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Unicorn metrics with almost vanishing H- and Ξ-curvatures

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Abstract: In this paper, we consider a class of almost regular (α, β)-metrics constructed by Shen called unicorn metrics. First, we prove that every unicorn metric with almost vanishing $\mathbf{H}$-curvature is a Berwald metric. Then we show that every unicorn metric with almost vanishing Ξ-curvature reduces to a Berwald metric.

Key words: Unicorn metric, the non-Riemannian quantity $\mathbf{H}$, almost vanishing $\mathbf{\Xi}$-curvature

1. Introduction

It is a long-existing open problem in Finsler geometry to find *unicorns*, i.e. Landsberg metrics that are not Berwaldian [3, 5]. For the sake of simpler prose, Bao referred to Landsberg metrics that are not of Berwald type as unicorns, by analogy with those mythical single-horned horse-like creatures for which no confirmed sighting is available. In [2], Asanov found a special family of unicorns that belongs to the class of nonregular (α, β)-metrics. Then Shen proved that there does not exist any unicorn in the class of regular (α, β)-metrics [10]. He found a more complicated family of unicorns in the class of nonregular (α, β)-metrics that contains the Asanov's metrics. Let us explain some details about the unicorns in Finsler geometry. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost regular (α, β)-metric defined by following:

$$\phi(s) = \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}} dt\right], \tag{1}$$

where $q > 0$ and k are real constants. Suppose that β satisfies

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \qquad s_{ij} = 0, \tag{2}$$

where $c = c(x)$ is a scalar function on M. If $c \neq 0$, then F is a Landsberg metric that is not Berwaldian. In this case, F is a unicorn metric [16]. If $c = 0$, then F reduces to a Berwald metric. If $k = 0$ and $c \neq 0$, then we obtain the family of unicorn metrics introduced by Asanov [2].

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost regular non-Berwaldian (α, β)-metric on a manifold M. Suppose that F is not a Finsler metric of Randers type. In [16], the first author with Sadeghi proved that F is a generalized Douglas–Weyl metric with vanishing S-curvature if and only if ϕ is given by (1). In this case, F is not a Douglas metric nor a Weyl metric.

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In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ obtained from the mean Berwald curvature by covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature, the flag curvature is a scalar function on the manifold if and only if $\mathbf{H} = \mathbf{0}$. It is remarkable that the quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics, where $H_{ij} := E_{ij|m}y^m$. A Finsler metric F is called of almost vanishing $\mathbf{H}$-curvature if

$$\mathbf{H} = \frac{n+1}{2}F^{-1}\theta\ \mathbf{h}, \tag{3}$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric. In [7], Najafi-Shen and the first author establish an equation between the the flag curvature and quantity $\mathbf{H}$ for Finsler metrics of scalar flag curvature. They generalized Akbar-Zadeh's theorem and proved that a Finsler metric has almost isotropic flag curvature $\mathbf{K} = 3\theta/F + \sigma$ if and only if it has almost vanishing $\mathbf{H}$-curvature $\mathbf{H} = (n+1)\theta/(2F)\mathbf{h}$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i(x)dx^i$ is a 1-form on M. Then Mo established a new fundamental equation between non-Riemannian quantity $\mathbf{H}$ and Riemannian quantities on a Finsler manifold [6]. This motivates us to consider unicorn metrics with almost vanishing $\mathbf{H}$-curvature.

Theorem 1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M defined by (1), where $q > 0$ and k are real constants such that $q \neq \sqrt{3}k$. Suppose that β satisfies (2) for some scalar function $c = c(x)$ on M. Then F has almost vanishing $\mathbf{H}$-curvature if and only if β is parallel with respect to α. In this case, F is a Berwald metric.*

Let (M,F) be an n-dimensional Finsler manifold. The non-Riemannian quantity Ξ-curvature $\Xi = \Xi_i dx^i$ on the tangent bundle TM is defined by

$$\Xi_i := \mathbf{S}_{.i|m}y^m - \mathbf{S}_{|i}, \tag{4}$$

where $\mathbf{S}$ denotes the S-curvature, and " . " and " | " denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F, respectively [11]. The Finsler metric F is said to be of almost vanishing Ξ-curvature if its Ξ-curvature can be expressed as follows:

$$\Xi_i = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^i}, \tag{5}$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M.

Theorem 2 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M defined by (1). Suppose that β satisfies (2) for some scalar function $c = c(x)$ on M. Then F has almost vanishing $\mathbf{\Xi}$-curvature if and only if β is parallel with respect to α. In this case, F is a Berwald metric.*

By Theorems 1 and 2, we conclude the following.

Corollary 3 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M defined by the following:*

$$\phi(s) = \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}}dt\right], \tag{6}$$

where $q > 0$ *and* k *are real constants such that* $q \neq \sqrt{3}k$. *Suppose that* β *satisfies* $r_{ij} = c(b^2 a_{ij} - b_i b_j)$ *and* $s_{ij} = 0$ *for some scalar function* $c = c(x)$ *on* M. *Then the following are equivalent:*

(i) F *has almost vanishing* $\mathbf{H}$*-curvature;*

(ii) F *has almost vanishing* $\mathbf{\Xi}$*-curvature.*

In this case, F *reduces to a Berwald metric.*

According to Corollary 3, every unicorn metric has almost vanishing $\mathbf{H}$-curvature if and only if it has almost vanishing $\mathbf{\Xi}$-curvature.

2. Preliminary

Given a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x,y)$, $\lambda > 0$, and given by

$$G^i = \frac{1}{4} g^{il}\left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\right].$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in M. F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$ [9, 17, 18].

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{L}_y(u,v,w) := L_{\ ijk}(y) u^i v^j w^k$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

$\mathbf{B}$ and $\mathbf{L}$ are called Berwald curvature and Landsberg curvature, respectively. Then F is called a Berwald metric and Landsberg metric if $\mathbf{B} = 0$ and $\mathbf{L} = 0$, respectively [12]. By the definition, every Berwald metric is a Landsberg metric. According to Shen's paper, the Finsler metric defined by (1) is a non-Berwaldian Landsberg metric.

For a Finsler metric F on an n-dimensional manifold M, the Busemann–Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in R^n \ \Big|\ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ cannot be expressed in terms of elementary functions, even F is locally expressed by elementary functions [4]. Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ [14, 15].

The E-curvature $\mathbf{E} = E_{ij}dx^i \otimes dx^j$ and Ξ-curvature $\Xi = \Xi_i dx^i$ are given by $E_{ij} = \frac{1}{2}\mathbf{S}_{.i.j}$ and $\Xi_i := \mathbf{S}_{.i|m}y^m - \mathbf{S}_{|i}$, where " . " and " | " denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F, respectively.

The quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics. More precisely, $H_{ij} := E_{ij|m}y^m$. In local coordinates,

$$H_{ij} = \frac{1}{2}\Big[y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^i}\frac{\partial^3 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j}\frac{\partial^4 G^k}{\partial y^i \partial y^k \partial y^m}\Big]. \tag{7}$$

A Finsler metric F is called of almost vanishing H-curvature if $\mathbf{H} = (n+1)/(2F)\theta\mathbf{h}$, where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric.

An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on M (see [13]). For an (α, β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}\Big[b_{i|j} + b_{j|i}\Big], \quad s_{ij} := \frac{1}{2}\Big[b_{i|j} - b_{j|i}\Big].$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

To satisfy that F is positive and strongly convex on TM_0, it is known that if and only if

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (B - s^2)\phi''(s) > 0, \quad |s|^2 \le B < b_0$$

where $B := b^2 = ||\beta||^2_\alpha$.

Let $G^i = G^i(x,y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. Then we have

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \Big(r_{00} - 2Q\alpha s_0\Big)\Big(\Theta\frac{y^i}{\alpha} + \Psi b^i\Big), \tag{8}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}.$$

3. Proof of Theorem 1

In this section, we are going to find the formula of $\mathbf{H}$-curvature of (α, β)-metrics. First, we remark on the formula of $\mathbf{E}$-curvature of (α, β)-metrics.

Lemma 4 [8] Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Put $\Omega := \Phi/(2\Delta^2)$, where

$$\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''.$$

Then the **E**-curvature of F is given by the following:

$$\begin{aligned} E_{ij} = \;& C_1 b_i b_j + C_2(b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5(r_{i0} b_j + r_{j0} b_i) + C_6(r_{i0} y_j + r_{j0} y_i) \\ & + C_7 r_{ij} + C_8(s_i b_j + s_j b_i) + C_9(s_i y_j + s_j y_i) + C_{10}(r_i b_j + r_j b_i) + C_{11}(r_i y_j + r_j y_i), \end{aligned} \tag{9}$$

where

$$\begin{aligned} C_1 :=\;& \frac{1}{2\alpha^3\Delta^2}\Big\{\Phi\alpha Q'' s_0 + 2\alpha\Delta^2\Psi'' r_0 - \Delta^2\Omega'' r_0 + 2\Delta^2\alpha\Omega'' Q s_0 + 4\Delta^2\alpha\Omega' Q' s_0 + 2\alpha\Delta^2\Psi'' s_0\Big\}, \\ C_2 :=\;& \frac{-1}{2\alpha^4\Delta^2}\Big\{2\alpha\Delta^2\Psi'' s_0 - 2\Omega'\Delta^2 r_0 + 2\Omega'\Delta^2\alpha Q s_0 - \Delta^2\Omega'' s r_0 + 2\Delta^2\alpha\Omega'' s Q s_0 \\ & + 4\Delta^2\alpha\Omega' Q' s_0 s + 2\alpha\Delta^2\Psi' r_0 + 2\alpha\Delta^2\Psi'' s r_0 + 2\alpha\Delta^2\Psi'' s s_0 + \Phi\alpha Q' s_0 + \Phi\alpha Q'' s_0 s\Big\}, \\ C_3 :=\;& \frac{1}{4\alpha^5\Delta^2}\Big\{4\Delta^2 s^2\Omega''\alpha Q s_0 - 2\Delta^2 s^2\Omega'' r_0 + 12\alpha\Delta^2\Psi' s r_0 + 12\alpha\Delta^2\Psi' s s_0 + 4\alpha\Delta^2\Psi'' s^2 r_0 \\ & + 4\alpha\Delta^2\Psi'' s^2 s_0 + 8\Delta^2 s^2\Omega'\alpha Q' s_0 + 2\Phi\alpha Q'' s_0 s^2 - 10\Omega'\Delta^2 s r_0 + 12\Omega'\Delta^2 s\alpha Q s_0 \\ & + 6\Phi\alpha Q' s_0 s - 3\Phi r_0\Big\}, \\ C_4 :=\;& \frac{-1}{4\alpha^3\Delta^2}\Big\{4\alpha\Delta^2\Psi' s s_0 - \Phi r_0 - 2\Omega'\Delta^2 s r_0 + 4\Omega'\Delta^2 s\alpha Q s_0 + 4\alpha\Delta^2\Psi' s r_0 + 2\Phi\alpha Q' s_0 s\Big\}, \\ C_5 :=\;& \frac{-\Omega'}{\alpha^2}, \quad C_6 := \frac{2\Delta^2 s\Omega' + \Phi}{2\alpha^3\Delta^2}, \quad C_7 := \frac{-\Phi}{2\alpha\Delta^2}, \\ C_8 :=\;& \frac{1}{2\alpha\Delta^2}\Big\{2\Omega'\Delta^2 Q + 2\Delta^2\Psi' + \Phi Q'\Big\}, \\ C_9 :=\;& \frac{-s}{\alpha}C_8, \quad C_{10} := \frac{\Psi'}{\alpha}, \quad C_{11} := \frac{-s}{\alpha}C_{10}. \end{aligned}$$

By Lemma 4, we get the following.

Proposition 5 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on a manifold M. Then the **H**-curvature of F is given by the following:*

$$\begin{aligned} H_{ij} = \;& C_1\Big[(r_{i0} + s_{i0})b_j + (r_{j0} + s_{j0})b_i\Big] + C_2 b_i b_j + C_3\Big[(r_{i0} + s_{i0})y_j + (r_{j0} + s_{j0})y_i\Big] \\ & + C_4(b_i y_j + b_j y_i) + C_5 y_i y_j + C_6 a_{ij} + C_7\Big[r_{i0|0} b_j + r_{j0|0} b_i + r_{i0}(r_{j0} + s_{j0}) + r_{j0}(r_{i0} + s_{i0})\Big] \\ & + C_8(r_{i0} b_j + r_{j0} b_i) + C_9\Big[r_{i0|0} y_j + r_{j0|0} y_i\Big] + C_{10}(r_{i0} y_j + r_{j0} y_i) + C_{11} r_{ij|0} + C_{12} r_{ij} \\ & + C_{13}\Big[s_{i|0} b_j + s_{j|0} b_i + s_i(r_{j0} + s_{j0}) + s_j(r_{i0} + s_{i0})\Big] + C_{14}(s_i b_j + s_j b_i) \\ & + C_{15}\Big[s_{i|0} y_j + s_{j|0} y_i\Big] + C_{16}(s_i y_j + s_j y_i) + C_{17}\Big[(r_{i|0} b_j + r_{j|0} b_i) + r_i(r_{j0} + s_{j0}) \\ & + r_j(r_{i0} + s_{i0})\Big] + C_{18}(r_i b_j + r_j b_i) + C_{19}(r_{i|0} y_j + r_{j|0} y_i) + C_{20}(r_i y_j + r_j y_i), \end{aligned} \tag{10}$$

where

$$C_1 := \frac{1}{2\alpha^3\Delta^2}\Big[\Phi\alpha Q_{ss}s_0 + \Delta^2(2\alpha\Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha\Omega_{ss}Qs_0 + 4\alpha\Omega_sQ_ss_0 + 2\alpha\Psi_{ss}s_0)\Big],$$

$$C_2 := \frac{-\Delta_{|0}}{\alpha^3\Delta^3}\Big[\Phi\alpha Q_{ss}s_0 + \Delta^2(2\alpha\Psi_{ss}r_0 - \Omega_{ss}r_0 + 2\alpha\Omega_{ss}Qs_0 + 4\alpha\Omega_sQ_ss_0 + 2\alpha\Psi_{ss}s_0)\Big],$$

$$+\frac{1}{2\alpha^3\Delta^2}\Big[\alpha(\Phi_{|0}Q_{ss}s_0 + \Phi Q_{ss|0}s_0 + \Phi Q_{ss}s_{0|0}) + 2\alpha\Delta\Big(2\Delta_{|0}\Psi_{ss}r_0 + \Delta\Psi_{ss|0}r_0 + \Delta\Psi_{ss}r_{0|0}\Big)$$

$$-\Delta(2\Delta_{|0}\Omega_{ss}r_0 - \Delta\Omega_{ss|0}r_0 - \Delta\Omega_{ss}r_{0|0}) + 2\alpha\Delta\Big(2\Delta_{|0}\Omega_{ss}Q + \Delta\Omega_{ss|0}Q + \Delta\Omega_{ss}Q_{|0}\Big)s_0$$

$$+2\alpha\Delta^2\Omega_{ss}Qs_{0|0} + 4\alpha\Big(2\Delta\Delta_{|0}\Omega_sQ_ss_0 + \Delta^2\Omega_{s|0}Q_ss_0 + \Delta^2\Omega_sQ_{s|0}s_0 + \Delta^2\Omega_sQ_ss_{0|0}\Big)$$

$$+2\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s_0 + \Delta^2\Psi_{ss|0}s_0 + \Delta^2\Psi_{ss}s_{0|0}\Big)\Big]$$

$$C_3 := \frac{-1}{2\alpha^4\Delta^2}\Big[2\alpha\Delta^2\Psi_{ss}s_0 - 2\Omega_s\Delta^2r_0 + 2\Omega_s\Delta^2\alpha Qs_0 - \Delta^2\Omega_{ss}sr_0 + 2\Delta^2\alpha\Omega_{ss}sQs_0$$

$$+4\Delta^2\alpha\Omega_sQ_ss_0s + 2\alpha\Delta^2\Psi_sr_0 + 2\alpha\Delta^2\Psi_{ss}sr_0 2\alpha\Delta^2\Psi_{ss}ss_0 + \Phi\alpha Q_ss_0 + \Phi\alpha Q_{ss}s_0s\Big],$$

$$C_4 := \frac{-1}{2\alpha^4\Delta^2}\Big[2\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s_0 + \Delta^2\Psi_{ss|0}s_0 + \Delta^2\Psi_{ss}s_{0|0} - 2\Omega_{s|0}\Delta^2r_0 - 4\Omega_s\Delta\Delta_{|0}r_0$$

$$-2\Omega_s\Delta^2r_{0|0} + 2\alpha(\Omega_{s|0}\Delta^2Qs_0 + 2\Omega_s\Delta\Delta_{|0}Qs_0 + \Omega_s\Delta^2Q_{|0}s_0 + \Omega_s\Delta^2Qs_{0|0}\Big)$$

$$-2\Delta\Delta_{|0}\Omega_{ss}sr_0 - \Delta^2\Omega_{ss|0}sr_0 - \Delta^2\Omega_{ss}\frac{r_{00}}{\alpha}r_0 - \Delta^2\Omega_{ss}sr_{0|0}$$

$$+2\alpha\Big(2\Delta\Delta_{|0}\Omega_{ss}sQs_0 + \Delta^2\Omega_{ss|0}sQs_0 + \Delta^2\Omega_{ss}sQ_{|0}s_0 + \Delta^2\Omega_{ss}sQs_{0|0}\Big) + \Delta^2\Omega_{ss}Qr_{00}s_0$$

$$+4\alpha\Big(2\Delta\Delta_{|0}\Omega_sQ_ss_0s + \Delta^2\Omega_{s|0}Q_ss_0s + \Delta^2\Omega_sQ_{s|0}s_0s + \Delta^2\Omega_sQ_ss_{0|0}s\Big) + \Delta^2\Omega_sQ_ss_0r_{00}$$

$$+2\alpha\Big(2\Delta\Delta_{|0}\Psi_sr_0 + \Delta^2\Psi_{s|0}r_0 + \Delta^2\Psi_sr_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}sr_0 + \Delta^2\Psi_{ss|0}sr_0$$

$$+\Delta^2\Psi_{ss|0}sr_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}ss_0 + \Delta^2\Psi_{ss|0}ss_0 + \Delta^2\Psi_{ss}ss_{0|0}\Big) + \Delta^2\Psi_{ss}r_{00}s_0 + \Delta^2\Psi_{ss}r_{00}r_0$$

$$+\alpha\Big(\Phi_{|0}Q_ss_0 + \Phi Q_{s|0}s_0 + \Phi Q_ss_{0|0} + \Phi_{|0}Q_{ss}s_0s + \Phi Q_{ss|0}s_0s + \Phi Q_{ss}s_{0|0}s\Big) + \Phi Q_{ss}s_0r_{00}\Big],$$

$$+\frac{\Delta_{|0}}{\alpha^4\Delta^3}\Big[2\alpha\Delta^2\Psi_{ss}s_0 + 2\Omega_s\Delta^2\alpha Qs_0 + 2\Delta^2\alpha\Omega_{ss}sQs_0 + 4\Delta^2\alpha\Omega_sQ_sss_0 + 2\alpha\Delta^2\Psi_{ss}ss_0$$

$$+\Phi\alpha Q_ss_0 + \Phi\alpha Q_{ss}ss_0 + 2\alpha\Delta^2\Psi_sr_0 + 2\alpha\Delta^2\Psi_{ss}sr_0 - \Delta^2\Omega_{ss}sr_0 - 2\Omega_s\Delta^2r_0\Big],$$

$$
\begin{aligned}
C_5 := \ & \frac{1}{4\alpha^5\Delta^2}\Bigg[4\alpha\Big(2\Delta\Delta_{|0}s^2\Omega_{ss}Qs_0 + 2\Delta^2 s\frac{r_{00}}{\alpha}\Omega_{ss}Qs_0 + \Delta^2 s^2\Omega_{ss|0}Qs_0 + \Delta^2 s^2\Omega_{ss}Q_{|0}s_0 \\
& +\Delta^2 s^2\Omega_{ss}Qs_{0|0}\Big) - 2\Big(2\Delta\Delta_{|0}s^2\Omega_{ss}r_0 + 2\Delta^2 s\frac{r_{00}}{\alpha}\Omega_{ss}r_0 + \Delta^2 s^2\Omega_{ss|0}r_0 + \Delta^2 s^2\Omega_{ss}r_{0|0}\Big) \\
& +12\alpha\Big(2\Delta\Delta_{|0}\Psi_s sr_0 + \Delta^2\Psi_{s|0}sr_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}r_0 + \Delta^2\Psi_s sr_{0|0} + 2\Delta\Delta_{|0}\Psi_s ss_0 \\
& +\Delta^2\Psi_{s|0}ss_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}s_0 + \Delta^2\Psi_s ss_{0|0}\Big) + 4\alpha\Big(2\Delta\Delta_{|0}\Psi_{ss}s^2r_0 + \Delta^2\Psi_{ss|0}s^2r_0 \\
& +2\Delta^2\Psi_{ss}s\frac{r_{00}}{\alpha}r_0 + \Delta^2\Psi_{ss}s^2r_{0|0} + 2\Delta\Delta_{|0}\Psi_{ss}s^2s_0 + \Delta^2\Psi_{ss|0}s^2s_0 + 2\Delta^2\Psi_{ss}s\frac{r_{00}}{\alpha}s_0 \\
& +\Delta^2\Psi_{ss}s^2s_{0|0}\Big) + 2\alpha s^2\Big(8\Delta\Delta_{|0}\Omega_sQ_ss_0 + 4\Delta^2\Omega_{s|0}Q_ss_0 + 4\Delta^2\Omega_sQ_{s|0}s_0 + 4\Delta^2\Omega_sQ_ss_{0|0} \\
& +\Phi_{|0}Q_{ss}s_0 + \Phi Q_{ss|0}s_0 + \Phi Q_{ss}s_{0|0}\Big) + 4ss_0\Big(4\Delta^2\Omega_sQ_s + \Phi Q_{ss}\Big)r_{00} - 10\Big(\Omega_{s|0}\Delta^2sr_0 \\
& +2\Omega_s\Delta\Delta_{|0}sr_0 + \Omega_s\Delta^2\frac{r_{00}}{\alpha}r_0 + \Omega_s\Delta^2sr_{0|0} + 6\alpha(2\Omega_{s|0}\Delta^2sQs_0 + 4\Omega_s\Delta\Delta_{|0}sQs_0 \\
& +2\Omega_s\Delta^2\frac{r_{00}}{\alpha}Qs_0 + 2\Omega_s\Delta^2sQ_{|0}s_0 + 2\Omega_s\Delta^2sQs_{0|0} + \Phi_{|0}Q_ss_0s + \Phi Q_{s|0}s_0s + \Phi Q_ss_{0|0}s \\
& -3\Phi_{|0}r_0 - 3\Phi r_{0|0}\Big) + \Phi Q_ss_0r_{00}\Bigg] - \frac{\Delta_{|0}}{2\alpha^5\Delta^3}\Bigg[4\Delta^2s^2\Omega_{ss}\alpha Qs_0 - 2\Delta^2s^2\Omega_{ss}r_0 + 12\alpha\Delta^2\Psi_ssr_0 \\
& +12\alpha\Delta^2\Psi_sss_0 + 4\alpha\Delta^2\Psi_{ss}s^2r_0 + 4\alpha\Delta^2\Psi_{ss}s^2s_0 + 8\Delta^2s^2\Omega_s\alpha Q_ss_0 + 2\Phi\alpha Q_{ss}s_0s^2 \\
& -10\Omega_s\Delta^2sr_0 + 12\Omega_s\Delta^2s\alpha Qs_0 + 6\Phi\alpha Q_ss_0s - 3\Phi r_0\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
C_6 := & \frac{-1}{4\alpha^3\Delta^2}\Bigg[4\alpha\Big(2\Delta\Delta_{|0}\Psi_sss_0 + \Delta^2\Psi_{s|0}ss_0 + \Delta^2\Psi_s\frac{r_{00}}{\alpha}s_0 + \Delta^2\Psi_sss_{0|0}\Big) - \Phi_{|0}r_0 - \Phi r_{0|0} \\
& -2\Big(\Omega_{s|0}\Delta^2sr_0 + 2\Omega_s\Delta\Delta_{|0}sr_0 + \Omega_s\Delta^2\frac{r_{00}}{\alpha}r_0 + \Omega_s\Delta^2sr_{0|0}\Big) + \alpha\Big[4\Big(\Omega_{s|0}\Delta^2sQs_0 \\
& +\Omega_s\Delta^2\frac{r_{00}}{\alpha}Qs_0 + \Omega_s\Delta^2sQ_{|0}s_0 + \Omega_s\Delta^2sQs_{0|0} + 2\Omega_s\Delta\Delta_{|0}sQs_0 + 2\Delta\Delta_{|0}\Psi_ssr_0 + \Delta^2\Psi_{s|0}sr_0 \\
& +\Delta^2\Psi_s\frac{r_{00}}{\alpha}r_0 + \Delta^2\Psi' sr_{0|0}\Big) + 2\Phi_{|0}Q_ss_0s + 2\Phi Q_{s|0}s_0s + 2\Phi Q_ss_{0|0}s\Big] + 2\Phi Q_ss_0r_{00}\Bigg] \\
& +\frac{\Delta|0}{4\alpha^3\Delta^3}\Bigg[4\alpha\Delta^2\Psi_sss_0 - \Phi r_0 - 2\Omega_s\Delta^2sr_0 + 4\Omega_s\Delta^2s\alpha Qs_0 + 4\alpha\Delta^2\Psi_ssr_0 + 2\Phi\alpha Q_ss_0s\Bigg],
\end{aligned}
$$

$$
C_7 := -\frac{1}{\alpha^2}\Omega_s, \qquad C_8 := -\frac{1}{\alpha^2}\Omega_{s|0}, \qquad C_9 := \frac{1}{2\alpha^3\Delta^2}(2\Delta^2s\Omega_s + \Phi),
$$

$$
C_{10} := \frac{4\Delta\Delta_{|0}s\Omega_s + 2\Delta^2s\Omega_{s|0} + \Phi_{|0}}{2\alpha^3\Delta^2} + \frac{r_{00}\Omega_s}{\alpha} - \frac{(2\Delta^2s\Omega_s + \Phi)\Delta_{|0}}{\alpha^3\Delta^3}
$$

$$
C_{11} := \frac{-\Phi}{2\alpha\Delta^2}, \qquad C_{12} := \frac{-\Phi_{|0}}{2\alpha\Delta^2} + \frac{\Phi\Delta_{|0}}{\alpha\Delta^3},
$$

$$
C_{13} := \frac{1}{2\alpha\Delta^2}\Big[2\Omega_s\Delta^2Q + 2\Delta^2\Psi_s + \Phi Q_s\Big],
$$

$$C_{14} := \frac{1}{2\alpha\Delta^2}\Big[2\Omega_{s|0}\Delta^2 Q + 4\Omega_s\Delta\Delta_{|0}Q + 2\Omega_s\Delta^2 Q_{|0} + 4\Delta\Delta_{|0}\Psi_s + 2\Delta^2\Psi_{s|0} + \Phi_{|0}Q_s + \Phi Q_{s|0}\Big],$$
$$-\frac{\Delta_{|0}}{\alpha\Delta^3}\Big[2\Omega_s\Delta^2 Q + 2\Delta^2\Psi_s + \Phi Q_s\Big],$$
$$C_{15} := -\frac{s}{\alpha}C_{13}, \qquad C_{16} := -\frac{s}{\alpha}C_{14} - \frac{r_{00}}{\alpha^2}C_{13}, \qquad C_{17} := \frac{1}{\alpha}\Psi_s, \qquad C_{18} := \frac{1}{\alpha}\Psi_{s|0},$$
$$C_{19} := -\frac{s}{\alpha}C_{17}, \qquad C_{20} := -\frac{r_{00}}{\alpha^2}C_{17} - \frac{s}{\alpha}C_{18},$$

and

$$\Delta_{|0} = Q_{|0}s + \frac{r_{00}}{\alpha}Q + 2\Big[r_0 + s_0 - \frac{sr_{00}}{\alpha}\Big]Q_s + (b^2 - s^2)Q_{s|0}$$
$$\Phi_{|0} = -\Big[n\Delta_0 + \frac{r_{00}}{\alpha}Q + sQ_{|0}\Big](Q - sQ_s) - (Q_{|0} - \frac{r_{00}}{\alpha}Q_s - sQ_{s|0})(n\Delta + 1 + sQ)$$
$$-2\Big[r_0 + s_0 - \frac{sr_{00}}{\alpha}\Big](1 + sQ)Q_{ss} - (b^2 - s^2)\Big[\frac{r_{00}}{\alpha}QQ_{ss} + sQ_{|0}Q_{ss} + (1 + sQ)Q_{ss|0}\Big].$$

Proof of Theorem 1: For the Finsler metric defined by (1), we get

$$Q := ks + q\sqrt{b^2 - s^2}, \qquad \Psi = \frac{k\sqrt{b^2 - s^2} - qs}{2(1 + kb^2)\sqrt{b^2 - s^2}}. \tag{11}$$

Using the Maple program, we can obtain the quantity $\mathbf{H}$ for the Finsler metric defined by (1). By using Proposition 5, putting $s_{ij} = 0$ and $r_{ij} = c(b^2 a_{ij} - b_i b_j)$ in (10) and decomposition of the rational (Rat) and irrational (Irrat) parts, we have the following:

$$H_{jk} = (Rat)_{jk} + \alpha(Irrat)_{jk}, \tag{12}$$

where Rat and $Irrat$ are listed in Appendix 4. It is remarkable that, if $\mathbf{H} = (n+1)/(2F)\theta\mathbf{h}$ then by equating the parts of rational and irrational parts in the two sides of the equation, we get two simple equations. By the obtained equations, one can get the desired result.

Now suppose that F has almost vanishing $\mathbf{H}$-curvature on a n-dimensional manifold M. Then

$$H_{ij} = \frac{n+1}{2}F^{-1}\theta h_{ij}, \tag{13}$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M. Substituting (12) in (13), we get

$$(n+1)\theta h_{jk} = 2F\big[(Rat)_{jk} + \alpha(Irrat)_{jk}\big]. \tag{14}$$

Multiplying (14) by $b^j b^k$ and using the Maple program, we get the following:

$$
\begin{aligned}
&A\Big[q\Big(12b^8c^2\beta^2-24b^{10}c^2\beta^2k\Big)\alpha^{14}+q\Big(12b^{10}c^2q^2\beta^4-24b^{10}c^2\beta^4k^2+90kb^{10}c^2\beta^4-106kb^8c^2\beta^4\\
&-45b^8c^2\beta^4+53b^6c^2\beta^4-8\beta^3\big(kb^8+b^6\big)c_{|0}\Big)\alpha^{12}+q\beta^5\Big(90b^{10}c^2\beta k^2-45b^{10}c^2\beta q^2-118b^8c^2\beta k^2\\
&+57b^8c^2\beta q^2-540b^8c^2\beta k+718b^6c^2\beta k+180b^6c^2\beta-181b^4c^2\beta+8\big(kb^6+2b^4-k^2b^8\big)c_{|0}\Big)\alpha^{10}\\
&+q\beta^7\Big(b^2c^2\beta\big(225q^2b^6-495k^2b^6+665k^2b^4-282q^2b^4+1260kb^4-1178kb^2-270b^2+155\big)\\
&+8\big(kb^4+2k^2b^6-b^2\big)c_{|0}\Big)\alpha^8+q\beta^9\Big(1080b^6c^2k^2\beta-450b^6c^2q^2\beta-997b^4c^2k^2\beta\\
&+384b^4c^2q^2\beta-1440b^4c^2k\beta+746b^2c^2k\beta+180b^2c^2\beta-39c^2\beta-8b^4c_{|0}k^2-8b^2c_{|0}k\Big)\alpha^6\\
&+q\beta^{12}\Big(450b^4c^2q^2-1170b^4c^2k^2+591b^2c^2k^2-210Bc^2q^2+810b^2c^2k-156c^2k-45c^2\Big)\alpha^4\\
&+qc^2\beta^{14}\Big(630b^2k^2-225b^2q^2-117k^2+39q^2-180k\Big)\alpha^2-45qc^2\beta^{16}(3k^2-q^2)\Big]=0,
\end{aligned}
\tag{15}
$$

where

$$
A:=-\frac{b^2q}{4\alpha^8(1+kb^2)^2(b^2\alpha^2-\beta^2)^2(\alpha^2+k\beta^2)^2}.
$$

The only equation that does not contain α^2 is $45c^2\beta^{16}q(3k^2-q^2)$. Since β^{18} is not divisible by α^2, then we have two cases as follows:

Case (b1): $c^2\beta^{18}q(-q^2+3k^2)$ contains α^2. Since q and k are real constants, then they cannot contain α. Thus this case will not happen.

Case (b2): $c^2\beta^{18}q(-q^2+3k^2)=0$. In this case, $c(x)=0$ or $q^2=3k^2$. Since $q\neq 3\sqrt{k}$, then $c=0$ and then $r_{ij}=0$. In this case, F reduces to a Berwald metric. □

4. Proof of Theorem 2

Now we are going to consider the Finsler metric defined by (1) with almost vanishing **Ξ**-curvature.

Proof of Theorem 2: Suppose that β satisfies

$$
r_{ij}=c(b^2a_{ij}-b_ib_j)\quad\text{and}\quad s_{ij}=0,
$$

where $c=c(x)$ is a scalar function on M. Using the Maple program, we can obtain the quantity Ξ for the unicorn metric. By putting $s_{ij}=0$ and decomposition of the rational (Rat) and irrational (Irrat) parts, we have the following:

$$
\Xi_i:=(Rat)_i+(Irrat)_i, \tag{16}
$$

where *Rat* and *Irrat* are listed in Appendix 4.

By assumption, F has almost vanishing Ξ-curvature. Thus there is a 1-form $\theta := \theta_i(x)y^i$ on M such that the following holds:

$$\Xi_i = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^i}, \tag{17}$$

By substituting (16) in (17), we get

$$(n+1)(F_i\ \theta - \theta_i F) = (Rat)_i + (Irrat)_i. \tag{18}$$

Multiplying (18) with b^i and using the Maple program, we get the following:

$$\begin{aligned} q^2\Bigg[& 2cn(kb^2+1)b^4\alpha^7 + \beta\Big(2nb^4c^2 + (knb^2 - knb^4 - kb^4 + kb^2 - nb^2 - b^2 + n + 1)c_{|i}b^i\Big)\alpha^6 \\ & +4nc\beta^2(k^2b^6 - kb^4 - 2b^2)\alpha^5 + \beta^2(knb^2c^2b^4\beta + 6knb^4c^2\beta - nc^2b^4\beta + knb^4c_{|0} - 2nb^2c^2\beta \\ & +knb^2c_{|i}b^i\beta + kb^2c_{|i}b^i\beta + nb^2c_{|0} + nc_{|i}b^i\beta + c_{|i}b^i\beta)\alpha^4 + cn\beta^4(6 - 16b^4k^2 - 10b^2k)\alpha^3 \\ & +n\beta^4\big[c^2\beta(1 - 2kb^4 - 11kb^2) - (kb^2+n)c_{|0}\big]\alpha^2 + 12knc(1+kb^2)\beta^6\alpha + 6knc^2\beta^7\Bigg] = 0. \end{aligned} \tag{19}$$

By assumption, $q \neq 0$. In the last sentence of (19), $6knc^2\beta^7$ contains α. This case cannot happen, because k is a real number. Thus k and β cannot contain α. Since $c = c(x)$, then the equation cannot contain α. Therefore, $c = 0$. In this case, F reduces to a Berwald metric. □

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Appendix A. Coefficients in (12)

$$
\begin{aligned}
(Rat)_{jk} = \ & \Lambda\Bigg[\beta\Big(6knqb^8c^2\beta\ a_{jk} + 10b^8c^2knqb_jy_k + 6b^6c^2\beta\ nqa_{jk} + 10b^8c^2knqb_ky_j - 26b^6\ c^2\beta knqb_jb_k \\
& -12b^8c^2kqb_jy_k - 12b^8c^2kqb_ky_j - 2b^8c_{|0}knqa_{jk} + 10b^6c^2nqb_jy_k + 10b^6c^2nqb_ky_j + 2b^6c_{|0}knqb_jb_k \\
& -26b^8c^2\beta nqb_jb_k + 6b^6c^2qb_jy_k + 6b^6c^2qb_ky_j - 2b^6c_{|0}nqa_{jk} + 2b^8c_{|0}nqb_jb_k\Big)\alpha^{14} + \Big(6b^8c^2\beta^3q^3b_ky_j \\
& +16b^8c^2\beta^3\ q^3b_jy_k - 16b^6c^2\beta^4kqa_{jk} - 45b^8c^2\beta^2\ qy_jy_k - 12b^8c^2\beta^4nqa_{jk} + 45b^6c^2\beta^2qy_jy_k \\
& -8b^6c_{|0}\beta^3kqa_{jk} + 12b^8c^2\beta^3qb_jy_k + 12b^8c^2\beta^3qb_ky_j + 4b^8c_{|0}\beta^3nqa_{jk} + 10b^8c^2\beta^3k^2nqb_jy_k \\
& -26b^6c^2\beta^4k^2nqb_jb_k - 42b^8c^2\beta^2knqy_jy_k + 32b^6c^2\beta^3knqb_jy_k + 32b^6c^2\beta^3knqb_ky_j - 8b^8c_{|0}\beta^3qa_{jk} \\
& +6b^6c_{|0}\beta^2knqb_jy_k + 6b^6c_{|0}\beta^2knqb_ky_j - 8b^8c_{|0}\beta^3knqb_jb_k + 2b^6c_{|0}\beta^3k^2nqb_jb_k - 16b^8c^2\beta^4qa_{jk} \\
& +6b^8c^2\beta^4k^2nqa_{jk} - 12b^8c^2\beta^3k^2qb_jy_k - 12b^8c^2\beta^3k^2qb_ky_j + 90b^{10}c^2\beta^2kqy_jy_k - 2b^8c_{|0}\beta^3k^2nqa_{jk} \\
& -6b^6c^2\beta^4knqa_{jk} - 90b^8c^2\beta^2kqy_jy_k - 42b^6c^2\beta^2nqy_jy_k + 2b^6c_{|0}\beta^3knqa_{jk} + 22b^8c^2\beta^3nqb_jy_k \\
& +22b^8c^2\beta^3nqb_ky_j + 10b^2c^2\beta^4nqb_jb_k - 6b^6c_{|0}\beta nqy_jy_k + 6b^8c_{|0}\beta^2nqb_jy_k - 16b^8c^2\beta^4knqb_jb_k \\
& +6b^8c_{|0}\ \beta^2\ nq\ b_ky_j - 10nqb^2c_{|0}\beta^3\ b_jb_k - 6b^8c_{|0}\beta\ knqy_jy_k + 10b^8c^2\beta^3\ k^2nqb_ky_j\Big)\alpha^{12} \\
& +\Big(45b^8c^2\beta^4q^3y_jy_k + 6b^6c^2\beta^5q^3b_ky_j - 45b^{10}c^2\beta^4q^3y_jy_k - 16b^6c^2\beta^6k^2qa_{jk} + 6b^6c^2\beta^5q^3b_jy_k \\
& -8b^6c_{|0}\beta^5k^2qa_{jk} + 16b^8c^2\beta^6\ kqa_{jk} + 180b^6c^2\beta^4\ qy_jy_k + 6b^2c^2\beta^6\ nqa_{jk} + 16c^2\beta^6nqb_jb_k \\
& +32b^2c^2\beta^6qa_{jk} + 8c_{|0}\beta^5nqb_jb_k - 42b^8c^2\beta^4k^2nqy_jy_k + 22b^6c^2\beta^5k^2nqb_jy_k + 22b^6c^2\beta^5k^2nqb_ky_j \\
& -6b^8c_{|0}\beta^3k^2nqy_jy_k + 6b^6c_{|0}\beta^4k^2nqb_jy_k - 10b^8c^2\beta^5knqb_jy_k + 6b^6c_{|0}\beta^4k^2nqb_ky_j + 26b^2c^2\beta^6knqb_jb_k \\
& -10b^8c_{|0}\beta^5k^2nqb_jb_k - 2b^2c_{|0}\beta^5knqb_jb_k + 90b^{10}c^2\beta^4k^2qy_jy_k - 12b^6c^2\beta^6k^2nqa_{jk} - 90b^8c^2\beta^4k^2qy_jy_k \\
& -6b^6c^2\beta^5k^2qb_jy_k - 540b^8c^2\beta^4kqy_jy_k + 4b^6c_{|0}\beta^5k^2nqa_{jk} - 6b^8c^2\beta^6knqa_{jk} + 438b^6c^2\beta^4kqy_jy_k \\
& +132b^8c^2\beta^5kqb_ky_j + 42b^8c^2\beta^4nqy_jy_k + 2b^8c_{|0}\beta^5knqa_{jk} - 32b^2c^2\beta^5nqb_jy_k - 10b^8c^2\beta^5knqb_ky_j \\
& +6b^8c_{|0}\beta^3nqy_jy_k - 6b^2c_{|0}\beta^4nqb_jy_k - 6b^2c_{|0}\beta^4nqb_ky_j + 10b^8c^2\beta^6k^2nqb_jb_k + 16b^2c_{|0}\beta^5qa_{jk} \\
& -32b^2c^2\beta^5nqb_ky_j + 132b^8c^2\beta^5kqb_jy_k - 6b^6c^2\beta^5k^2qb_ky_j\Big)\alpha^{10} + \Big(32b^8c^2\beta^8k^2qa_{jk} - 16c^2\beta^8qa_{jk} \\
& -54b^8c^2\beta^7q^3b_ky_j + 16b^8c_{|0}\beta^7k^2qa_{jk} - 8c_{|0}\beta^7qa_{jk} - 174b^6c^2\beta^6q^3y_jy_k - 54b^8c^2\beta^7q^3b_jy_k \\
& +8b^2c_{|0}\beta^7kqa_{jk} + 225b^8c^2\beta^6q^3y_jy_k + 16b^2c^2\beta^8kqa_{jk} - 270b^8c^2\beta^6qy_jy_k + 123b^2c^2\beta^6qy_jy_k \\
& +42b^6c^2\beta^6k^2nqy_jy_k - 32b^8c^2\beta^7k^2nqb_jy_k + 16b^2c^2\beta^8k^2nqb_jb_k + 42b^8c^2\beta^6knqy_jy_k + 24c^2\beta^7qb_ky_j \\
& -6b^8c_{|0}\beta^6k^2nqb_ky_j - 32b^2c^2\beta^7knqb_jy_k - 32b^2c^2\beta^7knqb_ky_j + 6b^6c_{|0}\beta^5k^2nqy_jy_k - 6b^8c_{|0}\beta^6k^2nqb_jy_k \\
& +8b^2c_{|0}\beta^7k^2nqb_jb_k + 6b^8c_{|0}\beta^5knqy_jy_k - 6b^2c_{|0}\beta^6knqb_jy_k - 6b^2c_{|0}\beta^6knqb_ky_j + 24c^2\beta^7qb_jy_k \\
& +6b^2c^2\beta^8knqa_{jk} + 393b^6c^2\beta^6k^2qy_jy_k + 120b^8c^2\beta^7k^2qb_jy_k + 120b^8c^2\beta^7k^2qb_ky_j + 1260b^6c^2\beta^6kqy_jy_k \\
& +8c_{|0}\beta^7knqb_jb_k - 495b^8c^2\beta^6k^2qy_jy_k + 16c^2\beta^8knqb_jb_k - 762b^8c^2\beta^6kqy_jy_k - 216b^2c^2\beta^7kqb_jy_k \\
& +6b^8c^2\beta^8k^2nqa_{jk} - 216b^2c^2\beta^7kqb_ky_j - 2b^2c_{|0}\beta^7knqa_{jk} - 32b^8c^2\beta^7k^2nqb_ky_j - 2b^8c_{|0}\beta^7k^2nqa_{jk}\Big)\alpha^8 \\
& +\Big(1080b^6c^2\beta^8k^2qy_jy_k - 450b^6c^2\beta^8q^3y_jy_k - 633b^8c^2\beta^8k^2qy_jy_k + 252b^8c^2\beta^8q^3y_jy_k + 66b^2c^2\beta^9q^3b_ky_j
\end{aligned}
$$

$$
\begin{aligned}
&-174b^2c^2\beta^9k^2qb_jy_k-174b^2c^2\beta^9k^2qb_ky_j+66b^2c^2\beta^9q^3b_jy_k+570b^2c^2\beta^8kqy_jy_k-8b^2c_{|0}\beta^9k^2qa_{jk}\\
&-16c^2\beta^10kqa_{jk}+96c^2\beta^9kqb_jy_k+96c^2\beta^9kqb_ky_j+180b^2c^2\beta^8qy_jy_k-39c^2\beta^8qy_jy_k-8c_{|0}\beta^9kqa_{jk}\\
&-16b^2c^2\beta^{10}k^2qa_{jk}-1440b^8c^2\beta^8kqy_jy_k\Big)\alpha^6+\Big(450b^8c^2\beta^{10}q^3y_jy_k-1170b^8c^2\beta^{10}k^2qy_jy_k\\
&+447b^2c^2\beta^{10}k^2qy_jy_k-162b^2c^2\beta^{10}q^3y_jy_k+72c^2\beta^{11}k^2qb_jy_k+72c^2\beta^{11}k^2qb_ky_j-24c^2\beta^{11}q^3b_jy_k\\
&-24c^2\beta^{11}q^3b_ky_j+810b^2c^2\beta^{10}kqy_jy_k-156c^2\beta^{10}kqy_jy_k-45c^2\beta^{10}qy_jy_k\Big)\alpha^4+c^2\beta^{12}y_jy_k\Big(630b^2k^2q\\
&-225b^2q^3-117k^2q+39q^3-180kq\Big)\alpha^2+qc^2\beta^{14}(45q^2-135k^2)y_jy_k\Big],
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
(Irrat)_{jk}=\;&\Pi\Big[\Big((3b^6c^2\beta kna_{jk}+5b^6c^2kn(b_jy_k+b_ky_j)-13b^4c^2\beta knb_jb_k-6b^6c^2kb_jy_k-6b^6c^2kb_ky_j\\
&-b^6c_{|0}kna_{jk}+b^4c_{|0}knb_jb_k+3b^4c^2\beta na_{jk}+5b^4c^2nb_jy_k+5b^4c^2nb_ky_j-13b^2c^2\beta nb_jb_k\\
&-b^4c_{|0}na_{jk}+b^2c_{|0}nb_jb_k)\mathbf{D}+(b^6\beta cknqa_{jk}+b^4\mathbf{D}ck^2nb_jb_k-b^4\beta cknqb_jb_k+b^6\beta ckqa_{jk}\\
&-b^2\beta cnqb_jb_k+b^4\mathbf{D}cka_{jk}+b^4\mathbf{D}ck^2b_jb_k-b^4\beta ckqb_jb_k+b^4\mathbf{D}ckna_{jk}+b^4\beta cnqa_{jk}\\
&+b^2\mathbf{D}cknb_jb_k+b^4\beta cqa_{jk}+b^2\mathbf{D}ckb_jb_k-b^2\beta cqb_jb_k+b^2\mathbf{D}cna_{jk}+B\mathbf{D}ca_{jk})\phi\Big)\alpha^{14}\\
&+\Big((6b^6c^2\beta^3k^2na_{jk}+3b^6c^2\beta^3nq^2a_{jk}-12b^6c^2\beta^2k^2b_jy_k-12b^6c^2\beta^2k^2b_ky_j+12b^6c^2\beta^2q^2b_jy_k\\
&+12b^6c^2\beta^2q^2b_ky_j+45b^4c^2\beta ky_jy_k-2b^6c_{|0}\beta^2k^2na_{jk}-b^6c_{|0}\beta^2nq^2a_{jk}+3b^4c^2\beta^3kna_{jk}\\
&-45b^6c^2\beta ky_jy_k-12b^4c^2\beta^2kb_jy_k-12b^4c^2\beta^2kb_ky_j-3b^6c_{|0}kny_jy_k-21b^4c^2\beta ny_jy_k\\
&+16b^2c^2\beta^2nb_jy_k+16b^2c^2\beta^2nb_ky_j-b^4c_{|0}\beta^2kna_{jk}+3b^2c_{|0}\beta nb_jy_k+3b^2c_{|0}\beta nb_ky_j\\
&-8b^2c^2\beta^3a_{jk}-4b^2c_{|0}\beta^2a_{jk}+5b^4c^2\beta^2knq^2b_jy_k+5b^4c^2\beta^2knq^2b_ky_j-13b^6c^2\beta^3knq^2b_jb_k\\
&+b^6c_{|0}\beta^2knq^2b_jb_k-8b^4c^2\beta^3ka_{jk}-3b^2c^2\beta^3na_{jk}-8c^2\beta^3nb_jb_k-4b^4c_{|0}\beta^2ka_{jk}\\
&-3b^4c_{|0}ny_jy_k+b^2c_{|0}\beta^2na_{jk}-4c_{|0}\beta^2nb_jb_k+b^4c_{|0}\beta^2nq^2b_jb_k+26b^4c^2\beta^2knb_jy_k\\
&+26b^4c^2\beta^2knb_ky_j-34b^2c^2\beta^3knb_jb_k+3b^4c_{|0}\beta knb_jy_k+3b^4c_{|0}\beta knb_ky_j-2b^2c_{|0}\beta^2knb_jb_k\\
&+10b^6c^2\beta^2k^2nb_jy_k+3b^4c^2\beta^3knq^2a_{jk}-6b^4c^2\beta^2kq^2b_jy_k-6b^4c^2\beta^2kq^2b_ky_j+5b^6c^2\beta^2nq^2b_ky_j\\
&-b^4c_{|0}\beta^2knq^2a_{jk}+10b^6c^2\beta^2k^2nb_ky_j+5b^6c^2\beta^2nq^2b_jy_k-26b^4c^2\beta^3k^2nb_jb_k-13b^4c^2\beta^3nq^2b_jb_k\\
&-21b^6c^2\beta kny_jy_k+2b^4c_{|0}\beta^2k^2nb_jb_k)\mathbf{D}+(b^4\mathbf{D}\beta^2k^2na_{jk}-b^4\mathbf{D}\beta k^2nb_jy_k-b^4\mathbf{D}\beta k^2nb_ky_j\\
&-b^6\beta knqy_jy_k-2b^4\beta^3knqa_{jk}+b^4\beta^2knqb_jy_k+b^4\beta^2knqb_ky_j-b^2\mathbf{D}\beta^2k^2nb_jb_k+b^2\beta^3knqb_jb_k\\
&-b^6\beta kqy_jy_k+b^4\mathbf{D}\beta^2k^2a_{jk}-b^4\mathbf{D}\beta k^2b_jy_k-b^4\mathbf{D}\beta k^2b_ky_j-2b^4\beta^3kqa_{jk}+b^4\beta^2kqb_jy_k\\
&+b^4\beta^2kqb_ky_j-b^2\mathbf{D}\beta^2k^2b_jb_k+b^2\beta^3kqb_jb_k-b^4\mathbf{D}ckny_jy_k-b^4\beta nqy_jy_k-b^2\mathbf{D}\beta knb_jy_k\\
&-b^2\mathbf{D}\beta knb_ky_j-2b^2\beta^3nqa_{jk}+b^2\beta^2nqb_jy_k+b^2\beta^2nqb_ky_j-\mathbf{D}\beta^2knb_jb_k+\beta^3nqb_jb_k\\
&-b^4\mathbf{D}cky_jy_k-b^4\beta qy_jy_k-b^2\mathbf{D}\beta kb_jy_k-b^2\mathbf{D}\beta kb_ky_j-2b^2\beta^3qa_{jk}+b^2\beta^2qb_jy_k+b^2\beta^2qb_ky_j\\
&-\mathbf{D}\beta^2kb_jb_k+\beta^3qb_jb_k-b^2\mathbf{D}cny_jy_k-\mathbf{D}\beta^2na_{jk}-b^2\mathbf{D}cy_jy_k-\mathbf{D}\beta^2a_{jk})\phi(s)\Big)\alpha^{12}\\
&+\Big((45b^{10}c^2\beta^3kq^2y_jy_k-6b^6c^2\beta^5knq^2a_{jk}+5b^6c^2\beta^4k^3nb_jy_k+5b^6c^2\beta^4k^3nb_ky_j+6b^6c^2\beta^4kq^2b_ky_j
\end{aligned}
$$

$$-13b^4c^2\beta^5k^3nb_jb_k-45b^8c^2\beta^3kq^2y_jy_k+6b^6c^2\beta^4kq^2b_jy_k-42b^6c^2\beta^3k^2ny_jy_k-21b^6c^2\beta^3nq^2y_jy_k$$
$$-2b^6c_{|0}\beta^4knq^2a_{jk}+37b^4c^2\beta^4k^2nb_jy_k+37b^4c^2\beta^4k^2nb_ky_j+11b^4c^2\beta^4nq^2b_jy_k+11b^4c^2\beta^4nq^2b_ky_j$$
$$+b^4c_{|0}\beta^4k^3nb_jb_k-29b^2c^2\beta^5k^2nb_jb_k+5b^2c^2\beta^5nq^2b_jb_k-21b^8c^2\beta^3knq^2y_jy_k+11b^6c^2\beta^4knq^2b_jy_k$$
$$+11b^6c^2\beta^4knq^2b_ky_j+5b^4c^2\beta^5knq^2b_jb_k-3b^8c_{|0}\beta^2knq^2y_jy_k+3b^6c_{|0}\beta^3knq^2b_jy_k$$
$$+3b^6c_{|0}\beta^3knq^2b_ky_j-5b^4c_{|0}\beta^4knq^2b_jb_k-16c^2\beta^5knb_jb_k+129b^4c^2\beta^3ky_jy_k+42b^2c^2\beta^4kb_jy_k$$
$$+42b^2c^2\beta^4kb_ky_j+2b^2c_{|0}\beta^4kna_{jk}-8c_{|0}\beta^4knb_jb_k-16b^4c^2\beta^5k^2a_{jk}-8b^4c^2\beta^5q^2a_{jk}$$
$$-8b^4c_{|0}\beta^4k^2a_{jk}-4b^4c_{|0}\beta^4q^2a_{jk}-8b^2c^2\beta^5ka_{jk}-4b^2c_{|0}\beta^4ka_{jk}-6b^6c_{|0}\beta^2k^2ny_jy_k$$
$$-3b^6c_{|0}\beta^2nq^2y_jy_k-42b^4c^2\beta^3kny_jy_k+6b^4c_{|0}\beta^3k^2nb_jy_k+6b^4c_{|0}\beta^3k^2nb_ky_j+3b^4c_{|0}\beta^3nq^2b_jy_k$$
$$+3b^4c_{|0}\beta^3nq^2b_ky_j+32b^2c^2\beta^4knb_jy_k+32b^2c^2\beta^4knb_ky_j-7b^2c_{|0}\beta^4k^2nb_jb_k-5b^4c_{|0}\beta^4nq^2b_jb_k$$
$$-6b^4c_{|0}\beta^2kny_jy_k+6b^2c_{|0}\beta^3knb_jy_k+6b^2c_{|0}\beta^3knb_ky_j+3b^6c^2\beta^5k^3na_{jk}-8b^6c^2\beta^5kq^2a_{jk}$$
$$-6b^6c^2\beta^4k^3b_jy_k-6b^6c^2\beta^4k^3b_ky_j+90b^8c^2\beta^3k^2y_jy_k-90b^8c^2\beta^3q^2y_jy_k-b^6c_{|0}\beta^4k^3na_{jk}$$
$$-3b^4c^2\beta^5k^2na_{jk}-6b^4c^2\beta^5nq^2a_{jk}-90b^6c^2\beta^3k^2y_jy_k+90b^6c^2\beta^3q^2y_jy_k-4b^6c_{|0}\beta^4kq^2a_{jk}$$
$$-24b^4c^2\beta^4k^2b_jy_k-24b^4c^2\beta^4k^2b_ky_j+24b^4c^2\beta^4q^2b_jy_k+24b^4c^2\beta^4q^2b_ky_j-180b^6c^2\beta^3ky_jy_k$$
$$+b^4c_{|0}\beta^4k^2na_{jk}+2b^4c_{|0}\beta^4nq^2a_{jk}-6b^2c^2\beta^5kna_{jk}+8c^2\beta^5a_{jk}+4c_{|0}\beta^4a_{jk})\mathbf{D}+(b^2\beta^3k^2nb_jy_k$$
$$-b^2\beta^4k^2na_{jk}+b^2\beta^3k^2nb_ky_j-b^2\beta^4k^2a_{jk}+b^2\beta^3k^2b_jy_k+b^2\beta^3k^2b_ky_j+b^2\beta^2kny_jy_k-\beta^4kna_{jk}$$
$$+\beta^3knb_jy_k+\beta^3knb_ky_j+b^2\beta^2ky_jy_k-\beta^4ka_{jk}+\beta^3kb_jy_k+\beta^3kb_ky_j+\beta^2ny_jy_k+\beta^2cy_jy_k)\mathbf{D}\phi(s)$$
$$+q\beta^3(b^4kny_jy_k+b^2\beta^2kna_{jk}-b^2\beta knb_jy_k-b^2\beta knb_ky_j+b^4ky_jy_k+b^2\beta^2ka_{jk}-b^2\beta kb_jy_k$$
$$-b^2\beta kb_ky_j+b^2ny_jy_k+\beta^2na_{jk}-\beta nb_jy_k-\beta nb_ky_j+b^2y_jy_k+\beta^2a_{jk}-\beta b_jy_k-\beta b_ky_j)\phi\Big)\alpha^{10}$$
$$+\Big(3b^4c^2\beta^7knq^2a_{jk}-315b^8c^2\beta^5kq^2y_jy_k-21b^6c^2\beta^5k^3ny_jy_k+16b^4c^2\beta^6k^3nb_jy_k-8b^2c^2\beta^7k^3nb_jb_k$$
$$+16b^4c^2\beta^6k^3nb_ky_j+264b^6c^2\beta^5kq^2y_jy_k+78b^4c^2\beta^6kq^2b_jy_k+78b^4c^2\beta^6kq^2b_ky_j-21b^4c^2\beta^5k^2ny_jy_k$$
$$-3b^6c_{|0}\beta^4k^3ny_jy_k+21b^4c^2\beta^5nq^2y_jy_k-b^4c_{|0}\beta^6knq^2a_{jk}+3b^4c_{|0}\beta^5k^3nb_jy_k+3b^4c_{|0}\beta^5k^3nb_ky_j$$
$$+16b^2c^2\beta^6k^2nb_jy_k+16b^2c^2\beta^6k^2nb_ky_j-16b^2c^2\beta^6nq^2b_jy_k-16b^2c^2\beta^6nq^2b_ky_j-4b^2c_{|0}\beta^6k^3nb_jb_k$$
$$-3b^2c_{|0}\beta^4k^2ny_jy_k+3b^4c_{|0}\beta^4nq^2y_jy_k+3b^2c_{|0}\beta^5k^2nb_jy_k+3b^2c_{|0}\beta^5k^2nb_ky_j-3b^2c_{|0}\beta^5nq^2b_jy_k$$
$$-3b^2c_{|0}\beta^5nq^2b_ky_j+21b^6c^2\beta^5knq^2y_jy_k-16b^4c^2\beta^6knq^2b_jy_k-16b^4c^2\beta^6knq^2b_ky_j+8b^2c^2\beta^7knq^2b_jb_k$$
$$+3b^6c_{|0}\beta^4knq^2y_jy_k-3b^4c_{|0}\beta^5knq^2b_jy_k-3b^4c_{|0}\beta^5knq^2b_ky_j+4b^2c_{|0}\beta^6knq^2b_jb_k+45b^8c^2\beta^5k^3y_jy_k$$
$$-3b^4c^2\beta^7k^3na_{jk}-45b^6c^2\beta^5k^3y_jy_k+16b^4c^2\beta^7kq^2a_{jk}-12b^4c^2\beta^6k^3b_jy_k-12b^4c^2\beta^6k^3b_ky_j$$
$$-360b^6c^2\beta^5k^2y_jy_k+360b^6c^2\beta^5q^2y_jy_k+b^4c_{|0}\beta^6k^3na_{jk}-3b^2c^2\beta^7k^2na_{jk}+3b^2c^2\beta^7nq^2a_{jk}$$
$$-8c^2\beta^7k^2nb_jb_k+8c^2\beta^7nq^2b_jb_k+258b^4c^2\beta^5k^2y_jy_k-258b^4c^2\beta^5q^2y_jy_k+8b^4c_{|0}\beta^6kq^2a_{jk}$$
$$+84b^2c^2\beta^6k^2b_jy_k+84b^2c^2\beta^6k^2b_ky_j-84b^2c^2\beta^6q^2b_jy_k-84b^2c^2\beta^6q^2b_ky_j+270b^4c^2\beta^5ky_jy_k$$
$$+b^2c_{|0}\beta^6k^2na_{jk}-b^2c_{|0}\beta^6nq^2a_{jk}-4c_{|0}\beta^6k^2nb_jb_k+4c_{|0}\beta^6nq^2b_jb_k-123b^2c^2\beta^5ky_jy_k-8b^4c^2\beta^7k^3a_{jk}$$
$$-4b^4c_{|0}\beta^6k^3a_{jk}+8b^2c^2\beta^7k^2a_{jk}+16b^2c^2\beta^7q^2a_{jk}+4b^2c_{|0}\beta^6k^2a_{jk}+8b^2c_{|0}\beta^6q^2a_{jk}-24c^2\beta^6kb_jy_k$$
$$-24c^2\beta^6kb_ky_j+16c^2\beta^7ka_{jk}+8c_{|0}\beta^6ka_{jk}\Big)\mathbf{D}\alpha^8+\Big(810b^6c^2\beta^7kq^2y_jy_k-510b^4c^2\beta^7kq^2y_jy_k$$

$$-150b^2c^2\beta^8kq^2b_jy_k - 150b^2c^2\beta^8kq^2b_ky_j + 39c^2\beta^7ky_jy_k - 180b^6c^2\beta^7k^3y_jy_k + 129b^4c^2\beta^7k^3y_jy_k$$
$$-8b^2c^2\beta^9kq^2a_{jk} + 42b^2c^2\beta^8k^3b_jy_k + 42b^2c^2\beta^8k^3b_ky_j + 540b^4c^2\beta^7k^2y_jy_k - 540b^4c^2\beta^7q^2y_jy_k$$
$$-246b^2c^2\beta^7k^2y_jy_k + 246b^2c^2\beta^7q^2y_jy_k - 4b^2c_{|0}\beta^8kq^2a_{jk} - 180b^2c^2\beta^7ky_jy_k + c^2\beta^9k^2a_{jk} - 8c^2\beta^9q^2a_{jk}$$
$$+4c_{|0}\beta^8k^2a_{jk} - 4c_{|0}\beta^8q^2a_{jk} + 8b^2c^2\beta^9k^3a_{jk} + 4b^2c_{|0}\beta^8k^3a_{jk} - 48c^2\beta^8k^2b_jy_k - 48c^2\beta^8k^2b_ky_j$$
$$+48c^2\beta^8q^2b_jy_k + 48c^2\beta^8q^2b_ky_j\Big)\mathbf{D}\alpha^6 + \Big(270b^4c^2\beta^9k^3y_jy_k - 990b^4c^2\beta^9kq^2y_jy_k - 123b^2c^2\beta^9k^3y_jy_k$$
$$+408b^2c^2\beta^9kq^2y_jy_k - 24c^2\beta^{10}k^3b_jy_k - 24c^2\beta^{10}k^3b_ky_j + 72c^2\beta^{10}kq^2b_jy_k + 72c^2\beta^{10}kq^2b_ky_j$$
$$-360b^2c^2\beta^9k^2y_jy_k + 360b^2c^2\beta^9q^2y_jy_k + 78c^2\beta^9k^2y_jy_k - 78c^2\beta^9q^2y_jy_k + 45c^2\beta^9ky_jy_k\Big)\mathbf{D}\alpha^4$$
$$+\Big[c^2y_jy_k\beta^{11}\Big(585b^2kq^2 - 180b^2k^3 + 39ck^3 - 117kq^2 + 90k^2 - 90q^2\Big)\alpha^2$$
$$+kc^2\beta^{13}(45k^2 - 135q^2)y_jy_k\Big]\mathbf{D}\Big]. \tag{A.2}$$

where $\mathbf{D} := \sqrt{b^2\alpha^2 - \beta^2}$ and

$$\Lambda := -\frac{b^2q}{4\alpha^8(b^2\alpha^2-\beta^2)^2(1+kb^2)^2(\beta^2k+\alpha^2)^2},$$
$$\Pi := -\frac{b^2q}{4\alpha^8(b^2\alpha^2-\beta^2)^2(1+kb^2)^2(\sqrt{b^2\alpha^2-\beta^2}\beta q+\beta^2k+\alpha^2)^2}.$$

Appendix B. Coefficients in (16)

$$(Rat)_i = \Gamma\Big[b^2(b^4knc_{|i} - b^2c^2nb_i + b^4kc_{|i} - b^2knc_{|i} - b^2kc_{|i} + b^2nc_{|i} + b^2c_{|i} - nc_{|i} - c_{|i})\alpha^8$$
$$-2b^4\beta cknb_i\,\mathbf{V}\alpha^7 - \beta\,\Big(2b^6y_ic^2k - 2b^4y_i\,c^2k + b^4y_ic^2n + 2\,kb^4c_{|i}\beta\, - 2b^2\,b_i\beta c^2$$
$$-2nb^2y_ic^2 + 2nb_i\beta c^2 - b^2c_{|i}\beta k + 2b^2c_{|i}\beta n + b^2c_{|0}b_in - c_{|i}\beta - 2b^4b_i\beta c^2k + nb^6y_ic^2k$$
$$-2nb^4y_ic^2k - 2knb^4c_{|i}\beta + 2b^2b_i\beta c^2k - 3b^2b_i\beta c^2n - b^2c_{|i}\beta kn + b^4c_{|0}b_ikn$$
$$+2nb^2c^2b_i\beta + b^4y_ib^4c^2kn + 2b^4b_i\beta c^2kn + 2knb^2b_i\beta c^2 + 2kn\,b^4c^2b_i\beta + 2b^4y_ic^2$$
$$-2b^2y_ic^2 + 2b_i\beta c^2 + 2b^2c_{|i}\beta - c_{|i}\beta n\Big)\alpha^6 + 2b^2\beta^2ckn\mathbf{V}\Big(b^2y_i + 4\beta b_i\Big)\alpha^5 + \beta^2\Big(2b^2y_ib^4\beta c^2kn$$
$$+2b^4c^2y_i\beta kn + b^4y_i\beta c^2k + 4b^4y_i\beta c^2kn + 2b^2c^2b_i\beta^2kn + 5b^2b_i\beta^2c^2kn - 2b^2b_i\beta^2c^2k$$
$$+b^4c_{|0}y_ikn + 2b^2c^2y_i\beta n + 2c^2b_i\beta^2n + b^2c_{|0}b_i\beta kn + b^2c_{|i}\beta^2kn + b^2y_i\beta c^2n - 2b_i\beta^2c^2n$$
$$+b^2c_{|i}\beta^2k + 2b^2y_i\beta c^2 - 2b_i\beta^2c^2 + b^2c_{|0}y_in + c_{|0}b_i\beta n + c_{|i}\beta^2n + c_{|i}\beta^2\Big)\alpha^4$$
$$-2\beta^4ckn\,\mathbf{V}\Big(4b^2y_i + 3\beta b_i\Big)\alpha^3 - n\beta^4\Big(y_ib^4\beta c^2k + 2b^2c^2y_i\beta k + 6b^2y_i\beta c^2k + 3b_i\beta^2c^2k$$
$$+b^2c_{|0}y_ik + 2c^2y_i\beta + c_{|0}y_i\Big)\alpha^2 + 6\beta^6ckny_i\mathbf{V}\alpha + 3\beta^7c^2kny_i\Big], \tag{B.1}$$

$$
\begin{aligned}
Irrat = \ & \Upsilon\Big[b^8c^2knqb_i\alpha^{10} + \Big(2b^8c\beta k^2qb_i - b^8c^2knqb_i + b^8c_{|i}knq - b^6c^2nqb_i + 2b^6c\beta kqb_i + b^8c_{|i}kq \\
& -b^6c_{|i}knq - b^6c_{|i}kq + b^6c_{|i}nq + b^6c_{|i}q - b^4c_{|i}nq - b^4c_{|i}q\Big)\alpha^9 + \Big(-2q\beta(b^6ck^2b_i + b^4ckb_i)\mathbf{D}^2 \\
& +(-b^6c^2b^4\beta knqy_i - 2b^8c\beta k^2nqb_i + b^6c^2b^4\beta kqy_i - 5b^6c^2\beta^2knqb_i - b^8c^2\beta kqy_i + 2b^6c^2\beta^2kqb_i \\
& -2b^6c\beta knqb_i\Big)\alpha^8 + q\Big(2b^6c^2\beta kny_i - b^8c^2\beta kny_i - 2b^4c^2\beta^2knb_i - b^6c_{|0}\beta knb_i - 2b^8c\beta^2k^2y_i \\
& -10b^6c\beta^3k^2b_i + 2b^6c^2\beta ky_i - b^6c^2\beta ny_i - 2b^6c\beta^2ky_i - 2b^6c_{|i}\beta^2kn - 2b^4c^2\beta^2kb_i + b^4c^2\beta^2nb_i \\
& -10b^4c\beta^3kb_i + 2b^4c^2\beta ny_i + knb^4c_{|i}\beta^2 - 2b^2c^2\beta^2nb_i - b^4c_{|0}\beta nb_i - 2b^6c^2\beta y_i - 2b^6c_{|i}\beta^2k \\
& +2b^4c^2\beta^2b_i + 2b^4c^2\beta y_i + b^4c_{|i}\beta^2k - 2b^4c_{|i}\beta^2n - b^8c^2\beta ky_i - b^6c^2b^4\beta ky_i + b^6c^2\beta^2knb_i \\
& -2b^2c^2\beta^2b_i + b^2c_{|i}\beta^2n - 2b^4c_{|i}\beta^2 + b^2c_{|i}\beta^2\Big)\alpha^7 + \Big((2b^6c\beta^2k^2qy_i + 8b^4c\beta^3k^2qb_i + 2b^4c\beta^2kqy_i \\
& +8b^2c\beta^3kqb_i)\mathbf{D}^2 + (2b^4c^2b^4\beta^3knqy_i + 2b^8c\beta^2k^2nqy_i + 3b^6c^2\beta^3knqy_i + 8b^6c\beta^3k^2nqb_i \\
& -2b^4c^2b^4\beta^3kqy_i + 7b^4c^2\beta^4knqb_i - 4b^4c^2\beta^4kqb_i + 2b^6c\beta^2knqy_i + 8b^4c\beta^3knqb_i)\Big)\alpha^6 \\
& +\Big(3b^6c^2\beta^3knqy_i + 10b^6c\beta^4k^2qy_i + 2b^4c^2b^4\beta^3kqy_i + 14b^4c\beta^5k^2qb_i + 2b^6c^2\beta^3kqy_i + 2b^4c^2\beta^4kqb_i \\
& +b^6c_{|0}\beta^2knqy_i + 3b^4c^2\beta^3nqy_i + 10b^4c\beta^4kqy_i + b^4c_{|0}\beta^3knqb_i + b^4c_{|i}\beta^4knq + 14b^2c\beta^5kqb_i \\
& +2b^4c^2\beta^3qy_i + b^4c_{|i}\beta^4kq - 2b^2c^2\beta^4qb_i + b^4c_{|0}\beta^2nqy_i + b^2c_{|0}\beta^3nqb_i + b^2c_{|i}\beta^4nq + b^2c_{|i}\beta^4q\Big)\alpha^5 \\
& +\Big((-8b^4c\beta^4k^2qy_i - 6b^2c\beta^5k^2qb_i - 8b^2c\beta^4kqy_i - 6c\beta^5kqb_i)\mathbf{D}^2 + (-b^2c^2b^4\beta^5knqy_i - 8b^6c\beta^4k^2nqy_i \\
& -6b^4c^2\beta^5knqy_i - 6b^4c\beta^5k^2nqb_i + b^2c^2b^4\beta^5kqy_i - 3b^2c^2\beta^6knqb_i + 3b^4c^2\beta^5kqy_i + 2b^2c^2\beta^6kqb_i \\
& -8b^4c\beta^4knqy_i - 6b^2c\beta^5knqb_i)\Big)\alpha^4 - \beta^4\Big(2b^4c^2\beta knqy_i + 14b^4c\beta^2k^2qy_i + b^2c^2b^4\beta kqy_i + 6b^2c\beta^3k^2qb_i \\
& +3b^4c^2\beta kqy_i + 2b^2c^2\beta^2kqb_i + b^4c_{|0}knqy_i + 2b^2c^2\beta nqy_i + 14b^2c\beta^2kqy_i + 6c\beta^3kqb_i + b^2c_{|0}nqy_i\Big)\alpha^3 \\
& +\Big(6\beta^6(b^2ck^2qy_i + ckqy_i)\mathbf{D}^2 + (6b^4c\beta^6k^2nqy_i + 3b^2c^2\beta^7knqy_i - 2b^2c^2\beta^7kqy_i + 6b^2c\beta^6knqy_i)\Big)\alpha^2 \\
& +\Big(6b^2c\beta^8k^2qy_i + 2b^2c^2\beta^7kqy_i + 6c\beta^8kqy_i\Big)\alpha\Big], \qquad \text{(B.2)}
\end{aligned}
$$

where $\mathbf{V} := b^2k + 1$ and

$$
\Gamma := -\frac{b^2q}{2(b^2\alpha^2 - \beta^2)^{\frac{3}{2}}(b^2k+1)^2\alpha^4}, \qquad \Upsilon := -\frac{\sqrt{b^2\alpha^2 - \beta^2}}{2(b^2\alpha^2 - \beta^2)^2(b^2k+1)^2\alpha^5}.
$$