Unicorn metrics with almost vanishing $H$- and $\Xi$-curvatures

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Abstract: In this paper, we consider a class of almost regular $(\alpha, \beta)$-metrics constructed by Shen called unicorn metrics. First, we prove that every unicorn metric with almost vanishing $H$-curvature is a Berwald metric. Then we show that every unicorn metric with almost vanishing $\Xi$-curvature reduces to a Berwald metric.

Key words: Unicorn metric, the non-Riemannian quantity $H$, almost vanishing $\Xi$-curvature

1. Introduction

It is a long-existing open problem in Finsler geometry to find unicorns, i.e. Landsberg metrics that are not Berwaldian [3, 5]. For the sake of simpler prose, Bao referred to Landsberg metrics that are not of Berwald type as unicorns, by analogy with those mythical single-horned horse-like creatures for which no confirmed sighting is available. In [2], Asanov found a special family of unicorns that belongs to the class of nonregular $(\alpha, \beta)$-metrics. Then Shen proved that there does not exist any unicorn in the class of regular $(\alpha, \beta)$-metrics [10]. He found a more complicated family of unicorns in the class of nonregular $(\alpha, \beta)$-metrics that contains the Asanov’s metrics. Let us explain some details about the unicorns in Finsler geometry. Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an almost regular $(\alpha, \beta)$-metric defined by following:

$$\phi(s) = \exp \left[ \int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} \, dt \right],$$

where $q > 0$ and $k$ are real constants. Suppose that $\beta$ satisfies

$$r_{ij} = c(b^2a_{ij} - b_ib_j), \quad s_{ij} = 0,$$

where $c = c(x)$ is a scalar function on $M$. If $c \neq 0$, then $F$ is a Landsberg metric that is not Berwaldian. In this case, $F$ is a unicorn metric [16]. If $c = 0$, then $F$ reduces to a Berwald metric. If $k = 0$ and $c \neq 0$, then we obtain the family of unicorn metrics introduced by Asanov [2].

Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an almost regular non-Berwaldian $(\alpha, \beta)$-metric on a manifold $M$. Suppose that $F$ is not a Finsler metric of Randers type. In [16], the first author with Sadeghi proved that $F$ is a generalized Douglas–Weyl metric with vanishing S-curvature if and only if $\phi$ is given by (1). In this case, $F$ is not a Douglas metric nor a Weyl metric.

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In [1], Akbar-Zadeh considered a non-Riemannian quantity $H$ obtained from the mean Berwald curvature by covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. Akbar-Zadeh proved that for a Finsler metric of scalar curvature, the curvature is a scalar function on the manifold if and only if $H = 0$. It is remarkable that the quantity $H = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of $E$ along geodesics, where $H_{ij} := E_{ij|m} y^m$. A Finsler metric $F$ is called of almost vanishing $H$-curvature if

$$H = \frac{n+1}{2} F^{-1} \theta \ h,$$

where $\theta := \theta_i(x) y^i$ is a 1-form on $M$ and $h = h_{ij} dx^i \otimes dx^j$ is the angular metric. In [7], Najafi-Shen and the first author establish an equation between the the curvature and quantity $H$ for Finsler metrics of scalar curvature. They generalized Akbar-Zadeh’s theorem and proved that a Finsler metric has almost isotropic curvature $K = \frac{3}{F} + \sigma$ if and only if $H = (n + 1)/2 F h$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i(x) dx^i$ is a 1-form on $M$. Then Mo established a new fundamental equation between non-Riemannian quantity $H$ and Riemannian quantities on a Finsler manifold [6]. This motivates us to consider unicorn metrics with almost vanishing $H$-curvature.

**Theorem 1** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ defined by (1), where $q > 0$ and $k$ are real constants such that $q \neq \sqrt{3k}$. Suppose that $\beta$ satisfies (2) for some scalar function $c = c(x)$ on $M$. Then $F$ has almost vanishing $H$-curvature if and only if $\beta$ is parallel with respect to $\alpha$. In this case, $F$ is a Berwald metric.

Let $(M, F)$ be an $n$-dimensional Finsler manifold. The non-Riemannian quantity $\Xi$-curvature $\Xi = \Xi_i dx^i$ on the tangent bundle $TM$ is defined by

$$\Xi_i := S_{ij|m} y^m - S_{ji},$$

where $S$ denotes the S-curvature, and “.” and “|” denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of $F$, respectively [11]. The Finsler metric $F$ is said to be of almost vanishing $\Xi$-curvature if its $\Xi$-curvature can be expressed as follows:

$$\Xi_i = -(n + 1) F^2 \left( \frac{\theta}{F} \right) y^i,$$

where $\theta := \theta_i(x) y^i$ is a 1-form on $M$.

**Theorem 2** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ defined by (1). Suppose that $\beta$ satisfies (2) for some scalar function $c = c(x)$ on $M$. Then $F$ has almost vanishing $\Xi$-curvature if and only if $\beta$ is parallel with respect to $\alpha$. In this case, $F$ is a Berwald metric.

By Theorems 1 and 2, we conclude the following.

**Corollary 3** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ defined by the following:

$$\phi(s) = \exp \left[ \int_0^s \frac{kt + q \sqrt{b^2 - t^2}}{1 + kt^2 + qt \sqrt{b^2 - t^2}} dt \right],$$

where $k$, $q$, $b$ are constants such that $k > 0$, $q > 0$, and $b > 0$. Then $F$ is a Berwald metric.
where \( q > 0 \) and \( k \) are real constants such that \( q \neq \sqrt{3}k \). Suppose that \( \beta \) satisfies \( r_{ij} = c(b^2a_{ij} - b_ib_j) \) and \( s_{ij} = 0 \) for some scalar function \( c = c(x) \) on \( M \). Then the following are equivalent:

(i) \( F \) has almost vanishing \( H \)-curvature;

(ii) \( F \) has almost vanishing \( \Xi \)-curvature.

In this case, \( F \) reduces to a Berwald metric.

According to Corollary 3, every unicorn metric has almost vanishing \( H \)-curvature if and only if it has almost vanishing \( \Xi \)-curvature.

2. Preliminary

Given a Finsler manifold \((M, F)\), a global vector field \( G \) is induced by \( F \) on \( TM_0 \), which in a standard coordinate \((x^i, y^i)\) for \( TM_0 \) is given by

\[
G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},
\]

where \( G^i(x, y) \) are local functions on \( TM_0 \) satisfying \( G^i(x, y) = \lambda^2 G^i(x, y) , \lambda > 0 \), and given by

\[
G^i = \frac{1}{4} g^i_{jkl} \left[ \frac{\partial^2 F^2}{\partial x^j \partial y^k} y^k - \frac{\partial F^2}{\partial x^j} \right].
\]

\( G \) is called the associated spray to \((M, F)\). The projection of an integral curve of the spray \( G \) is called a geodesic in \( M \). \( F \) is called a Berwald metric if \( G^i \) are quadratic in \( y \) for any \( x \in M \) \([9, 17, 18]\).

For \( y \in T_x M_0 \), define \( B_y : T_x M \otimes T_x M \otimes T_x M \to T_x M \) and \( L_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R} \) by

\[
B_y(u, v, w) := B^i_{jkl}(y) w^j v^k w^l \frac{\partial}{\partial x^i} |_x \quad \text{and} \quad L_y(u, v, w) := L_{ijk}(y) w^i v^j w^k,
\]

where

\[
B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad L_{ijk} := -\frac{1}{2} y^i B_{jkl}.
\]

\( B \) and \( L \) are called Berwald curvature and Landsberg curvature, respectively. Then \( F \) is called a Berwald metric and Landsberg metric if \( B = 0 \) and \( L = 0 \), respectively \([12]\). By the definition, every Berwald metric is a Landsberg metric. According to Shen’s paper, the Finsler metric defined by (1) is a non-Berwaldian Landsberg metric.

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann–Hausdorff volume form \( dV_F = \sigma_F(x) dx^1 \cdots dx^n \) is defined by

\[
\sigma_F(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\left\{ (y^i) \in \mathbb{R}^n \mid F(y^i \frac{\partial}{\partial x^i} ) < 1 \right\}}.
\]

In general, the local scalar function \( \sigma_F(x) \) cannot be expressed in terms of elementary functions, even \( F \) is locally expressed by elementary functions \([4]\). Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature can be defined by

\[
S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right],
\]

where \( y = y^i \frac{\partial}{\partial x^i} |_x \in T_x M \) \([14, 15]\).
The E-curvature $E = E_{ij}dx^i \otimes dx^j$ and $\Xi$-curvature $\Xi = \Xi_i dx^i$ are given by $E_{ij} = \frac{1}{2}S_{i,j}$ and $\Xi_i := S_{i|m}y^m - S_{i|i}$, where "\(\cdot\)" and "\(|\cdot\)" denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of $F$, respectively.

The quantity $H_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $E$ along geodesics. More precisely, $H_{ij} := E_{ij|m}y^m$. In local coordinates,

$$H_{ij} = \frac{1}{2} \left[ y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - 2G^m \frac{\partial^3 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^i} \frac{\partial G^m}{\partial y^j} \frac{\partial G^m}{\partial y^k} \frac{\partial G^m}{\partial y^l} \right].$$ \hspace{1cm} (7)

A Finsler metric $F$ is called of almost vanishing $H$-curvature if $H = (n+1)/(2F)\theta y$, where $\theta := \theta_i(x)y^i$ is a 1-form on $M$ and $\theta = h_{ij}dx^i \otimes dx^j$ is the angular metric.

An $(\alpha, \beta)$-metric is a Finsler metric of the form $F := \alpha \phi(\frac{\beta}{n})$, where $\phi = \phi(s)$ is a $C^\infty$ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $M$ (see [13]). For an $(\alpha, \beta)$-metric $F := \alpha \phi(s)$, $s = \beta/\alpha$, let us define $b_{ij}$ by $b_{ij} := dB_i - b_j \theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^k_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let

$$r_{ij} := \frac{1}{2} \left[ b_{ij} + b_{ji} \right], \quad s_{ij} := \frac{1}{2} \left[ b_{ij} - b_{ji} \right].$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

To satisfy that $F$ is positive and strongly convex on $TM_0$, it is known that if and only if

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (B - s^2)\phi''(s) > 0, \quad |s|^2 \leq B < b_0$$

where $B := b^2 = ||\beta||_\alpha^2$.

Let $G^i = G^i(x, y)$ and $G^i _\alpha = G^i _\alpha(x, y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. Then we have

$$G^i = G^i _\alpha + \alpha Q s_0^i + (r_{00} - 2Qs_0) \left( \frac{\theta^i_\alpha}{Q} + \Psi b^i \right),$$ \hspace{1cm} (8)

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}.$$

3. Proof of Theorem 1

In this section, we are going to find the formula of $H$-curvature of $(\alpha, \beta)$-metrics. First, we remark on the formula of $E$-curvature of $(\alpha, \beta)$-metrics.

Lemma 4 [8] Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric. Put $\Omega := \Phi/(2\Delta^2)$, where

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'.$$
Then the $E$-curvature of $F$ is given by the following:

\[ E_{ij} = C_1 b_i b_j + C_2 (b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5 (r_{i0} b_j + r_{j0} b_i) + C_6 (r_{i0} y_j + r_{j0} y_i) + C_7 r_{ij} + C_8 (s_i b_j + s_j b_i) + C_9 (s_i y_j + s_j y_i) + C_{10} (r_i b_j + r_j b_i) + C_{11} (r_i y_j + r_j y_i), \]

where

\[ C_1 := \frac{1}{2\alpha^4 \Delta^2} \left\{ \Phi \alpha Q'' s_0 + 2\alpha \Delta^2 \Psi'' r_0 - \Delta^2 \Omega'' r_0 + 2\Delta^2 \alpha \Omega'' Q s_0 + 4\Delta^2 \alpha \Omega' Q' s_0 + 2\alpha \Delta^2 \Psi'' s_0 \right\}, \]

\[ C_2 := -\frac{1}{2\alpha^4 \Delta^2} \left\{ 2\alpha \Delta^2 \Psi'' s_0 - 2\Omega' \Delta^2 r_0 + 2\Omega' \Delta^2 \alpha Q s_0 - \Delta^2 \Omega'' s_r 0 + 2\Delta^2 \alpha \Omega'' s Q s_0 \right\}, \]

\[ C_3 := \frac{1}{4\alpha^4 \Delta^2} \left\{ 4\Delta^2 s^2 \Omega'' \alpha Q s_0 - 2\Delta^2 s^2 \Omega'' r_0 + 12\alpha \Delta^2 \Psi' s r_0 + 12\alpha \Delta^2 \Psi' s s_0 + 4\alpha \Delta^2 \Psi'' s^2 r_0 \right\}, \]

\[ C_4 := -\frac{1}{4\alpha^4 \Delta^2} \left\{ 4\alpha \Delta^2 \Psi' s s_0 - \Phi r_0 - 2\Omega' \Delta^2 s r_0 + 4\Omega' \Delta^2 s a Q s_0 + 4\alpha \Delta^2 \Psi' s r_0 + 2\Phi \alpha Q' s s_0 \right\}, \]

\[ C_5 := -\frac{\Phi'}{2\alpha}, \quad C_6 := \frac{2\Delta^2 s \Psi + \Phi}{\alpha}, \quad C_7 := \frac{-\Phi}{2\alpha \Delta^2}, \]

\[ C_8 := \frac{1}{\alpha} \left\{ 2 \Omega' \Delta^2 Q + 2 \Delta^2 \Psi' + \Phi' Q' \right\}, \]

\[ C_9 := -\frac{s}{\alpha} C_8, \quad C_{10} := \frac{\Psi'}{\alpha}, \quad C_{11} := -\frac{s}{\alpha} C_{10}. \]

By Lemma 4, we get the following.

**Proposition 5** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$. Then the $H$-curvature of $F$ is given by the following:

\[ H_{ij} = C_1 \left[ (r_{i0} + s_{i0}) b_j + (r_{j0} + s_{j0}) b_i \right] + C_2 b_i b_j + C_3 \left[ (r_{i0} + s_{i0}) y_j + (r_{j0} + s_{j0}) y_i \right] + C_4 (b_i y_j + b_j y_i) + C_5 y_i y_j + C_6 a_{ij} + C_7 \left[ r_{i0} b_j + r_{j0} b_i + r_{i0} (r_{j0} + s_{j0}) + r_{j0} (r_{i0} + s_{i0}) \right] + C_8 (r_{i0} b_j + r_{j0} b_i) + C_9 \left[ r_{i0} y_j + r_{j0} y_i \right] + C_{10} (r_{i0} y_j + r_{j0} y_i) + C_{11} r_{ij} + C_{12} r_{ij} \]

\[ + C_{13} \left[ s_i b_j + s_j b_i + s_i (r_{j0} + s_{j0}) + s_j (r_{i0} + s_{i0}) \right] + C_{14} (s_i b_j + s_j b_i) + C_{15} \left[ s_{i0} y_j + s_{j0} y_i \right] + C_{16} (s_i y_j + s_j y_i) + C_{17} \left[ r_{i0} b_j + r_{j0} b_i + r_i (r_{j0} + s_{j0}) \right] + r_i (r_{i0} + s_{i0}) + C_{18} (r_i b_j + r_j b_i) + C_{19} (r_i y_j + r_j y_i) + C_{20} (r_i y_j + r_j y_i), \]

(10)
where

\[ C_1 := \frac{1}{2\alpha^3\Delta^2} \left[ \Phi_0 Q_{ss}s_0 + \Delta^2 (2\alpha \Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha \Omega_{ss}Q_{ss}s_0 + 4\alpha \Omega_{ss}Q_{ss}s_0 + 2\alpha \Psi_{ss}s_0) \right], \]

\[ C_2 := -\frac{\Delta^1}{\alpha^3\Delta^2} \left[ \Phi_0 Q_{ss}s_0 + \Delta^2 (2\alpha \Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha \Omega_{ss}Q_{ss}s_0 + 4\alpha \Omega_{ss}Q_{ss}s_0 + 2\alpha \Psi_{ss}s_0) \right], \]

\[ + \frac{1}{2\alpha^3\Delta^2} \left[ \alpha (\Phi_0^0 Q_{ss}s_0 + \Phi Q_{ss}s_0 + \Phi Q_{ss}s_0^0) + 2\alpha \Delta \left( 2\alpha \Psi_{ss}r_0 + \Delta \Psi_{ss}r_0 + \Delta \Psi_{ss}r_0^0 \right) \right] \]

\[-\Delta (2\Delta_{ss}Q_{ss}s_0 - \Delta \Omega_{ss}s_0 - \Delta \Omega_{ss}s_0^0) + 2\alpha \Delta \left( 2\Delta_{ss}Q_{ss}Q_{ss}s_0 + \Delta \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0^0 \right) \]

\[ + 2\alpha^2 \Omega_{ss}s_0 + 4\alpha \left( 2\Delta_{ss}Q_{ss}s_0 + \Delta^2 \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0^0 \right) \]

\[ + 2\alpha \left( 2\Delta_{ss} \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0^0 \right) \right] \]

\[ C_3 := -\frac{1}{2\alpha^3\Delta^2} \left[ 2\alpha \Delta^2 \Psi_{ss}s_0 - 2\Omega_{ss} \Delta^2 r_0 + 2\Omega_{ss} \Delta^2 \alpha Q_{ss}s_0 - \Delta^2 \Omega_{ss}s_0 + 2\Delta^2 \alpha \Omega_{ss}s_0 \right] \]

\[ + 4\Delta^2 \alpha \Omega_{ss}s_0r_0 + 2\alpha^2 \Delta^2 \Psi_{ss}s_0 + 2\alpha^2 \Psi_{ss}s_0r_0 + 2\alpha \Psi_{ss}s_0 \right] \];

\[ C_4 := -\frac{1}{2\alpha^3\Delta^2} \left[ 2\alpha \left( 2\Delta_{ss} \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0 + 2\alpha \Omega_{ss}s_0 - 2\Omega_{ss} \Delta^2 r_0 - 4\Omega_{ss} \Delta_{ss} \right) \right] \]

\[-2\Omega_{ss} \Delta^2 r_0 + 2\alpha \left( \Omega_{ss}s_0 + 2\Omega_{ss} \Delta_{ss}Q_{ss}s_0 + \Omega_{ss} \Delta^2 Q_{ss}s_0 + \Omega_{ss} \Delta^2 Q_{ss}s_0^0 \right) \]

\[-2\Delta_{ss} \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0^0 - \Delta^2 \Omega_{ss}s_0^0 \]

\[ + 2\alpha \left( 2\Delta_{ss} \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0 + \Delta^2 \Omega_{ss}s_0^0 + \Delta^2 \Omega_{ss}s_0^0 \right) \]

\[ + 2\alpha \left( 2\Delta_{ss} \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0^0 + \Delta^2 \Psi_{ss}s_0^0 \right) \]

\[ + \Delta^2 \Psi_{ss}s_0 + \Delta^2 \Psi_{ss}s_0^0 + \Delta^2 \Psi_{ss}s_0^0 \right] 

\[ + \alpha \left( \Phi_0 Q_{ss}s_0 + \Phi Q_{ss}s_0 + \Phi Q_{ss}s_0^0 + \Phi Q_{ss}s_0^0 + \Phi Q_{ss}s_0^0 + \Phi Q_{ss}s_0^0 \right) \]

\[ + \Delta_{ss}^3 \left[ 2\alpha \Delta^2 \Psi_{ss}s_0 + 2\Omega_{ss} \Delta^2 \alpha Q_{ss}s_0 + 2\alpha \Omega_{ss} \Delta^2 Q_{ss}s_0 + 4\Delta^2 \Omega_{ss} \Delta^2 Q_{ss}s_0 + 2\alpha \Delta^2 \Omega_{ss} \Delta^2 Q_{ss}s_0 \right] \]
\[ C_5 := \frac{1}{4\alpha^3\Delta^2} \left[ 4\alpha \left( 2\Delta\Delta_0^2\Omega_s Q_s s_0 + 2\Delta^2 \frac{r_{00}}{\alpha} \Omega_{ss} Q_s s_0 + \Delta^2 \frac{\Omega_{ss}}{\alpha} Q_s s_0 + \Delta^2 s^2 \Omega_{ss} Q_{s0} s_0 \right) + \Delta^2 \Omega_{ss} Q_{s0} s_0 - 2 \left( 2\Delta\Delta_0^2 \Omega_{ss} r_0 + 2\Delta^2 \frac{r_{00}}{\alpha} \Omega_{ss} r_0 + \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 r_0 + \Delta^2 s^2 \Omega_{ss} r_0 \right) \right. \\
+ 12\alpha \left( 2\Delta\Delta_0^2 \Psi_s s_0 r_0 + 2\Delta^2 \frac{r_{00}}{\alpha} \Psi_s s_0 r_0 + 2\Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 r_0 + 2\Delta\Delta_0^2 \Psi_s s_0 r_0 \right) \\
+ \Delta^2 \frac{\Psi_s s_0}{\alpha} + \Delta^2 \frac{\Psi_s s_0}{\alpha} r_0 + \Delta^2 \Psi_s s_0 s_0 r_0 + 4\alpha \left( 2\Delta\Delta_0^2 \Psi_s s_0^2 r_0 + 2\Delta^2 \frac{\Omega_{ss}}{\alpha} s_0^2 r_0 \right) \\
+ 2\Delta^2 \frac{\Psi_s s_0^2}{\alpha} r_0 + 2\Delta^2 \frac{\Psi_s s_0^2}{\alpha} s_0^2 r_0 + 2\Delta^2 \Psi_s s_0^2 s_0 + 2\Delta^2 \frac{\Omega_{ss}}{\alpha} s_0^2 s_0 + 2\Delta^2 \Psi_s s_0^2 s_0 \\
+ \Delta^2 \Psi_s s_0^2 s_0 r_0 + 2\alpha s_0 \left( 8\Delta\Delta_0^2 \Omega_s Q_s s_0 + 4\Delta^2 \frac{\Omega_{ss}}{\alpha} Q_s s_0 + 4\Delta^2 \frac{\Omega_{ss}}{\alpha} Q_s s_0 + 4\Delta^2 \frac{\Omega_{ss}}{\alpha} Q_s s_0 \right) \\
+ \Phi_0 Q_s s_0 + \Phi Q_s s_0 s_0 + \Phi Q_s s_0 s_0 r_0 + 4\alpha s_0 \left( 4\Delta^2 \Omega_s Q_s + \Phi Q_s s_0 \right) r_{00} - 10 \left( \Omega_s s_0 \right) \right] \\
+ 2\Delta^2 \frac{\Psi_s s_0^2}{\alpha} r_0 + 2\Delta^2 \frac{\Psi_s s_0^2}{\alpha} s_0^2 r_0 + 2\Delta^2 \Psi_s s_0^2 s_0 + 2\Phi_0 Q_s s_0 s_0 + 2\Phi Q_s s_0 s_0 s_0 \\
+ 2\Delta^2 \Psi_s s_0 + 4\alpha \Delta^2 \frac{\Psi_s s_0^2}{\alpha} r_0 + 4\alpha \Delta^2 \Psi_s s_0^2 s_0 + 8\Delta^2 \frac{\Omega_{ss}}{\alpha} Q_s s_0 + 2\Phi_0 Q_s s_0 s_0 \\
+ 10 \left( \Omega_s s_0 \right) \Delta^2 s_0 r_0 + 12 \Omega_s s_0 \Delta^2 s_0 \alpha Q_s s_0 + 6\Phi_0 Q_s s_0 s_0 s_0 - 3\Phi r_0 \right] \\
+ \frac{1}{4\alpha^3\Delta^2} \left[ 4\alpha \left( 2\Delta\Delta_0^2 \Psi_s s_0 + \Delta^2 \frac{r_{00}}{\alpha} \Psi_s s_0 + \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 \right) - \Phi_0 r_0 - \Phi r_0 \right] \\
- 2 \left( \Omega_s s_0 \right) \Delta^2 s_0 r_0 + 2\Omega_s \Delta\Delta_0^2 s_0 r_0 + \Omega_s \Delta^2 \frac{r_{00}}{\alpha} r_0 + \Omega_s \Delta^2 s_0 r_0 \right] + \alpha \left[ 4 \left( \Omega_s s_0 \right) \Delta^2 s_0 \right] \\
+ \Delta^2 \frac{r_{00}}{\alpha} Q_s s_0 + \Omega_s \Delta^2 s_0 Q_s s_0 + \Omega_s \Delta^2 s_0 Q_s s_0 + 2\Omega_s \Delta\Delta_0^2 s_0 Q_s s_0 + 2\Delta\Delta_0^2 \Psi_s s_0 r_0 + \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 r_0 \\
+ \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 r_0 + \Delta^2 \Psi_s s_0 r_0 + 2\Phi_0 Q_s s_0 s_0 + 2\Phi Q_s s_0 s_0 s_0 + 2\Phi Q_s s_0 s_0 s_0 \\
+ \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 r_0 + \Delta^2 \Psi_s s_0 r_0 \right] \\
+ \frac{1}{4\alpha^3\Delta^3} \left[ 4\alpha \Delta^2 \Psi_s s_0 - \Phi r_0 - 2\Omega_s \Delta^2 s_0 r_0 + 4\Omega_s \Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 + 4\alpha \Delta^2 \Psi_s s_0 + 2\Phi_0 Q_s s_0 s_0 \\
+ \frac{1}{2\alpha^3\Delta} \left( \frac{2\Delta^2 s_0 \alpha + \Phi}{\alpha} - \frac{\Omega_{ss}}{\alpha} \right) \Delta_0 \right], \\
C_7 := -\frac{1}{\alpha^2} \Omega_s s_0, \quad C_8 := -\frac{1}{\alpha^2} \Omega_{ss} s_0, \quad C_9 := \frac{1}{2\alpha^3\Delta} \left( \frac{2\Delta^2 s_0 \alpha + \Phi}{\alpha} \right), \\
C_{10} := \frac{4\Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 + 2\Delta^2 \frac{\Omega_{ss}}{\alpha} s_0 + \Phi s_0 + \frac{r_{00}}{\alpha} \Omega_{ss}}{2\alpha^3\Delta^2} + \frac{\Delta \left( \frac{2\Delta^2 s_0 \alpha + \Phi}{\alpha} \right)}{\alpha^3\Delta^3} \\
C_{11} := \frac{-\Phi}{2\alpha^3\Delta}, \quad C_{12} := \frac{-\Phi s_0}{2\alpha^3\Delta} + \frac{\Phi \Delta_0}{\alpha^3\Delta}, \\
C_{13} := \frac{1}{2\alpha^3\Delta} \left[ 2\Omega_s \Delta^2 Q + 2\Delta^2 \Psi_s + \Phi Q_s \right].
\[
C_{14} := \frac{1}{2a\Delta^2} \left[ 2\Omega_s|0\Delta^2Q + 4\Omega_s\Delta\Delta_0Q + 2\Omega_s\Delta^2Q_0 + 4\Delta\Delta_0\Psi_s + 2\Delta^2\Psi_s|0 + \Phi_0Q_s + \Phi\Phi Q_s|0 \right],
\]
\[
- \frac{\Delta|0}{\alpha\Delta^3} \left[ 2\Omega_s\Delta^2Q + 2\Delta^2\Psi_s + \Phi\Phi Q_s \right],
\]
\[
C_{15} := -\frac{s}{\alpha} C_{13}, \quad C_{16} := -\frac{s}{\alpha} C_{14} - \frac{r_{00}}{\alpha^2} C_{13}, \quad C_{17} := \frac{1}{\alpha}\Psi_s, \quad C_{18} := \frac{1}{\alpha}\Phi_s|0,
\]
\[
C_{19} := -\frac{s}{\alpha} C_{17}, \quad C_{20} := -\frac{r_{00}}{\alpha^2} C_{17} - \frac{s}{\alpha} C_{18},
\]

and
\[
\Delta_0 = Q_0 s + \frac{r_{00}}{\alpha} Q + 2 \left[ r_0 + s_0 - \frac{s r_{00}}{\alpha} \right] Q_s + (b^2 - s^2) Q_s|0
\]
\[
\Phi_0 = - \left[ n\Delta_0 + \frac{r_{00}}{\alpha} Q + sQ_0 \right] (Q - sQ_s) - (Q_0 - \frac{r_{00}}{\alpha} Q_s - sQ_s|0)(n\Delta + 1 + sQ)
\]
\[
- 2 \left[ r_0 + s_0 - \frac{s r_{00}}{\alpha} \right] (1 + sQ) Q_{ss} - (b^2 - s^2) \left[ \frac{r_{00}}{\alpha} QQ_{ss} + sQ_0 Q_{ss} + (1 + sQ) Q_{ss|0} \right].
\]

**Proof of Theorem 1**: For the Finsler metric defined by (1), we get
\[
Q := ks + q\sqrt{b^2 - s^2}, \quad \Psi = \frac{k\sqrt{b^2 - s^2} - qs}{2(1 + kb^2)\sqrt{b^2 - s^2}}. \tag{11}
\]

Using the Maple program, we can obtain the quantity \( H \) for the Finsler metric defined by (1). By using Proposition 5, putting \( s_{ij} = 0 \) and \( r_{ij} = c(b^2a_{ij} - b_ib_j) \) in (10) and decomposition of the rational (Rat) and irrational (Irrat) parts, we have the following:
\[
H_{jk} = (\text{Rat})_{jk} + \alpha(\text{Irrat})_{jk}, \tag{12}
\]

where \( \text{Rat} \) and \( \text{Irrat} \) are listed in Appendix 4. It is remarkable that, if \( H = (n + 1)/(2F)\theta h \) then by equating the parts of rational and irrational parts in the two sides of the equation, we get two simple equations. By the obtained equations, one can get the desired result.

Now suppose that \( F \) has almost vanishing \( H \)-curvature on a \( n \)-dimensional manifold \( M \). Then
\[
H_{ij} = \frac{n + 1}{2} F^{-1} \theta h_{ij}, \tag{13}
\]

where \( \theta := \theta_i(x)y^i \) is a 1-form on \( M \). Substituting (12) in (13), we get
\[
(n + 1)\theta h_{jk} = 2F [(\text{Rat})_{jk} + \alpha(\text{Irrat})_{jk}], \tag{14}
\]
Multiplying (14) by $b^ib^k$ and using the Maple program, we get the following:

$$
A \left[ q \left( 12b^8c^2\beta^2 - 24b^{10}c^2\beta^2k \right) \alpha^{14} + q \left( 12b^{10}c^2q^2\beta^4 - 24b^{10}c^2\beta^4k^2 + 90kb^{10}c^2\beta^4 - 106kb^8c^2\beta^4 - 45b^8c^2\beta^4 + 53b^6c^2\beta^4 - 8\beta^5(kb^8 + b^6)c_{i0} \right) \alpha^{12} + q\beta^5 \left( 90b^{10}c^2\beta k^2 - 45b^{10}c^2\beta q^2 - 118b^8c^2\beta^2k^2 + 57b^8c^2\beta q^2 - 540b^8c^2\beta k + 718b^6c^2\beta k + 180b^6c^2\beta - 181b^4c^2\beta + 8(kb^6 + 2b^4 - k^2b^8)c_{i0} \right) \alpha^{10}
+ q\beta^7 \left( b^2c^2\beta(225q^2b^6 - 495k^2b^6 + 665k^2b^4 - 282q^2b^4 + 1260kb^4 - 1178kb^2 - 270b^2 + 155) + 8(kb^4 + 2k^2b^6 - b^2)c_{i0} \right) \alpha^8 + q\beta^9 \left( 1080b^6c^2\beta k^2 - 450b^6c^2q^2\beta - 9976c^2k^2\beta - 384b^6c^2\beta - 1440b^6c^2k\beta + 746b^6c^2k^2 + 180b^2c^2\beta - 39c^2\beta - 8b^4c_0k^2 - 8b^2c_0k \right) \alpha^6
+ q\beta^{12} \left( 450b^4c^2q^2 - 1170b^4c^2k^2 + 591b^2c^2k^2 - 210Bc^2q^2 + 810b^2c^2k - 156c^2k - 45c^2 \right) \alpha^4
+ q\beta^{14} \left( 630b^2k^2 - 225b^2q^2 - 117k^2 + 39q^2 - 180k \right) \alpha^2 - 45qc^2\beta^{16}(3k^2 - q^2) \right] = 0,
$$

(15)

where

$$
A := -\frac{b^2q}{4\alpha^8(1 + kb^2)^2(b^2\alpha^2 - \beta^2)^2(\alpha^2 + k\beta^2)^2}.
$$

The only equation that does not contain $\alpha^2$ is $45c^2\beta^{16}q(3k^2 - q^2)$. Since $\beta^{18}$ is not divisible by $\alpha^2$, then we have two cases as follows:

**Case (b1):** $c^2\beta^{18}q(-q^2 + 3k^2)$ contains $\alpha^2$. Since $q$ and $k$ are real constants, then they cannot contain $\alpha$. Thus this case will not happen.

**Case (b2):** $c^2\beta^{18}q(-q^2 + 3k^2) = 0$. In this case, $c(x) = 0$ or $q^2 = 3k^2$. Since $q \neq 3\sqrt{k}$, then $c = 0$ and then $r_{ij} = 0$. In this case, $F$ reduces to a Berwald metric. $\square$

### 4. Proof of Theorem 2

Now we are going to consider the Finsler metric defined by (1) with almost vanishing $\Xi$-curvature.

**Proof of Theorem 2:** Suppose that $\beta$ satisfies

$$
 r_{ij} = c(b^2a_{ij} - b_ib_j) \quad \text{and} \quad s_{ij} = 0,
$$

where $c = c(x)$ is a scalar function on $M$. Using the Maple program, we can obtain the quantity $\Xi$ for the unicorn metric. By putting $s_{ij} = 0$ and decomposition of the rational (Rat) and irrational (Irrat) parts, we have the following:

$$
\Xi_i := (\text{Rat})_i + (\text{Irrat})_i,
$$

(16)

where *Rat* and *Irrat* are listed in Appendix 4.
By assumption, $F$ has almost vanishing $\Xi$-curvature. Thus there is a 1-form $\theta := \theta_i(x)y^i$ on $M$ such that the following holds:

$$\Xi_i = -(n+1)F^2 \left( \frac{\theta}{F} \right)_y,$$

(17)

By substituting (16) in (17), we get

$$(n+1)(F_i \theta - \theta_i F) = (\text{Rat})_i + (\text{Irrat})_i.$$  

(18)

By substituting (16) in (17), we get

$$(n+1)(F_i \theta - \theta_i F) = (\text{Rat})_i + (\text{Irrat})_i.$$  

(18)

Multiplying (18) with $b^i$ and using the Maple program, we get the following:

$$q^2 \left[ 2c(c(kb^2 + 1)b^4 + \beta \left( 2nb^2 c^2 + (knb^2 - knb^4 - kb^4 + kb^2 - nb^2 + n + 1)c_i b^i \right) \alpha^6 
+ 4nc\beta^2(k2b^6 - kb^2 - 2b^2) + \beta^2(knb^2 c^2 b^4 + 6knb^2 c^2 \beta - nc^2 b^4 \beta + knb^4 c_i - 2nb^2 c^2 \beta 
+ knb^2 c_i b^i \beta + kb^2 c_i b^i \beta + nb^2 c_i b^i \beta + c_i b_i \beta) \alpha^4 + c\beta^4 (6b^4 k^2 - 10b^2 k)^3 
+ n\beta^4 \left( c^2 \beta (1 - 2kb^4 - 11kb^2) - (kb^2 + n)c_i (0) \right) \alpha^2 + 12nc(1 + kb^2)\beta^6 \alpha + 6nc^2 \beta^7 \right] = 0.$$  

(19)

By assumption, $q \neq 0$. In the last sentence of (19), $6nc^2 \beta^7$ contains $\alpha$. This case cannot happen, because $k$ is a real number. Thus $k$ and $\beta$ cannot contain $\alpha$. Since $c = c(x)$, then the equation cannot contain $\alpha$. Therefore, $c = 0$. In this case, $F$ reduces to a Berwald metric. 

References


Appendix A. Coefficients in (12)

\[(\text{Rat})_{jk} = \mathbf{A} \left[ \beta \left( 6knq^b c^2 \beta \ a_{jk} + 10^8 c^2 \beta nqa_{jk} + 6^6 c^2 \beta nqy_{jk} + 10^8 c^2 \beta knq_{jk} + 26^6 c^2 \beta knqb_{jk} \right) 
- 12^8 c^2 \beta kqb_{jk} + 6^6 c^2 \beta qy_{jk} - 26^8 c^2 \beta knqb_{jk} + 6^6 c^2 \beta qy_{jk} - 26^8 c^2 \beta knqb_{jk} \right] \alpha_{14} + \left( 6^8 c^2 \beta^3 q^3 b_{jk} \right) \]
\[-174b^2c^2\beta^4k^2q_b y_k - 174b^2c^2\beta^4k^2q_b y_{j} + 66b^2c^2\beta^3 q^3_b y_k + 570b^2c^2\beta^3k^2q_{yj} - 86b^2c_{10}\beta^3k^2q_{ajk} \]
\[-16c^2\beta^3kq_a y_k + 96c^2\beta^3kq_y y_k + 96c^2\beta^3kq_{by_k} + 1805c^2\beta^3q_{yj} - 39c^2\beta^3q_{yj} y_k - 8c_{10}\beta^3kq_{ajk} \]
\[-16b^2c^2\beta^10k^2q_a y_k - 1440b^2c^2\beta^8kq_{yj} y_k \alpha^4 + \left( 450b^8c^2\beta^{10}q^3 y_k - 1170b^8c^2\beta^{10}k^2q_{yj} y_k \right) \]
\[+ 447b^2c^2\beta^9k^2q_{yj} y_k \]
\[-24c^2\beta^311q^3_b y_k + 810b^2c^2\beta^310kq_y y_k - 156b^2c^2\beta^11kq_{yj} y_k - 45c^2\beta^11q_{yj} y_k \alpha^4 + c^2\beta^12y_k y_k \left( 630b^2k^2q \right) \]
\[-225b^2q^3 - 117k^2q + 39q^3 - 180kq \right) \alpha^4 + q c^2\beta^14\left( 45q^2 - 135k^2 \right) q_{yj} \right]. \] (A.1)
Appendix B. Coefficients in (A.2)

\[-150b^2c^7\beta \phi^2 c_i j k n + 150b^2c^2 \beta^3 k q^2 b_i j c_i + 39c^3 \beta^7 k y j k y c_i - 180b^5c^2 \beta^7 k^3 j y j y k - 8b^5c^2 \beta^2 k q^2 a_j k + 42b^2c^2 \beta^8 k^3 b_i j y + 42b^2c^2 \beta^8 k^3 b_i j y - 540b^5c^2 \beta^7 k^2 j y j y k - 540b^5c^2 \beta^7 q^2 j y j y k - 24b^2c^2 \beta^7 k^3 y j y k + 24b^2c^2 \beta^7 q^2 j y j y k - 4b^2c^2 \beta^3 k q^2 a_j k - 180b^5c^2 \beta^7 k^3 j y j y k + c^3 \beta^3 k^2 a_j k - 8c^3 \beta^3 q^2 a_j k + 4c^2 \beta^2 q^2 a_j k - 4c^2 \beta^2 q^2 a_j k + 4b^2c^2 \beta^3 k^3 a_j k + 4b^2c^2 \beta^3 k^3 a_j k - 48c^2 \beta^2 q^2 b_i j y - 48c^2 \beta^2 q^2 b_i j y + 270b^5c^2 \beta^3 k^3 j y j y k - 990b^4c^2 \beta^3 k q^2 j y j y k - 123b^2c^2 \beta^3 k^3 j y j y k + 40b^2c^2 \beta^3 q^2 j y j y k - 24b^2c^3 k^3 b_i j y - 24b^2c^3 k^3 b_i j y + 72b^2c^3 k q^2 b_i j y + 72b^2c^3 k q^2 b_i j y - 36b^2c^2 \beta^3 k^2 j y j y k + 36b^2c^2 \beta^3 q^2 j y j y k + 78b^2c^2 \beta^3 k^2 j y j y k - 78b^2c^2 \beta^3 q^2 j y j y k + 45c^2 \beta^3 k q^2 j y j y k D^6 + \left(585b^2k^2 - 180b^2k^3 + 39ck^2 - 117kq^2 + 90k^2 - 90q^2\right) a^2 + k c^2 \beta^3 (45k^2 - 135q^2) y j y k D \right].

where \(D := \sqrt{b^2a^2 - \beta^2}\) and

\[
\begin{align*}
\Lambda &:= -\frac{b^2q}{4\alpha^8(b^2a^2 - \beta^2)^2(1 + kb^2)^2(\beta^2 k + \alpha^2)^2}, \\
\Pi &:= -\frac{b^2q}{4\alpha^8(b^2a^2 - \beta^2)^2(1 + kb^2)^2(\sqrt{b^2a^2 - \beta^2} \beta q + \beta^2 k + \alpha^2)^2}.
\end{align*}
\]

Appendix B. Coefficients in (16)

\[
(Rat)_i = \Gamma \left[ b^7b^4 c k n c_i - b^2c^2 n b_i + b^4 c & k c_i - b^2 k^2 c i c_i + b^2 c n c_i + b^2 c i - n c_i - c_i) \alpha^8 
- 2b^4 \beta c k n b_i V a \beta - \beta \left( 2b^6 y c^2 k - 2b^4 y c^2 k + b^4 y c^2 n + 2 b^2 \gamma c_i \beta - 2b^2 b_i \beta c^2 
- 2b^2 y c^2 + 2b n b_i \beta c^2 - b^2 c_i \beta k + 2b^2 c_i \beta n + b^2 c n b_i - c_i \beta - 2b^2 b_i \beta c^2 k + n b^2 y c^2 k 
- 2b^4 y c^2 k - 2 b n b^2 c_i \beta k + 2b^2 b_i \beta c^2 k - 3b^2 \gamma b_i \beta c^2 n - b^2 c_i \beta n + b^2 c n b_i k n 
+ 2 b n b^2 c_i \beta + b^4 y c^2 c k + 2b^4 b_i \beta c^2 k + 2 b n b^2 b_i \beta c^2 + 2 b n c^2 c_i \beta + 2 b^4 y c^2 
- 2b^2 y c^2 + 2b n c^2 c_i \beta - c_i \beta n) \alpha^5 + 2b^2 \beta^2 c k n V \left( b^2 y c^2 + 4 b^2 \gamma c b_i \right) \alpha^5 + \beta^2 \left( 2b^2 y b^4 \beta c^2 k n 
+ 2b^4 \gamma y c^2 k n + b^4 \gamma y c^2 c k + 2b^2 \gamma c_i \beta k + 2b^2 b_i \beta c^2 k n + 5b^2 b_i \beta c^2 k n - 2b^2 b_i \beta c^2 k 
+ b^2 c_i \beta k n + 2b^2 c_i \beta n + 2b^2 c n b_i \beta k n + b^2 c n b_i \beta n + b^2 c n b_i \beta n + b^2 c_i \beta k n + b^2 c_i \beta n - 2b \gamma b^2 c^2 n + b^2 c_i \beta k n + 2b^2 c_i \beta n + c_i \beta n + c_i \beta^2 n \right) \alpha^4 
- 2\beta^4 c k n V \left( 4b^2 y c^2 + 3 b^2 \gamma b_i \right) \alpha^3 - n b^4 \left( y b^4 \beta c^2 k + 2b^2 y c^2 \beta k + 6b^2 y c^2 k + 3b^2 y c^2 k 
+ b^2 c n b_i + 2 c^2 y c \beta + c_i y c \right) \alpha^2 + 6b^4 c k n y_i V \alpha + 3b^2 c n k y_i \right],
\end{align*}
\]
\[ \text{Irrat} = \mathbf{T} \left[ b^6 c^2 knqb_0 \alpha^{10} + \left( 2b^6 c^3 k^2 qb_i - b^6 c^2 knqb_i + b^6 c_0 knq - b^6 c^2 nqb_i + 2b^6 c \beta kqb_i + b^6 c_0 kq \\
\quad - b^6 c_i knq - b^6 c_0 kq + b^6 c_{0i} q - b^6 c_i q - b^6 c_{0i} q - b^6 c_i q \right) \alpha^9 + \left( -2q \beta (b^6 c k^2 b_i + b^6 c kq) \mathbf{D}_0^2 \right) + b^6 c_i \beta knq_i - 2b^6 c_i \beta kq_i - 2b^6 c^2 \beta^2 knb_i - 2b^6 c_i \beta knb_i - 2b^6 c_i \beta^2 kq_i \right] \cdot (\alpha^6 + r) + \left( 2b^6 c^2 \beta q_{yi} + 8b^6 c^3 k^2 qb_i + 2b^6 c^3 kq_i y_i + 8b^6 c^3 kq_i (b^6 c_0 kq) \mathbf{D}_0^2 + (2b^6 c^3 k^2 qy_i + 8b^6 c^3 kq_i + 8b^6 c^3 kq_i nq_i + 2b^6 c^2 \beta^2 q_{yi} + 2b^6 c^2 \beta^2 q_{yi} + 2b^6 c^2 \beta^2 knq_i + 2b^6 c^2 \beta^2 knq_i) \alpha^6 \right) \cdot \left( \left[ b^6 c_0 \beta^3 knq_i + 10b^6 c_0 \beta^3 q_{yi} + 2b^6 c_0 \beta^3 q_{yi} + 14b^6 c_0 \beta^3 q_{yi} + 2b^6 c_0 \beta^3 q_{yi} + 2b^6 c_0 \beta^3 q_{yi} + 2b^6 c_0 \beta^3 q_{yi} \right] \alpha^5 \right) + \left( \left( -2b^6 c_0 \beta^2 kq_i - 6b^6 c^2 \beta^2 q_{yi} - 6b^6 c_{0i} \beta^3 q_{yi} - 8b^6 c^2 \beta^2 nq_{yi} - 8b^6 c^2 \beta^2 nq_{yi} \right) \mathbf{D}_0^2 + \left( b^6 c_0 \beta^3 kq_i - 6b^6 c_0 \beta^3 q_{yi} + b^6 c_0 \beta^3 kq_i - 3b^6 c^2 \beta^2 knq_i + 3b^6 c^2 \beta^2 knq_i + 2b^6 c^2 \beta^2 kq_i - 3b^6 c^2 \beta^2 knq_i - 8b^6 c^2 \beta^2 knq_i \right) \alpha^4 - \left( b^6 c^2 \beta kqi + 14b^6 c^2 \beta kqi + b^6 c^2 \beta kqi + 6b^6 c^3 k^2 q_{yi} + 2b^6 c^2 \beta kqi + 6b^6 c^3 k^2 q_{yi} + 3b^6 c^3 k^2 q_{yi} + 2b^6 c^3 k^2 q_{yi} + 3b^6 c^3 k^2 q_{yi} + 3b^6 c^3 k^2 q_{yi} \right) \alpha^3 \right) + \left( 6b^6 c^2 \beta^2 q_{yi} + c_{0yi} \mathbf{D}_0^2 + (6b^6 c^3 k^2 q_{yi} + 2b^6 c^3 \beta^2 q_{yi} + 2b^6 c^3 \beta^2 q_{yi} + 6b^6 c^3 q_{yi} + 2b^6 c^3 q_{yi} + 2b^6 c^3 q_{yi} + 2b^6 c^3 q_{yi} \right) \alpha^2 \right)
\right],
\]