On tetravalent normal edge-transitive Cayley graphs on the modular group

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Abstract: A Cayley graph $\Gamma = \text{Cay}(G, S)$ on a group $G$ with respective to a subset $S \subseteq G$, $S = S^{-1}$, $1 \notin S$, is said to be normal edge-transitive if $N_{\text{Aut}(\Gamma)}(\rho(G))$ is transitive on edges of $\Gamma$, where $\rho(G)$ is a subgroup of $\text{Aut}(\Gamma)$ isomorphic to $G$. We determine all connected tetravalent normal edge-transitive Cayley graphs on the modular group of order $8n$ in the case that every element of $S$ is of order $4n$.

Key words: Cayley graph, edge-transitive, modular group

1. Introduction

Let $G$ be a group and $S$ a subset of $G$ such that $1 \notin S$. The Cayley graph $\text{Cay}(G, S)$ is the graph with vertex set $V(\text{Cay}(G, S)) = G$ and edge set $E(\text{Cay}(G, S)) = \{(u, v)|vu^{-1} \in S\}$. The edge set can be identified with set of ordered pairs $\{(g, sg)|g \in G, s \in S\}$. If $S = S^{-1}$, that is, closed under taking the inverse, then $\text{Cay}(G, S)$ is an undirected graph. The degree of each vertex is $|S|$ and it is obvious that $\text{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$.

A graph $\Gamma$ is called vertex-transitive or edge-transitive if the automorphism group $\text{Aut}(\Gamma)$ acts transitively on the vertex-set or edge-set of $\Gamma$, respectively. Now let $\Gamma = \text{Cay}(G, S)$.

For $g \in G$, let $\rho_g : G \rightarrow G$ given by $\rho_g(x) = xg$. The set of all $\rho_g$, $g \in G$, forms the subgroup $\rho(G)$ (isomorphic to $G$) of $\text{Aut}(\Gamma)$. Since $\rho(G) \leq \text{Aut}(\Gamma)$ acts right regularly on the vertices of $\Gamma$, by definition, $\Gamma$ is vertex-transitive, while $\Gamma$ is not edge-transitive in general.

In 1999, Praeger [9] introduced the concept of normal edge-transitive Cayley graphs, which plays an important role for understanding Cayley graphs. The graph $\Gamma$ is called normal edge-transitive if $N_{\text{Aut}(\Gamma)}(\rho(G))$ is transitive on the edges of $\Gamma$.

The research on edge-transitive Cayley graphs is an active area of research. One of the standard problems in this respect is the study of normal edge-transitive Cayley graphs of small valencies. Here we mention some references on research about edge-transitive Cayley graphs. In [7] the edge-transitive tetravalent Cayley graphs on groups of square-free order are recognized. In [4] the authors characterized all nonnormal Cayley digraphs of outvalency 2 of all nonabelian groups of order $2p^2$, where $p$ is an odd prime. In [1] the author found normal edge-transitive Cayley graphs of abelian groups. In [6] all the tetravalent edge-transitive Cayley graphs on the group $\text{PSL}_2(p)$ and in [2] the normal edge-transitive Cayley graphs of Frobenius groups of order $pq$, where $p$
and \( q \) are different primes, are determined. In [3] the authors studied normal edge-transitive Cayley graphs of order \( 4p \) where \( p \) is an odd prime.

Our aim in this paper is to study connected tetravalent normal edge-transitive Cayley graphs of a certain group of order \( 8n \), \( n \in \mathbb{N} \). According to [8], up to isomorphism there are four nonabelian groups of order \( 8n \) with a cyclic subgroup of order \( 4n \), if \( n \) is a power of 2. One of these groups is called the modular group, with the following presentation:

\[
M_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n+1} \rangle.
\]

In the following we work with the modular group \( M_{8n} \) without assuming that \( n \) is a power 2.

We employ the following notation and terminology. The notation \( G = K \rtimes H \) is used to indicate that \( G \) is a semidirect product of \( K \) by \( H \). We denote by \( Aut(G; S) \) the subgroup of \( Aut(G) \) consisting of all \( \sigma \in Aut(G) \) such that \( \sigma(S) = S \). It is easy to see that \( Aut(G; S) \) is a subgroup of the automorphisms group of \( Cay(G, S) \). \( \mathbb{Z}_n \) denotes a cyclic group of order \( n \), and \( S_4 \) denotes for a the symmetric group on four letters. \( D_8 \) is employed to denote the dihedral group of order 8.

The following theorem is the main result of this paper.

**Main Theorem** Let \( G = M_{8n} \) and \( S \) be a symmetric subset of \( M_{8n} \) with cardinality 4 such that each element of \( S \) has order \( 4n \) and \( G = \langle S \rangle \). If \( \Gamma = Cay(G, S) \) is a normal edge-transitive Cayley graph, then \( N_{Aut(\Gamma)}(\rho(G)) \cong \rho(G) \rtimes \mathbb{Z}_2. \)

2. Preliminaries

We start with a famous lemma.

**Lemma 2.1** ([5, Lemma 2.1] or [9]) For a Cayley graph \( \Gamma = Cay(G, S) \), we have \( N_{Aut(\Gamma)}(\rho(G)) = \rho(G) \rtimes Aut(G; S) \).

Therefore, \( \Gamma \) is normal edge-transitive when \( \rho(G) \rtimes Aut(G; S) \) is transitive on the edge-set of \( \Gamma \).

Xu in [10] defined a Cayley graph \( \Gamma = Cay(G, S) \) to be normal if \( \rho(G) \) is a normal subgroup of \( Aut(\Gamma) \), i.e. \( N_{Aut(\Gamma)}(\rho(G)) = Aut(\Gamma) \).

The following lemma is very useful in this paper.

**Lemma 2.2** ([9, Proposition 1(c)]) Consider the Cayley graph \( \Gamma = Cay(G, S) \). Then the following are equivalent:

(i) \( \Gamma \) is normal edge-transitive;

(ii) \( S = T \cup T^{-1} \), where \( T \) is an \( Aut(G, S) \)-orbit in \( G \);

(iii) There exists \( H \leq Aut(G) \) and \( g \in G \) such that \( S = g^H \cup g^{-H} \), where \( g^H = \{ g^h \mid h \in H \} \).

Moreover, \( \rho(G) \rtimes Aut(G, S) \) is transitive on the arcs of \( \Gamma \) if and only if \( Aut(G, S) \) is transitive on \( S \).

3. Proof of the main theorem

First we are going to specify the automorphism group of \( M_{8n} \).
Elements of $M_{8n}$ are of the form $a^k$ or $a^k b$, $0 \leq k < 4n$. Using the defining relations of $M_{8n}$ we can find the orders of elements in $M_{8n}$ as follows: $o(a^k) = \frac{4n}{(k,4n)}$ and

$$o(a^k b) = \begin{cases} \frac{4n}{(k,2n)}, & \text{if } k \text{ is even,} \\ \frac{4n}{(n+k,2n)}, & \text{if } k \text{ is odd,} \end{cases}$$

where $0 \leq k < 4n$.

Elements of order 2 in $M_{8n}$ are of the form $a^{2n}, a^{2n} b, b$ and if $n$ is odd in addition to the above elements, $a^n b$ and $a^{3n} b$ are also of order 2.

Elements of order $4n$ in $M_{8n}$ are of the form $a^k$, $(k,4n) = 1$, and $a^k b$, $k$ odd, $(n+k,2n) = 1$, $0 \leq k < 4n$. Of course in the latter case $n$ must be even.

**Lemma 3.1** $|\mathbb{A}ut(M_{8n})| = 4\varphi(4n)$, where $\varphi$ refers to the Euler phi function.

**Proof** $f \in \mathbb{A}ut(M_{8n})$ is completely ascertained by $f(a)$ and $f(b)$. The elements $f(a)$ and $f(b)$ have orders $4n$ and 2, respectively.

Case(1). $n$ is odd. By what we mentioned earlier we must have $f(a) = a^k, (k,4n) = 1, 1 \leq k < 4n$ and $f(b) \in \{a^{2n}, a^{2n} b, a^n b, a^{3n} b\}$. The case $f(b) = a^{2n}$ is impossible and it verified that all other possibilities can happen. Therefore, $|\mathbb{A}ut(M_{8n})| = 4\varphi(4n)$.

Case(2). $n$ is even. In this case $f(a) = a^k, (k,4n) = 1, 1 \leq k < 4n$, or $f(a) = a^l b, l$ odd, $(n+l,2n) = 1, 0 \leq l < 4n$, and $f(b) \in \{a^{2n}, a^{2n} b, b\}$. The automorphisms of $M_{8n}$ are of two kinds. One kind is defined by $f(a) = a^k, (k,4n) = 1, 1 \leq k < 4n$ and $f(b) = a^{2n} b$ or $b$. The number of these automorphisms is $2\varphi(4n)$.

The other kind of automorphisms of $M_{8n}$ is defined by $f(a) = a^l b, l$ odd, $(n+l,2n) = 1, 0 \leq l < 4n$, and $f(b) \in \{a^{2n}, a^{2n} b, b\}$. However, $Z(M_{8n}) = \langle a^2 \rangle$ and hence $f(a^2) = a^{2l}$ and $f(b) = a^{2n}$ make a contradiction. Therefore, $f(b) = a^{2n} b$ or $b$.

However, it is easy to see that $(n+l,2n) = 1$ if and only if $(l,n) = 1$ (note that $n$ is even and $l$ is odd), and $(l,n) = 1$ if and only if $(l,4n) = 1$. Therefore, the number of automorphisms $f$ is equal to $2\varphi(4n)$ and altogether we have $4\varphi(4n)$ possibilities for elements of $\mathbb{A}ut(M_{8n})$. This completes the proof.

Let us consider the Cayley graph $\Gamma = Cay(M_{8n}, S)$ where $|S| = 4$ and $M_{8n} = \langle S \rangle$. We are interested in the case where $\Gamma$ is normal edge-transitive. By Lemma 2.2 elements of $S$ have the same order and $\mathbb{A}ut(M_{8n}, S)$ on $S$ is either transitive or has two orbits, $T$ and $T^{-1}$.

We are interested in the case where each element of $S$ has order $4n$. Therefore, elements of $S$ are of the form $a^k, (k,4n) = 1, 0 \leq k < 4n$ or $a^l b, (n+l,2n) = 1, l$ odd, $0 \leq k < 4n$. It is obvious that $n$ must be even. Therefore, from now on, we will assume that $n$ is even.

**Theorem 3.1** Let $n$ be an even number and $\Gamma = Cay(M_{8n}, S)$ be a normal connected edge-transitive Cayley graph where $|S| = 4$ and each element of $S$ has order $4n$. Then $S$ is of the following form: $\{a, zab, a^{-1}, z^{-1}b^{-1}a^{-1}\}$, where $z \in Z(M_{8n})$.

**Proof** Elements of order $4n$ in $M_{8n}$, $n$ even, are of the following types:
Type I: \(a^k, 0 \leq k < 4n, (k, 4n) = 1\).

Type II: \(a^l b, 0 \leq l < n, l \text{ odd}, (n + l, 2n) = 1\).

Let \(S\) be a generating set for \(M_{8n}\) such that \(o(x) = 4n, \forall x \in S, \) and \(|S| = 4, S = S^{-1}\). Since \(a' b a' b = a^{l+t(2n+1)}\) is a central element of \(M_{8n}\), two elements of the same type can not generate \(M_{8n}\). Therefore, we have to choose one element from each type. Let \(S = \{x, y, x^{-1}, y^{-1}\}, M_{8n} = \langle x, y \rangle\).

Let \(x = a^k, 0 \leq k < 4n, (k, 4n) = 1\), and \(y = a^l b, 0 \leq l < 4n, l \text{ odd}, (n + l, 2n) = 1\). From \(a^k \in S\) it is easy to deduce that \(a \in \langle S \rangle\); hence, \(b \in \langle S \rangle\). Therefore, for any \(x\) and \(y\) with the above conditions \(S\) is a generating set for \(M_{8n}\).

If we take the automorphism \(f \in Aut(M_{8n})\) with \(f(a) = a^{k'}, f(b) = b, \) and choose \(k'\) in such a way that \(k k' \equiv 1 (\text{mod} \ 4n)\), then \(f(a^k) = a\) and \(f(a^l b) = a^{k' l} b\). Since \(k'\) and \(l\) are odd, we can write \(k' l = 1 + 2t\), and hence \(a^{k' l} b = a^{1+2t} b = a^{2t} b\). However, \(Z(M_{8n}) = \langle a^2 \rangle\), and we see that \(a^{2t} = z \in Z(M_{8n})\) and \(f(S) = \{a, zab, a^{-1}, z^{-1} b^{-1} a^{-1}\}\), and the theorem is proved.

Now we are going to prove the main theorem.

By Theorem 3.1, \(S\) is equivalent to \(\{a, z a b, a^{-1}, (z a b)^{-1}\}\), and by Lemma 2.1 we have \(N_{\text{Aut}(\langle T \rangle)}(\rho(G)) = \rho(G) \rtimes \text{Aut}(G, S)\). It is enough to find \(\text{Aut}(G, S)\). Because of \(G = \langle S \rangle\), we have \(\text{Aut}(G, S) \leq S_4\). The group \(\text{Aut}(G, S)\) does not contain elements of order 3 because if \(\sigma \in \text{Aut}(G, S)\) fixes \(x \in S\), then it will fix \(x^{-1}\) as well. Therefore, \(|\text{Aut}(G, S)| \equiv 8\), and \(\text{Aut}(G, S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8\).

We consider the following cases:

Case I. \(\text{Aut}(G, S)\) does not contain elements of order 4.

Let \(\sigma \in \text{Aut}(G, S)\) be of order 4. Then \(\sigma\) induces a cycle of length 4 on \(S\). If \(x \in S\), obviously \(\sigma(x) = x^{-1}\) is impossible because then \(\sigma\) would be the product of two cycles. Therefore, we may assume that \(\sigma = (a, z a b, a^{-1}, (z a b)^{-1})\). Since \(z \in Z(M_{8n}) = \langle a^2 \rangle\), we set \(z = a^{2t}, t \in \mathbb{N}\).

From \(\sigma(a) = z a b, \sigma(z a b) = a^{-1}\) we obtain:

\[a^{-1} = \sigma(z a b) = \sigma(z) \sigma(a) \sigma(b) = \sigma(z) a b \sigma(b) \Rightarrow \sigma(z) = z^{-1} a^{-2} \text{ or } z^{-1} a^{-2-2n}.
\]

However, \(\sigma(a)\) can only be of the form \(\sigma(a) = a^l b\) where \(l\) is odd and hence \(a^l b = z a b, \) from which it follows that \(l = 2t + 1\).

Now:

\[\sigma(z) = \sigma(a^{2t}) = \sigma(a)^{2t} = (a^l b)^{2t} = a^{2(t-1)l} (ab)^{2t} = a^{2((t-1)l) 2^{t}} = a^{2^{l+1} t} = z^{n+l} \]

If \(\sigma(z) = z^{-1} a^{-2} = z^{n+l}, \) then \(z^{n+l+1} a^{-2} = 1, \) from which we obtain \(2t(n + l + 1) + 2 = 4mn\) for some \(m \in \mathbb{N}\). It follows that \(t(n + l + 1) = 2mn - 1, \) but the left-hand side of the last equality is even whereas its right-hand side is odd, a contradiction.

Similarly, the case \(\sigma(z) = z^{-1} a^{-2-2n}\) results in a contradiction. Therefore, \(\text{Aut}(G, S)\) cannot be isomorphic to \(\mathbb{Z}_4, D_8\).
Case II. $\text{Aut}(G, S)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is enough to prove that $\text{Aut}(G, S)$ does not contain an element $\sigma$ with $\sigma(a) = zab$ and $\sigma(zab) = a$.

From the form of the automorphism of $\text{Aut}(G)$ we have $\sigma(a) = a^k$ for some $k$, $(k, 4n) = 1$. If $\sigma(a) = zab$, then $a^k = zab$, from which we obtain $b = a^{-2t+k-1}$, which is not the case because $a$ and $b$ are independent generators of $G$.

Case III. $\text{Aut}(G, S)$ contains an element of order 2.

If we define $\sigma(a) = a^{-1}$, $\sigma(b) = a^{2nb}$, we see that the cycle structure of $\sigma \in \text{Aut}(G, S)$ on $S$ is $(a, a^{-1})(zab, (zab)^{-1})$.

Therefore, $\text{Aut}(G, S)$ is isomorphic to $\mathbb{Z}_2$. This completes the proof.

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