$ in $\mathfrak{B}$ and $Z \in \mathfrak{X}/_K$ there is co-Cartesian morphism $\psi : Z \to Z'$ over v: such a ψ is called a co-Cartesian lifting of Z along v.*

If $\Phi : \mathfrak{X} \to \mathfrak{B}$ is a cofibration, for every morphism $v : K \to J$ in $\mathfrak{B}$ and an object $Z \in \mathfrak{X}/_K$, a co-Cartesian lifting of Z

$$v_Z : Z \to v_*(Z)$$

along v can be selected. Under these conditions, the functor v_* is said to give the objects induced by v. The first construction of an induced crossed module was given in [5]. By following [5], examples of induced crossed modules over groups were developed in [7]. The construction of induced crossed modules over groupoids was given by Brown and Sivera in [6].

Proposition 5.3 ([6]) Let $\Phi : \mathfrak{X} \to \mathfrak{B}$ be a fibration of categories. Then $\psi : Z \to Y$ in $\mathfrak{X}$ over $v : K \to J$ in $\mathfrak{B}$ is co-Cartesian if and only if for all $\theta' : Z \to X'$ over v there is a unique morphism $\psi' : Y \to X'$ in $\mathfrak{X}/_J$ with $\theta' = \psi'\psi$.

Remark: Brown and Sivera showed that the functor from the category of groupoids to the category of sets is a fibration. The proof that this functor is also a cofibration is given in essence in [2]. In Section 3.1, we have already shown that the forgetful functor $\Phi : \mathfrak{Rgrpd} \to \mathfrak{Grpd}$ is a fibration. By using a similar construction method, it can be proved that this functor is also a cofibration.

5.1. Modules over regular groupoids

A module over a regular groupoid $\mathfrak{C} := (C_1, C_0)$ is a pair $(M, \mathfrak{C})$ where M is a totally disconnected regular groupoid over C_0 with an action of C_1 on M. This action is given by a family of maps

$$M(x) \times C_1(x, y) \to M(y)$$

for all $x, y \in C_0$. The action of $p \in C_1(x, y)$ on $m \in M(x)$ is denoted by m^p and satisfies the usual properties of a groupoid action, and each action of C_0 on C_1 and M is compatible with the action of C_1 on M. A morphism of modules over regular groupoids is a pair

$$(\theta, f) : (M, \mathfrak{C}) \to (N, \mathfrak{H})$$

where $f : \mathfrak{C} \to \mathfrak{H}$ and $\theta : M \to N$ are morphisms of regular groupoids and they preserve the action. This defines the category $\mathfrak{ModReg}$ having modules over regular groupoids as objects.

We have a forgetful functor $\Phi : \mathfrak{ModReg} \to \mathfrak{Rgrpd}$ from the category of modules over regular groupoids to the category of regular groupoids, which sends a module over a regular groupoid $(M, \mathfrak{C})$ to the regular groupoid $\mathfrak{C}$.

Proposition 5.4 *The forgetful functor $\Phi : \mathfrak{ModReg} \to \mathfrak{Rgrpd}$ is fibered and cofibered.*

Proof Brown and Sivera in Proposition 6.2 of [6] proved that the functor from the category of modules over groupoids to the category of groupoids is a fibration and a cofibration. Using the same construction method with some additional conditions, we will prove that the functor Φ_M is a fibration and a cofibration.

Let $(N, \mathfrak{H})$ be a module over the regular groupoid $\mathfrak{H} := (H_1, H_0)$ and let

$$v = (v_1, v_0) : \mathfrak{C} \to \mathfrak{H}$$

be a morphism from a regular groupoid $\mathfrak{C} := (C_1, C_0)$ to the regular groupoid $\mathfrak{H}$. The module $(M, \mathfrak{C}) = v^*(N, \mathfrak{H})$ was defined in [6] as follows. For $x \in C_0$, set $M(x) = \{x\} \times N(v_0(x))$ with addition given by that in $N(v_0(x))$. The operation is given by

$$(x, n)^p = (y, n^{v_1(p)})$$

for $p \in C_1(x, y)$. We can define the actions of C_0 as follows. For $u \in C_0$ and $(x, n) \in M(x)$, the actions are

$$(x, n) \otimes u = (x \otimes u, n \otimes v_0(u)) \in M(x \otimes u)$$

$$u \otimes (x, n) = (u \otimes x, v_0(u) \otimes n).$$

According to these actions, we obtain for $p \in C_1(x, y)$ and $u \in C_0$

$$u \otimes ((x, n)^p) = (u \otimes (x, n))^{u \otimes p}$$

and

$$((x, n)^p) \otimes u = ((x, n) \otimes u)^{p \otimes u}.$$

Thus, M is a module over the regular groupoid (C_1, C_0). Now we show that the functor Φ is a cofibration.

Let M be a module over the regular groupoid $\mathfrak{C} := (C_1, C_0)$ and $v = (v_1, v_0)$ be a morphism from the regular groupoid to a regular groupoid $\mathfrak{H} = (H_1, H_0)$. We use the construction of $N = v_*(M, \mathfrak{C})$ from [6]. For any $y \in H_0$, the abelian group $N(y)$ is generated by pairs (m,q) with $m \in M$, $q \in H_1(v_0(t(m)),y)$, so that $N(y)=0$ if no such pairs exist. The action of $q' \in H_1(y,u)$ on $(m,q) \in N(y)$ is given by

$$(m,q)^{q'} = (m,qq')$$

and the composition is defined by $(m,q)(m',q)=(mm',q)$. The relation given for $N(y)$ in [6] is

$$(m^p,q) = (m,v_1(p)q)$$

for $p \in C_1$. To get a module over regular groupoids, we must define the left and right actions of H_0 and we need to add new relations to $N(y)$. The actions are defined by

$$u \otimes (m,q) = (m,u\otimes q) \text{ and } (m,q)\otimes u = (m,q\otimes u)$$

for $u \in H_0$. The new relations imposed to $N(y)$ are

$$(x\otimes m,q) = (m,v_0(x)\otimes q) \text{ and } (m\otimes x,q) = (m,q\otimes v_0(x))$$

for $x \in C_0$. Then we obtain, for $(m,q)\in N$ and $u \in H_0$,

$$\begin{aligned} (m,q)^{q'}\otimes u &= (m,qq')\otimes u \\ &= (m,qq'\otimes u) \\ &= (m,(q\otimes u)(q'\otimes u)) \\ &= (m,q\otimes u)^{q'\otimes u} \\ &= ((m,q)\otimes u)^{q'\otimes u} \end{aligned} \qquad \begin{aligned} u\otimes (m,q)^{q'} &= u\otimes (m,qq') \\ &= (m,u\otimes (qq')) \\ &= (m,(u\otimes q)(u\otimes q')) \\ &= (m,u\otimes q)^{u\otimes q'} \\ &= (u\otimes (m,q))^{u\otimes q'}. \end{aligned}$$

Thus, we have a module over the regular groupoid $\mathfrak{H}$. The co-Cartesian morphism over $v=(v_1,v_0)$ is given by $\psi : M \to N$, $\psi(m) = (m, e_{v_0 t(m)})$. Using the new relations added to N, for $m\in M$ and $x \in C_0$, we have

$$\psi(m\cdot x) = (m\otimes x, e_{v_0(t(m\cdot x))}) = (m, e_{v_0(tm)}\otimes v_0(x)) = (m,e_{v_0(tm)})\otimes v_0(x) = \psi(m)\otimes v_0(x)$$

and similarly $\psi(x\otimes m) = v_0(x)\otimes\psi(m)$. That is, the morphism ψ preserves the left and right actions of C_0.
$\square$

6. Braided regular crossed modules cofibered over regular groupoids

In Section 4, we showed that the functor Φ_1 from the category of braided regular crossed modules to that of regular groupoids is a fibration. In this section we will show that this functor is also a cofibration. In [6], Brown and Sivera proved that the functor $\Phi_1 : \mathfrak{XMod} \to \mathfrak{Gpd}$ from the category of crossed modules of groupoids to the category of groupoids is a cofibration. In the following proposition we use this result. We must add some extra relations to obtain an induced braided regular crossed module.

Proposition 6.1 *The forgetful functor* $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ *is a cofibration.*

Proof Using the construction of an induced crossed module of groupoids (cf. [6]) and Proposition 5.3, and by adding new relations, we will give a construction of an induced braided regular crossed module.

Let

$$\mathfrak{M} := \left(M \xrightarrow{\delta} P_1 \overset{s}{\underset{t}{\rightrightarrows}} P_0 \right)$$

be a braided regular crossed module with braiding map $\{-,-\} : P_1 \times P_1 \to M$. Let $f = (f_1, f_0)$ be a morphism from the regular groupoid $\mathfrak{P} := (P_1, P_0)$ to a regular groupoid $\mathfrak{Q} := (Q_1, Q_0)$. First we recall the crossed module construction from [6]. Let $y \in Q_0$. If there is no $q \in Q_1$ from a point of $f_0(P_0)$ to y, then $N(y)$ is a trivial group. Otherwise, define $F(y)$ to be the free group on the set of pairs (m, q) such that $m \in M(x)$ for some $x \in P_0$ and $q \in Q_1(f_0(x), y)$. If $q' \in Q_1(y, y')$, the action is defined by $(m,q)^{q'} = (m, qq')$. Now, if $u \in Q_0$, we set the biaction (whiskering operations) of Q_0 on $F(y)$ by

$$(m, q) \otimes u = (m, q \otimes u)$$

and

$$u \otimes (m, q) = (m, u \otimes q).$$

The morphism ∂' from $F(y)$ to $Q_1(y)$ is defined by $(m, q) \mapsto q^{-1} f_1 \delta(m) q$. This gives a free precrossed module with function $i : M \to F$ given by $m \mapsto (m, 1)$ where $m \in M(x)$ and then $1 \in Q_1(f_0(x))$ is the identity.

To make $i : M \to F$ an operator morphism in $\mathfrak{Rgrpd}$, we must factor F out by relations:

(i) $(m, q)(m', q) = (mm', q)$

(ii) $(m^p, q) = (m, f_1(p)q)$

(iii) $(m \otimes a, q) = (m, q \otimes f_0(a))$

(iv) $(a \otimes m, q) = (m, f_0(a) \otimes q)$

for $a \in P_0$.

Our aim is to obtain a braided regular crossed module over (Q_1, Q_0). To get it, we consider the free product $N * \langle Q_1 \times Q_1 \rangle$ where $\langle Q_1 \times Q_1 \rangle$ is the free group generated by the set $Q_1 \times Q_1$ on generators $\langle q_1, q_2 \rangle : t(q_1)t(q_2) \to t(q_1)t(q_2)$.

The left and right actions of Q_0 on $\langle Q_1 \times Q_1 \rangle$ are given on generators by

$$u \otimes \langle q_1, q_2 \rangle = \langle u \otimes q_1, q_2 \rangle \text{ and } \langle q_1, q_2 \rangle \otimes u = \langle q_1, q_2 \otimes u \rangle$$

with the relation $\langle q_1 \otimes u, q_2 \rangle = \langle q_1, u \otimes q_2 \rangle$ for $u \in Q_0, q_1, q_2 \in Q_1$.

We have a morphism $\delta_* : N * \langle Q_1 \times Q_1 \rangle \to Q$ induced on N by ∂' and given on $\langle Q_1 \times Q_1 \rangle$ by Relation **B4** in the definition of a braided regular crossed module. We factor the group $N * \langle Q_1 \times Q_1 \rangle$ out by the Relations **B2**, **B3**, **B5**, and **B6**. Relation **B7** is given by the definition of the right and left actions of Q_0 on $\langle Q_1 \times Q_1 \rangle$. Finally, to get the braided regular crossed module induced by (f_1, f_0), we need to add the relation

$$\{p_1, p_2\} = \langle f_1(p_1), f_1(p_2) \rangle$$

for $p_1, p_2 \in P_1$ and where $\{-,-\}$ is the braiding map of δ. This gives a braided regular crossed module morphism

$$(\varphi, f_1, f_0) : \left(M \xrightarrow{\delta} P_1 \overset{s}{\underset{t}{\rightrightarrows}} P_0 \right) \longrightarrow \left(N * \langle Q_1 \times Q_1 \rangle \xrightarrow{\delta_*} Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0 . \right)$$

Now we show that this morphism is co-Cartesian in $\mathfrak{BRCM}$ over the morphism $\Phi_1(\varphi, f_1, f_0) = (f_1, f_0)$. Since $\Phi_1 : \mathfrak{BRCM} \to \mathfrak{Rgrpd}$ is a fibration, to show that the morphism (φ, f_1, f_0) is co-Cartesian, we can use Proposition 5.3.

Let

$$X \xrightarrow{\delta'} Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0$$

be a braided regular crossed module with a morphism (θ', f_1, f_0) given by

$$\begin{tikzcd}
M \arrow[r, "\theta'"] \arrow[d, "\delta"'] & X \arrow[d, "\delta'"] \\
P_1 \arrow[r, "f_1"'] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & Q_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
P_0 \arrow[r, "f_0"'] & Q_0
\end{tikzcd}$$

with $\delta'\theta' = f_1\delta$.

Then there exists a unique morphism $\theta_* : N * \langle Q_1 \times Q_1 \rangle \to X$ such that the diagram

$$\begin{tikzcd}
M \arrow[d, "\delta"'] \arrow[dr, "\varphi"] \arrow[drr, bend left, "\theta'"] & & \\
P_1 \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, "f_1"'] \arrow[drr, "f_1"] & N * \langle Q_1 \times Q_1 \rangle \arrow[r, dotted, "\theta_*"] \arrow[d, "\delta_*"'] & X \arrow[d, "\delta'"] \\
P_0 \arrow[dr, "f_0"'] \arrow[drr, "f_0"] & Q_1 \arrow[r, equal] \arrow[d, shift right] \arrow[d, shift left] & Q_1 \arrow[d, shift right] \arrow[d, shift left] \\
& Q_0 \arrow[r, equal] & Q_0
\end{tikzcd}$$

is commutative. The necessary unique morphism $\theta_* : N * \langle Q_1 \times Q_1 \rangle \to X$ can be defined as follows. For any $y \in Q_0$, $x \in P_0, m \in M(x), q \in Q_1(f_0(x), y)$, we define θ_* on the generator (m, q) of N by $\theta_*(m, q) = \theta'(m)^q$, where $\theta(m) \in X(f_0(x))$. Thus, $\theta_*(m, q) = \theta(m)^q \in X(y)$. Similarly, θ_* is defined on generators $\langle q_1, q_2 \rangle$ by

$$\theta_*(\langle q_1, q_2 \rangle) = \{q_1, q_2\}'$$

where $\{-,-\}'$ is the braiding map of δ' from $Q_1 \times Q_1$ to X. Then we obtain

$$\begin{aligned}\delta'\theta_*(m,q) &= \delta'(\theta'(m)^q)\\ &= (q)^{-1}\delta'\theta'(m)(q)\\ &= (q)^{-1}f_1\delta(m)(q)\\ &= \delta_*(m,q)\end{aligned}$$

and

$$\begin{aligned}\delta'\theta_*(\langle q_1,q_2\rangle) &= \delta'\{q_1,q_2\}\\ &= [q_1,q_2]\\ &= \delta_*\langle q_1,q_2\rangle.\end{aligned}$$

Finally we have for $m \in M$

$$\theta_*\varphi(m) = \theta_*(m,1) = \theta(m)$$

and

$$\delta_*\varphi(m) = \delta_*(m,1) = f_1\delta(m).$$

Thus, we proved that the morphism (φ, f_1, f_0) is a co-Cartesian morphism in $\mathfrak{BRCM}$ over $\Phi_1((\varphi, f_1, f_0)) = (f_1, f_0)$. □

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