Depth and Stanley depth of the path ideal associated to an $n$-cyclic graph

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Abstract: We compute the depth and Stanley depth for the quotient ring of the path ideal of length 3 associated to an $n$-cyclic graph, given some precise formulas for the depth when $n \not\equiv 1 \pmod{4}$, tight bounds when $n \equiv 1 \pmod{4}$, and for Stanley depth when $n \equiv 0,3 \pmod{4}$, tight bounds when $n \equiv 1,2 \pmod{4}$. We also give some formulas for the depth and Stanley depth of a quotient of the path ideals of length $n-1$ and $n$.

Key words: Stanley depth, Stanley inequality, path ideal, cyclic graph

1. Introduction
Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over a field $K$ and $M$ a finitely generated $Z^n$-graded $S$-module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \ldots, x_n\}$, $uK[Z]$ denotes the $K$-subspace of $M$ generated by all the homogeneous elements of the form $uv$, where $v$ is a monomial in $K[Z]$. The $Z^n$-graded $K$-subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where, as usual, $|Z|$ denotes the number of elements of $Z$. A Stanley decomposition of $M$ is a decomposition of $M$ as a finite direct sum of $Z^n$-graded $K$-vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

where each $u_i K[Z_i]$ is a Stanley space of $M$. The number $sdepth_S(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$ is called the Stanley depth of decomposition $\mathcal{D}$ and the quantity $sdepth(M) := \max\{sdepth(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}$ is called the Stanley depth of $M$. Stanley [13] conjectured that

$$sdepth(M) \geq \text{depth}(M)$$

for all $Z^n$-graded $S$-modules $M$. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals; see [6].

Herzog et al. [8] introduced a method to compute the Stanley depth of a factor of a monomial ideal, which was later developed into an effective algorithm by Rinaldo [12], implemented in CoCoA [5]. However, it

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is difficult to compute this invariant, even in some very particular cases. For instance, in [1] Biró et al. proved that $sdepth(m) = \left\lceil \frac{n}{3} \right\rceil$ where $m = (x_1, \ldots, x_n)$ is the graded maximal ideal of $S$ and $\left\lceil \frac{n}{3} \right\rceil$ denotes the smallest integer $\geq \frac{n}{3}$. For an introduction to Stanley depth, we refer the reader to [7].

Let $I_{n,m}$ and $J_{n,m}$ be the path ideals of length $m$ associated to the $n$-line, respectively $n$-cyclic, graph. Cimpoeaş [3] proved that depth $(S/J_{n,2}) = \left\lceil \frac{n+1}{2} \right\rceil$ and when $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$, $sdepth(S/J_{n,2}) = \left\lceil \frac{n+1}{3} \right\rceil$ and when $n \equiv 1 \pmod{3}$, $\left\lceil \frac{n+1}{3} \right\rceil \leq sdepth(S/J_{n,2}) \leq \left\lceil \frac{n}{3} \right\rceil$. In [4], he also showed that $sdepth(S/I_{n,m}) = depth(S/I_{n,m}) = n + 1 - \left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil$, where $\left\lfloor \frac{n+1}{m+1} \right\rfloor$ denotes the biggest integer $\leq \frac{n+1}{m+1}$. Using similar techniques, we prove that $sdepth(S/J_{n,3}) = n - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$ for $n \equiv 0 \pmod{4}$ or $n \equiv 3 \pmod{4}$ and $n - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor \leq sdepth(S/J_{n,3}) \leq n + 1 - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$ for $n \equiv 1 \pmod{4}$ or $n \equiv 2 \pmod{4}$.

Also, we prove that depth $(S/J_{n,3}) = n - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$ for $n \not\equiv 1 \pmod{4}$ and $n - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor \leq depth(S/J_{n,3}) \leq n + 1 - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$ for $n \equiv 1 \pmod{4}$. In Proposition 2.14, we prove that $sdepth(J_{n,3}/I_{n,3}) = n + 1 - \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$ for all $n \geq 4$. In the third section, we prove that $sdepth(S_{n,m-1}) = depth(S_{n,m-1}) = n - 2$ and $n - 3 \leq sdepth(S_{n,m-1})$, $\text{depth}(S_{n,m-1}) \leq n - 2$.

2. Depth and Stanley depth of the quotient of the path ideal with length 3

In this section, we will give some formulas for the depth and Stanley depth of quotient of the path ideals of length 3. We first recall some definitions about graphs and their path ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [16, 17].

**Definition 2.1** Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E$. Then $G = (V, E)$ is called an $n$-line graph, denoted by $L_n$, if its edge set is given by $E = \{x_ix_{i+1} \mid 1 \leq i \leq n-1\}$. Similarly, if $n \geq 3$, then $G = (V, E)$ is called an $n$-cyclic graph, denoted by $C_n$, if its edge set is given by $E = \{x_ix_{i+1} \mid 1 \leq i \leq n-1\} \cup \{x_nx_1\}$.

**Definition 2.2** Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$. A path of length $m$ in $G$ is an alternating sequence of vertices and edges $w = \{x_i, e_i, x_{i+1}, \ldots, x_{i+m-2}, e_{i+m-2}, x_{i+m-1}\}$, where $e_j = x_jx_{j+1}$ is the edge joining $x_j$ and $x_{j+1}$. A path of length $m$ may also be denoted $\{x_i, \ldots, x_{i+m-1}\}$, the edges being evident from the context.

**Definition 2.3** Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$. Then the path ideal of length $m$ associated to $G$ is the squarefree monomial ideal $I = (x_i \cdots x_{i+m-1} \mid \{x_i, \ldots, x_{i+m-1}\}$ is a path of length $m$ in $G)$ of $S$.

In this paper, we set $n \geq 3$ and consider the $n$-line graph $L_n$ and $n$-cyclic graph $C_n$; their path ideals of length $m$ are denoted by $I_{n,m}$ and $J_{n,m}$, respectively. Thus, we obtain that

$$I_{n,m} = (x_i \cdots x_{i+m-1} \mid 1 \leq i \leq n-m+1),$$

and

$$J_{n,m} = I_{n,m} + (x_{n-m} \cdots x_nx_1, x_{n-m+1} \cdots x_nx_1x_2, \ldots, x_nx_1 \cdots x_{m-1}).$$
Definition 2.4 Let $(S, m)$ be a local ring (or a Noetherian graded ring with $(S_0, m_0)$ local) and $M$ a finite generated $S$-module with the property that $mM \subseteq M$ (or a finite generated graded $S$-module with the property that $(m_0 \oplus \bigoplus_{i=1}^{\infty} S_i)M \subseteq M$). Then the depth of $M$ is defined as

$$\text{depth}(M) = \min \{ i \mid \text{Ext}^i(S/m, M) \neq 0 \}$$

(or $\text{depth}(M) = \min \{ i \mid \text{Ext}^i(S/(m_0 \oplus \bigoplus_{i=1}^{\infty} S_i), M) \neq 0 \}$).

We recall the well-known depth lemma; see, for instance, [16, Lemma 1.3.9] or [15, Lemma 3.1.4].

Lemma 2.5 (Depth lemma) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of modules over a local ring $S$, or a Noetherian graded ring with $S_0$ local; then:

(i) $\text{depth}(M) \geq \min \{ \text{depth}(L), \text{depth}(N) \}$;

(ii) $\text{depth}(L) \geq \min \{ \text{depth}(M), \text{depth}(N) + 1 \}$;

(iii) $\text{depth}(N) \geq \min \{ \text{depth}(L) - 1, \text{depth}(M) \}$.

Most of the statements of the above depth lemma are wrong if we replace depth by Stanley depth. Some counterexamples are given in [11, Example 2.5 and 2.6]. Rauf [11] proved the analog of Lemma 2.5 (i) for Stanley depth.

Lemma 2.6 Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded $S$-modules. Then

$$\text{sdepth}(M) \geq \min \{ \text{sdepth}(L), \text{sdepth}(N) \}.$$ 

In [3], Cimpoeaş computed depth and Stanley depth for $S/J_{n,2}$.

Lemma 2.7

(1) $\text{depth}(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$;

(2) $\text{sdepth}(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$ for $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$;

(3) $\lfloor \frac{n-1}{3} \rfloor \leq \text{sdepth}(S/J_{n,2}) \leq \lceil \frac{n}{3} \rceil$ for $n \equiv 1 \pmod{3}$.

In [4], Cimpoeaş computed depth and Stanley depth for $S/I_{n,m}$, which generalizes [9, Lemma 2.8] and [14, Lemma 4].

Lemma 2.8 $\text{sdepth}(S/I_{n,m}) = \text{depth}(S/I_{n,m}) = n + 1 - \lfloor \frac{n+1}{m+1} \rfloor - \lceil \frac{n+1}{m+1} \rceil$. In particular, $\text{sdepth}(S/J_{n,2}) = \text{depth}(S/I_{n,2}) = \lceil \frac{n}{3} \rceil$.

Using these lemmas, we are able to prove the main result of this section.

Theorem 2.9

(1) $\text{depth}(S/J_{n,3}) \geq n - \lceil \frac{n}{4} \rceil - \lceil \frac{n}{4} \rceil$;

(2) $\text{sdepth}(S/J_{n,3}) \geq n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$.
We denote \( \varphi(n) = n - \left\lfloor \frac{n}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil \). One can easily see that
\[
\varphi(n) = \begin{cases} 
  n - 2k, & \text{if } n \equiv 0 \pmod{4}; \\
  n - 2k + 1, & \text{otherwise}.
\end{cases}
\]

We assume that \( n \geq 6 \). Let \( k = \left\lfloor \frac{n}{4} \right\rfloor \) and \( \varphi(n) = n - \left\lfloor \frac{n}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil \). We only prove that \( s\text{depth} (S) \geq n - \left\lfloor \frac{n}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil \).

We may assume that \( n \geq 6 \). Let \( k = \left\lfloor \frac{n}{4} \right\rfloor \) and \( \varphi(n) = n - \left\lfloor \frac{n}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil \). We only prove that \( s\text{depth} (S) \geq n - \left\lfloor \frac{n}{4} \right\rfloor - \left\lceil \frac{n}{4} \right\rceil \).

We consider the following three cases:

1. If \( n = 4k \) or \( n = 4k - 1 \), we denote \( L_j = (L_j : x_{4j}) \) and \( U_j = (U_j : x_{4j}) \) for \( j = 1, 2, \ldots, k - 1 \). We conclude \( L_k \simeq J_{n-k,2}S \), \( U_k = (x_{4(k-1)}, V_k) \) where \( V_k = \left( \frac{u_1}{x_4}, \frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_5}{x_4}, \frac{u_6}{x_4}, \frac{u_7}{x_4}, \frac{u_8}{x_4}, \ldots, \frac{u_{4(k-2)}}{x_4}, \frac{u_{4(k-1)}}{x_4} \right) \). Note that
\[
V_k \simeq \begin{cases} 
  I_{n-k-2,2}, & \text{if } n = 4k; \\
  I_{n-k-1,2}, & \text{if } n = 4k - 1.
\end{cases}
\]

Thus, by Lemmas 2.7, 2.8, and [8, Lemma 3.6], it follows that
\[
s\text{depth} (S/L_k) = k + s\text{depth} (S_{n-k}/J_{n-k,2}) = k + k = \varphi(n),
\]
and
\[
s\text{depth} \left( \frac{S}{U_k} \right) = \begin{cases} 
  (k + 1) + s\text{depth} \left( \frac{S_{n-k-2}}{J_{n-k-2,2}} \right) = (k + 1) + k = 1 + \varphi(n), & \text{if } n = 4k; \\
  k + s\text{depth} \left( \frac{S_{n-k-1}}{J_{n-k-1,2}} \right) = k + k = \varphi(n), & \text{if } n = 4k - 1.
\end{cases}
\]

2. If \( n = 4k - 2 \), we denote \( L_k = (L_k : x_{4(k-2)}) \), \( U_k = (U_k : x_{4(k-2)}) \), \( L_k = (L_k : x_{4(k-1)}) \), \( U_k = (U_k : x_{4(k-1)}) \). We have \( L_k \simeq J_{n-k,2}S \), \( U_k = (x_{4(k-1)}, V_k) \) where \( V_k = \left( \frac{u_1}{x_4}, \frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_5}{x_4}, \frac{u_6}{x_4}, \frac{u_7}{x_4}, \frac{u_8}{x_4}, \ldots, \frac{u_{4(k-2)}}{x_4}, \frac{u_{4(k-1)}}{x_4} \right) \approx I_{n-k,2}S \). Thus, by Lemma 2.8 and [8, Lemma 3.6], we obtain
\[
s\text{depth} (S/U_k) = (k - 1) + s\text{depth} (S_{n-k}/I_{n-k,2}) = (k - 1) + k = \varphi(n).
\]

Applying Lemma 2.7 and [8, Lemma 3.6], we get
\[
s\text{depth} (S/L_k) = k + s\text{depth} (S_{n-k}/J_{n-k,2}) \geq k + (k - 1) = \varphi(n).
\]
and
\[ \text{sdepth}(S/L_k) = k + \text{sdepth}(S_{n-k}/J_{n-k,2}) \leq k + 1 + \varphi(n). \]

(3) If \( n = 4k - 3 \), we denote \( L_{k-1} = (L_{k-2} : x_{4(k-2)-1}), \) \( U_{k-1} = (L_{k-2}, x_{4(k-2)-1}), \) \( L_k = (L_{k-1} : x_{4(k-1)-2}) \), and \( U_k = (U_{k-1}, x_{4(k-1)-2}) \). We have \( L_k \cong J_{n-k,2}S \), \( U_k = (x_{4(k-1)-2}, V_k) \) where \( V_k = (u_2, u_3, u_4, x_{4}, x_3, \ldots, u_4(k-3), x_{4(k-3)-1}, u_{4k-2}u_3, x_{4k-3-2}, x_{4k-3-1}, u_{4n-2}, u_{4n-1}, u_{4n}) \cong I_{n-k,2}S \). Therefore, by Lemmas 2.7, 2.8, and [8, Lemma 3.6], we have
\[ \text{sdepth}(S/L_k) = k + \text{sdepth}(S_{n-k}/J_{n-k,2}) = k + (k - 1) = 1 + \varphi(n), \]
and
\[ \text{sdepth}(S/U_k) = (k - 1) + \text{sdepth}(S_{n-k}/I_{n-k,2}) = (k - 1) + (k - 1) = \varphi(n). \]
This shows that \( \varphi(n) \leq \text{sdepth}(S/L_k) \leq 1 + \varphi(n) \) and \( \text{sdepth}(S/U_k) \geq \varphi(n) \) \((*)\).

Consider the following short exact sequences:
\[
\begin{align*}
0 & \rightarrow \frac{S}{L_1} \rightarrow \frac{S}{L_0} \rightarrow \frac{S}{U_1} \rightarrow 0 \\
0 & \rightarrow \frac{S}{L_2} \rightarrow \frac{S}{L_1} \rightarrow \frac{S}{U_2} \rightarrow 0 \\
& \vdots \quad \vdots \\
0 & \rightarrow \frac{S}{L_{k-1}} \rightarrow \frac{S}{L_{k-2}} \rightarrow \frac{S}{U_{k-1}} \rightarrow 0 \\
0 & \rightarrow \frac{S}{L_k} \rightarrow \frac{S}{L_{k-1}} \rightarrow \frac{S}{U_k} \rightarrow 0.
\end{align*}
\]

By Lemma 2.6 and \((*)\), we have
\[
\text{sdepth}\left(\frac{S}{J_{n,3}}\right) = \text{sdepth}\left(\frac{S}{L_0}\right) \geq \min\{\text{sdepth}\left(\frac{S}{L_1}\right), \text{sdepth}\left(\frac{S}{U_1}\right)\} \\
\geq \min\{\text{sdepth}\left(\frac{S}{L_2}\right), \text{sdepth}\left(\frac{S}{U_2}\right), \text{sdepth}\left(\frac{S}{U_1}\right)\} \\
\geq \cdots \\
\geq \min\{\text{sdepth}\left(\frac{S}{L_k}\right), \text{sdepth}\left(\frac{S}{U_k}\right), \text{sdepth}\left(\frac{S}{U_{k-1}}\right), \ldots, \text{sdepth}\left(\frac{S}{U_1}\right)\} \\
\geq \min\{\varphi(n), \text{sdepth}\left(\frac{S}{U_{k-1}}\right), \ldots, \text{sdepth}\left(\frac{S}{U_2}\right), \text{sdepth}\left(\frac{S}{U_1}\right)\}.
\]

To show \( \text{sdepth}\left(\frac{S}{J_{n,3}}\right) \geq \varphi(n) \), it is enough to prove the claim below.

Claim: \( \text{sdepth}(S/U_{j+1}) \geq \varphi(n) \) for all \( 1 \leq j \leq k - 2 \).

For any \( 1 \leq j \leq k - 3 \), we set \( V_{j+1} = (x_2, x_3, x_4, \ldots, u_{4j}, x_{4j+1}, \ldots, u_{4n-2}, x_{4n-1}, u_{4n}) \) and \( W_{j+1} = (u_{4j+1}, \ldots, u_{4n-4}) \) where \( x_0 = 1 \) and \( u_j = 0 \) for \( j \leq 0 \). We have \( \frac{S}{U_{j+1}} \cong \frac{S}{V_{j+1} \oplus S/W_{j+1}} \). Since \( x_{4j} \) is regular on \( S/V_{j+1} \oplus S/W_{j+1} \), by [10, Theorem 1.1] and [2, Theorem 1.3], we have
\[
\text{sdepth}\left(\frac{S}{U_{j+1}}\right) = \text{sdepth}\left(\frac{S}{V_{j+1}} \oplus \frac{S}{W_{j+1}}\right) - 1 \geq \text{sdepth}\left(\frac{S}{V_{j+1}}\right) + \text{sdepth}\left(\frac{S}{W_{j+1}}\right) - n - 1.
\]
On the other hand, \( V_{j+1} \simeq I_{3j+1,2}S \), \( W_{j+1} \simeq I_{n-4(j+1)+2,3}S \). Thus, by Lemma 2.8, we have

\[
\text{sdepth } (S/V_{j+1}) = [n - (3j + 1)] + \left\lceil \frac{3j + 1}{3} \right\rceil = n - 2j
\]

and

\[
\text{sdepth } (S/W_{j+1}) = [4(j + 1) - 2] + [n - 4(j + 1) + 3] - \left\lfloor \frac{n - 4(j + 1) + 3}{4} \right\rfloor - \left\lceil \frac{n - 4j - 1}{4} \right\rceil
\]

\[
= n + 1 - \left\lfloor \frac{n - 4j - 1}{4} \right\rfloor - \left\lceil \frac{n - 1}{4} \right\rceil
\]

By some simple computations, we conclude that

\[
\text{sdepth } (S/U_{j+1}) \geq (n - 2j) + (n + 1 + 2j) - \left\lfloor \frac{n - 1}{4} \right\rfloor - \left\lceil \frac{n - 1}{4} \right\rceil - n - 1
\]

\[
= n - \left\lfloor \frac{n - 1}{4} \right\rfloor - \left\lceil \frac{n - 1}{4} \right\rceil \geq \varphi(n).
\]

If \( n \neq 4k - 3 \), we have \( V_{k-1} = (u_{4(k-2)+1}, \ldots, u_{n-4}) \) and \( W_{k-1} = (u_{4(k-2)+1}, \ldots, u_{n-4}) \).

It follows from similar arguments as above.

If \( n = 4k - 3 \), we have \( V_{k-1} = (u_{4(k-2)+1}, \ldots, u_{n-4}) \) and \( W_{k-1} = (u_{4(k-2)+1}, \ldots, u_{n-4}) \).

Note that \( V_{k-1} \simeq I_{3(k-2)+1,2}S \) and \( W_{k-1} \simeq I_{n-4(k-1)+3,3}S \). Thus, by Lemma 2.8, we obtain

\[
\text{sdepth } (S/V_{k-1}) = (n - (3(k - 2) + 1)) + \left\lceil \frac{3(k - 2) + 1}{3} \right\rceil = n - 2k + 4 = 2k + 1
\]

and

\[
\text{sdepth } (S/W_{k-1}) = (4(k - 1) - 3) + (n - 4(k - 1) + 4) - \left\lfloor \frac{n - 4(k - 1) + 4}{4} \right\rfloor
\]

\[
= n + 1 - \left\lfloor \frac{n - 4k + 8}{4} \right\rfloor - \left\lceil \frac{n - 4k + 8}{4} \right\rceil
\]

\[
= n + 1 - \left\lfloor \frac{5}{4} \right\rfloor - \left\lceil \frac{5}{4} \right\rceil = n - 2.
\]
One can easily see that \( \frac{S}{U_{k-1}} \simeq \frac{S/V_{k-1} \oplus S/W_{k-1}}{S/V_{k-1} \oplus S/W_{k-1}} \). Since \( x_{x_4(k-2)-1} \) is regular on \( S/V_{k-1} \oplus S/W_{k-1} \), by [10, Theorem 1.1] and [2, Theorem 1.3], we have

\[
\text{sdepth} \left( \frac{S}{U_{k-1}} \right) = \text{sdepth} \left( \frac{S}{V_{k-1}} \oplus \frac{S}{W_{k-1}} \right) - 1 
\geq \text{sdepth} \left( \frac{S}{V_{k-1}} \right) + \text{sdepth} \left( \frac{S}{W_{k-1}} \right) - n - 1 
= 2k + 1 + (n - 2) - n - 1 
= 2k - 2 = \varphi(n).
\]

This completes the proof.

Example 2.10 Let \( J_{4,3} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_1, x_4x_1x_2) \subset S = K[x_1, \ldots, x_4] \). Note that \( 4 - \left[ \frac{4}{4} \right] - \left[ \frac{4}{4} \right] = 2 \).

Set \( L_1 = (J_{4,3} : x_4) \) and \( U_1 = (J_{4,3}, x_4) \). Since \( L_1 = (x_1x_2, x_2x_3, x_3x_4) = J_{3,2}S \) and \( U_1 = (x_1x_2x_3, x_4) \), thus \( S/U_1 = K[x_1, x_2, x_3]/(x_1x_2x_3, x_4) \). By Lemmas 2.7, 2.8, and [8, Lemma 3.6], we have \( \text{sdepth}(S/L_1) = \text{depth}(S/L_1) = 1 + \left[ \frac{3 - 1}{3} \right] = 2 \) and \( \text{sdepth}(S/U_1) = 2 \). Applying Lemma 2.6 to the short exact sequence

\[
0 \to S/L_1 \to S/J_{4,3} \to S/U_1 \to 0,
\]

we obtain \( \text{sdepth}(\frac{S}{J_{4,3}}) = 2 \) and \( \text{sdepth}(\frac{S}{J_{4,3}}) \geq 2 \). By [2, Proposition 2.7], it follows that \( \text{sdepth}(\frac{S}{J_{4,3}}) \leq \text{sdepth}(S/L_1) = 2 \). Thus, \( \text{sdepth}(S/J_{4,3}) = 2 \).

Example 2.11 Let \( J_{5,3} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2) \subset S = K[x_1, \ldots, x_5] \). Note that \( 5 - \left[ \frac{5}{4} \right] - \left[ \frac{5}{4} \right] = 2 \).

Set \( L_1 = (J_{5,3} : x_5) \) and \( U_1 = (J_{5,3}, x_5) \). Since \( L_1 = (x_3x_4, x_4x_1, x_1x_2) \simeq I_{4,2}S \) and \( U_1 = (x_1x_2x_3, x_2x_3x_4, x_5, x_4x_1x_2) \), thus \( S/U_1 = S_4/I_{4,3} \), and by Lemma 2.8 and [8, Lemma 3.6], we have \( \text{sdepth}(S/L_1) = \text{depth}(S/L_1) = 1 + \left[ \frac{4}{4} \right] = 3 \) and \( \text{sdepth}(S/U_1) = 5 - \left[ \frac{5}{4} \right] - \left[ \frac{5}{4} \right] = 2 \). Using Lemmas 2.5 and 2.6 on the short exact sequence

\[
0 \to S/L_1 \to S/J_{5,3} \to S/U_1 \to 0,
\]

we obtain \( \text{sdepth}(S/J_{5,3}) \geq 2 \) and \( \text{sdepth}(S/J_{5,3}) \geq 2 \).

As a consequence of Theorem 2.9, one has the following results.

Corollary 2.12

1. \( \text{sdepth}(S/J_{n,3}) \leq n + 1 - \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] \) for \( n \equiv 1 \pmod{4} \) or \( n \equiv 2 \pmod{4} \);

2. \( \text{sdepth}(S/J_{n,3}) = n - \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] \) for \( n \equiv 0 \pmod{4} \) or \( n \equiv 3 \pmod{4} \).

Proof Set \( \varphi(n) = n - \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] \). From the proof of Theorem 2.9, we see that \( \text{sdepth}(S/L_k) \leq 1 + \varphi(n) \) for \( n \equiv 1 \pmod{4} \) or \( n \equiv 2 \pmod{4} \), and otherwise \( \text{sdepth}(S/L_k) = \varphi(n) \). These are direct consequences of [2, Proposition 2.7].

Corollary 2.13

1. \( \text{depth}(S/J_{n,3}) \leq n + 1 - \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] \) for \( n \equiv 1 \pmod{4} \);

2. \( \text{depth}(S/J_{n,3}) = n - \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] \) for \( n \not\equiv 1 \pmod{4} \).
Proof. Set \( \varphi(n) = n - \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor \). Replacing the Stanley depth by depth in the proof of Theorem 2.9, we see that depth \((S/L_k) = 1 + \varphi(n)\) for \( n \equiv 1 (mod\ 4)\), and otherwise depth \((S/L_k) = \varphi(n)\). These are direct consequences of [11, Corollary 1.3].

Proposition 2.14. \( \text{sdepth}(J_{n,3}/I_{n,3}) = n + 1 - \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor \) for all \( n \geq 4 \).

Proof. One can easily check that \( J_{3,3}/I_{3,3} \cong x_1 x_3 x_4 K[x_1, x_3, x_4] \oplus x_1 x_2 x_4 x_5 K[x_1, x_2, x_4, x_5] \). Thus, \( \text{sdepth}(J_{4,3}/I_{4,3}) = 3\), as required. Similarly, for \( n = 5 \), we have \( J_{5,3}/I_{5,3} \cong x_1 x_4 x_5 K[x_1, x_4, x_5] \oplus x_1 x_2 x_5 x_6 K[x_1, x_2, x_5, x_6] \); for \( n = 6 \), we have \( J_{6,3}/I_{6,3} \cong x_1 x_5 x_6 K[x_1, x_5, x_6] \oplus x_1 x_2 x_6 K[x_1, x_2, x_6] \); and for \( n = 7 \), we get \( J_{7,3}/I_{7,3} \cong x_1 x_6 x_7 K[x_1, x_6, x_7] \). Replacing the Stanley depth by depth in the proof of Theorem 3.6, Lemmas 3.6, we obtain \( J_{n,3}/I_{n,3} \cong 1 + \ldots + 1 \). Assume now that \( u \notin I_{n,3} \). It follows that \( u = x_1 x_{n-1} x_n v_1 \) or \( u = x_1 x_2 x_n v_2 \), with \( v_1 \in K[x_1, \ldots, x_{n-3}, x_{n-1}, x_n] \) and \( v_2 \in K[x_1, x_2, x_4, \ldots, x_n] \). We can write \( v_1 = x_1^{a_1} x_{n-1}^{a_{n-1}} x_n^{a_n} w \) with \( w \in K[x_2, \ldots, x_{n-3}] \). Since \( u \notin I_{n,3} \), it follows that \( w \notin (x_2 x_3, x_3 x_4 x_5, \ldots, x_{n-5} x_{n-4} x_{n-3}) \). Similarly, we can write \( v_2 = x_1^{a_1} x_2^{a_2} x_n^{a_n} w \) with \( w \in K[x_4, \ldots, x_{n-1}] \). Since \( u \notin I_{n,3} \), it follows that \( w \notin (x_4 x_5 x_6, \ldots, x_{n-4} x_{n-3} x_{n-2}, x_{n-2} x_{n-1}) \). Therefore, we have the \( S \)-module isomorphism:

\[
J_{n,3}/I_{n,3} \cong x_1 x_2 x_n K[x_4, \ldots, x_{n-2}] \oplus x_1 x_2 x_3 x_4 x_5 K[x_3, x_4, x_5, x_6] \oplus x_1 x_2 x_3 x_4 x_5 x_6 K[x_4, x_5, x_6]
\]

Therefore, by Lemma 2.8 and [8, Lemma 3.6], we obtain

\[
\text{sdepth}(J_{n,3}/I_{n,3}) = \min\{3+(n-4) - \left\lceil \frac{n-4}{4} \right\rceil - \left\lfloor \frac{n-4}{4} \right\rfloor, 4+(n-5) - \left\lceil \frac{n-5}{4} \right\rceil - \left\lfloor \frac{n-5}{4} \right\rfloor\}
\]

\[
= \min\{n + 1 - \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor, n + 1 - \left\lceil \frac{n-1}{4} \right\rceil - \left\lfloor \frac{n-1}{4} \right\rfloor\}
\]

\[
= n + 1 - \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.
\]

\[\square\]

3. Depth and Stanley depth of the quotient of the path ideal of length \( n-1 \) or \( n-2 \)

In this section, we will give some formulas for depth and Stanley depth of the quotient of the path ideal of length \( n-1 \) or \( n-2 \).

Proposition 3.1. \( \text{sdepth}(S/J_{n,n-1}) = \text{depth}(S/J_{n,n-1}) = n - 2 \).

Proof. We apply induction on \( n \). The case \( n = 3 \) follows from Lemma 2.7. Assume now that \( n \geq 4 \). Since \( J_{n,n-1} = (\prod_{i=1}^{n-1} x_i, \prod_{i=2}^{n-1} x_i, (\prod_{i=3}^{n-1} x_i)x_1, \ldots, (\prod_{i=k}^{n-1} x_i)(\prod_{i=1}^{k} x_i), \ldots, x_n \prod_{i=1}^{n-2} x_i) \), we obtain

\[
(J_{n,n-1} : x_n) = (\prod_{i=1}^{n-2} x_i, \prod_{i=2}^{n-2} x_i, (\prod_{i=3}^{n-2} x_i)x_1, \ldots, (\prod_{i=k}^{n-2} x_i)(\prod_{i=1}^{k-2} x_i), \ldots, x_{n-1} \prod_{i=1}^{n-3} x_i) = J_{n-1,n-2} S.
\]
(J_{n,n-1}, x_n) = \left( \prod_{i=1}^{n-1} x_i, x_n \right). Hence, we get \( S/(J_{n,n-1} : x_n) = (S_{n-1}/J_{n-1,n-2})[x_n] \). Using the induction hypothesis and [8, Lemma 3.6], we conclude
\[
sdepth(S/(J_{n,n-1} : x_n)) = 1 + sdepth(S_{n-1}/J_{n-1,n-2}) = n - 2,
\]
and
\[
depth(S/(J_{n,n-1} : x_n)) = 1 + depth(S_{n-1}/J_{n-1,n-2}) = n - 2.
\]
On the other hand, we obtain \( sdepth(S/J_{n,n-1}) = n - 2 \) by [10, Theorem 1.1]. By applying Lemmas 2.5 and 2.6 to the exact sequence
\[
0 \to S/(J_{n,n-1} : x_n) \xrightarrow{x_n} S/J_{n,n-1} \to S/(J_{n,n-1}, x_n) \to 0,
\]
we obtain \( \text{depth}(S/J_{n,n-1}) \geq n - 2 \) and \( \text{sdepth}(S/J_{n,n-1}) \geq n - 2 \). Therefore, it follows that \( \text{sdepth}(S/J_{n,n-1}) = n - 2 \) by [2, Proposition 2.7].

Proposition 3.2  
(1) \( n - 3 \leq \text{sdepth}(S/J_{n,n-2}) \leq n - 2 \).

(2) \( n - 3 \leq \text{depth}(S/J_{n,n-2}) \leq n - 2 \).

Proof  The case \( n = 3 \) is trivial. The case \( n = 4 \) follows from Lemma 2.7. We may assume that \( n \geq 5 \). Set \( L_0 = J_{n,n-2} \), \( L_j = (L_{j-1} : x_{n-j+1}) \) and \( U_j = (L_{j-1}, x_{n-j+1}) \) for all \( 1 \leq j \leq n - 4 \). We conclude that
\[
L_0 = \left( \prod_{i=1}^{n-2} x_i, \prod_{i=2}^{n-1} x_i, \prod_{i=3}^{n} x_i, (\prod_{i=4}^{n} x_i)x_1, \ldots, (\prod_{i=4}^{n} x_i)(\prod_{i=5}^{n} x_i), \ldots, x_n \prod_{i=1}^{n} x_i \right),
\]
\[
L_1 = \left( \prod_{i=3}^{n-1} x_i, (\prod_{i=4}^{n} x_i)x_1, \ldots, (\prod_{i=4}^{n} x_i)(\prod_{i=5}^{n} x_i), \ldots, x_n \prod_{i=1}^{n} x_i \right),
\]
and \( U_j = (U_{j-1}, x_{n-j+1}) = \left( \prod_{i=1}^{n-j-1} x_i, x_{n-j+1} \right) \). In particular, \( L_{n-4} = (x_3x_4, x_4x_1, x_1x_2) \) and \( S/L_{n-4} \cong (S_1/I_{4,2})[x_5, \ldots, x_n] \). Therefore, by Lemma 2.8 and [8, Lemma 3.6], we get \( \text{sdepth}(S/L_{n-4}) = \text{depth}(S/L_{n-4}) = (n - 4) + 5 - \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{n}{2} \right\rceil = n - 2 \). On the other hand, we obtain \( \text{sdepth}(S/U_j) = \text{depth}(S/U_j) = n - 2 \) by [10, Theorem 1.1]. By applying Lemmas 2.5 and 2.6 on the exact sequences
\[
0 \to S/L_j \xrightarrow{x_{n-j+1}} S/L_{j-1} \to S/U_j \to 0 \quad \text{for } 1 \leq j \leq n - 4,
\]
we conclude \( \text{depth}(S/J_{n,n-2}) \geq n - 3 \) and \( \text{sdepth}(S/J_{n,n-2}) \geq n - 3 \).

On the other hand, by [10, Theorem 1.1] and [2, Proposition 2.7], we have \( \text{depth}(S/J_{n,n-2}) \leq \text{depth}(S/L_{n-4}) \) and \( \text{sdepth}(S/J_{n,n-2}) \leq \text{sdepth}(S/L_{n-4}) \). This completes the proof.

\(\square\)
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