Permutation groups with cyclic-block property and $MNFC$-groups

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Abstract: This work continues the investigation of perfect locally finite minimal non-$FC$-groups in totally imprimitive permutation $p$-groups. At present, the class of totally imprimitive permutation $p$-groups satisfying the cyclic-block property is known to be the only class of $p$-groups having common properties with a hypothetical minimal non-$FC$-group, because a totally imprimitive permutation $p$-group satisfying the cyclic-block property cannot be generated by a subset of finite exponent and every non-$FC$-subgroup of it is transitive, which are the properties satisfied by a minimal non-$FC$-group. Here a sufficient condition is given for the nonexistence of minimal non-$FC$-groups in this class of permutation groups. In particular, it is shown that the totally imprimitive permutation $p$-group satisfying the cyclic-block property that was constructed earlier and its commutator subgroup cannot be minimal non-$FC$-groups. Furthermore, some properties of a maximal $p$-subgroup of the finitary symmetric group on $N^*$ are obtained.

Key words: Finitary permutation, totally imprimitive, cyclic-block property, homogeneous permutation, $FC$-group

1. Introduction

Let $\Omega$ be a nonempty (infinite) set. A permutation $g$ on $\Omega$ is called finitary if its support $supp(g)$ is finite. The set of all the finitary permutations on $\Omega$ forms a normal subgroup of the symmetric group $Sym(\Omega)$ and is called the restricted symmetric group on $\Omega$. It is denoted by $FSym(\Omega)$. A subgroup of $FSym(\Omega)$ is called a finitary permutation group on $\Omega$. Let $G$ be a transitive finitary permutation group on $\Omega$, where $\Omega$ is infinite. If $G$ has no proper blocks or has a maximal proper block, then $G$ is called primitive or almost primitive, respectively, and then $G$ has a homomorphic image that is isomorphic to one of $Alt(\Omega)$ or $Fsym(\Omega)$ by [10, p.261] (see also [9, Corollary 6.9]). Note that if $\Delta$ is a proper block for $G$, then there exists a $g \in G$ with $g(\Delta) \cap \Delta = \emptyset$ since two blocks are either equal or disjoint and then $\Delta$ must be finite since $supp(g)$ is finite. In the remaining case $G$ is called totally imprimitive. In this case, $G$ has an infinite ascending chain of proper blocks and their union is an infinite block for $G$, which must be equal to $\Omega$ since $G$ is transitive. Thus, $\Omega$ and $G$ are countably infinite. It is well known that a finitary permutation group $G$ has only finite orbits if and only if one of the following holds:

$G$ is solvable, hypercentral, an $FC$-group, or residually finite by [23, Theorems 1,2] or [10, Lemma 8.3D]. If $G$ is locally solvable, then $G$ is totally imprimitive and hyperabelian of height at most $\omega$ by [18, Theorem 2].

Let $G$ be a totally imprimitive subgroup of $FSym(\Omega)$, where $\Omega$ is infinite. It is well known that set-wise stabilizers of finite sets are $FC$-groups and they are hypercentral when $G$ is a $p$-group by [23, Theorem 1] or

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Subgroups having infinite orbits of $G$ are non-$FC$-subgroups $(NFC$-subgroups for short). In general an $NFC$-group is called a minimal $NFC$-group $(MNFC$-group for short) if every proper subgroup of this group is an $FC$-group.

The structure of an imperfect $MNFC$-group was determined in [6, 7] (see also [22, Theorem 8.13]). In this group the commutator subgroup is a divisible abelian $q$-group of finite rank (Chernikov $q$-group) and the commutator quotient is a finite $p$-group, where $p, q$ are primes. On the other hand, it is still unknown whether or not a perfect $MNFC$-group exists. If a perfect $MNF$-group exists, then it is a $p$-group for a prime $p$ by [7, Theorem 2] and [14, Theorem], and it has a nontrivial representation in the group of finitary permutations on some infinite set by the characterizations given in [8, 15]. (Some partial results in this direction are contained in [1–5].)

Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$, where $\Omega$ is infinite. An element $g$ of $G$ is said to satisfy the cyclic-block property if the support of each cycle in the cycle decomposition of $g$ is a block for $G$, and a subset $Y$ of $G$ satisfies the cyclic-block property if every element of $Y$ satisfies this property. Now suppose in addition that $G$ satisfies the cyclic-block property. By [4, Lemma 2.2] two blocks for $G$ are either disjoint or one is contained in the other one. This implies that $G$ must be a $p$-group for a prime $p$. Furthermore, every $NFC$-group of $G$ is transitive and a subset of finite exponent of $G$ generates a subgroup of finite exponent and so cannot be equal to $G$ (see Lemma 3.1(b) below). These are properties satisfied by a perfect $MNFC$-group. (A perfect $MNFC$-group cannot be generated by a subset of finite exponent (see [1, Remark 1.10]).) There are no known other types of $p$-groups that share common properties with a perfect $MNFC$-group. For this reason it is a rather crucial step to settle the existence problem of $MNFC$-groups in the class of permutation groups satisfying the cyclic-block property. In this work a new result (Theorem 1.1) is obtained in this direction. This result is a considerable generalization of [1, Theorem 1.5] (see below). In particular, if a group in this class is generated by homogeneous elements and satisfies $(\ast)$ (see below), then the group cannot be $MNFC$ (Corollary 1.2). Furthermore, Theorem 1.1 provides a short proof for [4, Theorem 1.2] (Corollary 1.3). (Another proof of [4, Theorem 1.2] is contained in [5, Theorem 1.6].) The group given in [4, Theorem 1.1] satisfies the cyclic-block property, it has an easily defined generating set, and all of its blocks of $p$-power size are easily described, but it is not known whether or not it contains an $MNFC$-subgroup. (This group satisfying the cyclic-block property is a transitive subgroup of the maximal $p$-subgroup, denoted by $W$ here, of $FSym(\mathbb{N}^\ast)$ constructed in [22]; see Proposition 2.1 for some properties of $W$.) [5] contains new properties of $NFC$-subgroups of a perfect totally imprimitive $p$-subgroup of $FSym(\Omega)$. Among other things it is shown there that the normalizer of an $NFC$-subgroup is self-normalizing and a self-normalizing subgroup is closed in the topology of point-wise convergence (see also [15]). It follows from [5] that a group of finitary permutations contains an $MNFC$-subgroup if and only if the set of self-normalizing subgroups contains minimal elements.

Let $G$ be a subgroup of $FSym(\Omega)$ and let $g \in G$. The minimum of the lengths of the cycles in the cycle decomposition of $g$ is denoted by $m(g)$. $g$ is called homogeneous if every cycle of $g$ has equal length. An infinite subset $Y$ of $G$ is called ascending if $Y$ has an infinite exponent and is not contained in a set stabilizer of a finite set. We say that $Y$ satisfies the property $(\ast)$ if for every $y, z \in Y$, and for all cycles $c_y, c_z$ in the cycle decompositions of $y$ and $z$, respectively, the following holds. Put $\text{supp}(c_y) = \Delta$ and suppose that $\Delta \subseteq \text{supp}(c_z)$. Then
\[ [c_z^{s(c_z, \Delta)}|_{\Delta}, c_y] = 1 \]

where \( s(c_z, \Delta) \) is the smallest positive integer such that \( c_z^{s(c_z, \Delta)} \in G(\Delta) \).

It is well known that this condition is equivalent to
\[ c_z^{s(c_z, \Delta)}|_{\Delta} = c_y^k \]
for a \( k \geq 1 \) by [13, Lemma 1]. (The centralizer of a cycle is generated by the cycle itself and permutations disjoint with it.)

Let \( \Delta \) be a block for \( G \) and put \( \Sigma = \{ x(\Delta) : x \in G \} \). Then the kernel of the natural permutation representation of \( G \) into \( Sym(\Sigma) \) is denoted by \( Ker_G(\Delta) \) and is called the kernel subgroup of \( G \) with respect to \( \Delta \). Since \( Ker_G(\Delta) \) fixes \( x(\Delta) \) for every \( x \in G \) it follows that \( Ker_G(\Delta) \) is isomorphic to a subgroup of the direct product of copies of a finite group, and so \( Ker_G(\Delta) \) is an FC-group of finite exponent.

For a nonempty subset \( X \) of \( G \), \( \exp(X) \) denotes the maximum of the set \( \{ m(x) : x \in X \} \) if it exists; otherwise, it is equal to \( \infty \).

**Theorem 1.1** Let \( G \) be a perfect totally imprimitive \( p \)-subgroup of \( FSym(\Omega) \), where \( \Omega \) is infinite. Suppose that \( G \) contains an ascending subset \( X \) satisfying the cyclic-block property such that the following properties hold.

(a) \( X \) satisfies \((*)\). Thus for all \( x, y \in X \) and for all cycles \( c_x, c_y \) in the cycle decompositions of \( x \) and \( y \), respectively, the following holds. If \( supp(c_x) \subseteq supp(c_y) \), then
\[ [c_y^{s(c_y, supp(c_x))}|_{supp(c_x)}, c_x] = 1, \]

where \( supp(c_x) \) and \( supp(c_y) \) are blocks for \( G \), which is equivalent to
\[ c_y^{s(c_y, supp(c_x))}|_{supp(c_x)} = c_x^q \]
for a \( q(c_x) \geq 1 \).

(b) For every \( x \in X \) there exists a \( y \in X \) so that \( m(x) < m(y) \).

Then \( G \) cannot be an MNFC-group.

Theorem 1.1 is a considerable generalization of [1, Theorem 1.5]. In [1, Theorem 1.5] if \( F \) is a finite subgroup of \( G \) and \( supp(F) \subseteq \Delta \) for a finite block \( \Delta \), then there exists \( y \in G \setminus G(\Delta) \) so that \( y^{s(y, \Delta)} \in C_G(F) \). In particular \( [F^y, F] = 1 \) since \( supp(F^y) \cap \Delta = 1 \). This leads to the existence of an ascending subgroup \( H \) of \( G \) for a given \( a \in G \) with \( \langle a^G \rangle \) nonabelian so that \( \langle a^H \rangle \) is abelian, which gives a contradiction. On the other hand, in Theorem 1.1, there is information only about the centralizer of a cycle, namely \( c_x \) of \( x \in X \), but \( X \) is required to satisfy the additional property called the cyclic-block property. (Also in the proof of Theorem 1.1 \( \langle c_x^G \rangle \) is not abelian, but there will exist an ascending subgroup, say \( X^* \) of \( G \), so that \( \langle c_x^{X^*} \rangle \) is abelian, which gives a contradiction.) It is not known yet whether condition (b) of Theorem 1.1 is indispensable.
Corollary 1.2 Let $G$ be a perfect totally imprimitive $p$-subgroup of $FSym(\Omega)$, where $\Omega$ is infinite. Suppose that $G$ contains an ascending subset $X$ of homogeneous elements satisfying the cyclic-block property and the $(\ast)$ condition. Then $G$ cannot be an MNFC-group.

Corollary 1.3 The totally imprimitive $p$-subgroup of $FSym(\mathbb{N}^*)$ given in [4, Theorem 1.1] and its commutator subgroup cannot be MNFC-groups.

For definitions, notations, and basic properties the reader is referred to [9, 10, 21, 22].

**Question.** Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$ satisfying the cyclic-block property where $\Omega$ is infinite. Does $G$ contain a minimal non-FC subgroup?

2. A finitary permutation group with cyclic-block property

In this section the finitary permutation $p$-group given in [4] and satisfying the cyclic-block property is described briefly for the convenience of the reader. This group is a subgroup of the example given in [23] by Wiegold.

For each $k, n \geq 1$ define

$$x_{k,n} = \prod_{i=1}^{p-1} (i + (n-1)p^k, i + (n-1)p^k + p^k, \ldots, i + (n-1)p^k + (p-1)p^k).$$

Each $x_{k,n}$ is a disjoint product of $p^{k-1}$ cycles, each of which has length $p$.

For each $k \geq 1$ define

$$T_k = \{x_{k,n} : n \geq 1\} \text{ and } T_k^* = \langle T_i : 1 \leq i \leq k \rangle.$$

Wiegold’s group, denoted here by $W$, is defined as $W = \langle T_k : k \geq 1 \rangle$. $T_k$ is a set of pairwise disjoint permutations of order $p$ and it is easy to check that $T_k^* \triangleleft W$ and $T_{k+1}^*/T_k^*$ is elementary abelian for every $k \geq 0$, where $T_0 = 1$. $W$ is a totally imprimitive $p$-subgroup of $FSym(\mathbb{N}^*)$ since every element of every $T_k^*$ has finite support.

For $p = 2$,

$$T_1 = \{(1,2),(3,4),(5,6),\ldots\}, T_2 = \{(1,3)(2,4),(5,7)(6,8),(9,11)(10,12),\ldots\}$$

$$T_3 = \{(1,5)(2,6)(3,7)(4,8),(9,13)(10,14)(11,15)(12,16),\ldots\}.$$

For all $k, n \geq 1$ the sets

$$\Delta_{k,n} = \{1 + (n-1)p^k, 2 + (n-1)p^k, \ldots, p^k + (n-1)p^k\}$$

are blocks for $W$ and $|\Delta_{k,n}| = p^k$. We may show that each $\Delta_{k,1}$ is a block. We may put $\Delta_k = \Delta_{k,1}$ when no confusion arises. Thus, $\Delta_k = \{1,2,\ldots,p^k\}$ for $k \geq 1$. It suffices to show that $T_k^*(1) = \Delta_k$ for all $k \geq 1$ by [10, Theorem 1.6A(i)]. For $k = 1$ $T_1^*(1) = \{1,2,\ldots,p\} = \Delta_1$. Assume that $T_k^*(1) = \Delta_k$. Now

$$x_{k+1,1} = (1,1+p^k,1+2p^k,\ldots,1+(p-1)p^k)\cdots(p^k,p^k+p^k,p^k+2p^k,\ldots,p^k+(p-1)p^k)$$

$$= (1,1+p^k,1+2p^k,\ldots,1+(p-1)p^k)\cdots(p^k,2p^k,3p^k,\ldots,p^{k+1}).$$
Hence, it is easy to see that
\[
\langle x_{k+1,1} \rangle (\Delta_k) = \{1, 2, \ldots, p^k\} \cup \{1 + p^k, 2 + p^k, \ldots, 2p^k\} \cup \cdots \cup \{1 + (p - 1)p^k, 2 + (p - 1)p^k, \ldots, p^{k+1}\}
\]
\[
= \Delta_{k+1}
\]
since the sets in the union are pairwise disjoint and are contained in \( \Delta_{k+1} \). In particular, it is easy to see that \( x_{k+1,1} \) permutes the sets
\[
\{1, 2, \ldots, p^k\}, \{1 + p^k, 2 + p^k, \ldots, 2p^k\}, \ldots, \{1 + (p - 1)p^k, 2 + (p - 1)p^k, \ldots, p^{k+1}\}
\]
among themselves. Since \( \langle x_{k+1,1} \rangle (\Delta_k) = \langle x_{k+1,1} \rangle (T_k^*(1)) = ((\langle x_{k+1,1} \rangle T_k^*)(1) = T_{k+1}^*(1) \) it follows that \( T_{k+1}^*(1) = \Delta_{k+1} \), which was to be shown. It can be shown that the finite blocks of \( p \)-power size for \( W \) consist of
\[
\Delta_{k,n} = \{1 + (n - 1)p^k, 2 + (n - 1)p^k, \ldots, p^k + (n - 1)p^k\}
\]
for \( k, n \geq 1 \).
Define
\[
u_k = x_{k,1}x_{k-1,1} \cdots x_{1,1}
\]
for all \( k \geq 1 \). Then \( u_k \in T_k^* \) and \( u_k = (a_1, a_2, \ldots, a_{p^k}) \), where \( 1 \leq a_i \leq p^k \) by [4, Lemma 3.2(a)]. Next define
\[
\nu_k = x_{k,x_{k,1}} \cdots x_{k,x_{k,1}}.
\]
Then \( v_k = u_k^{x_{k,1}} \times \ldots \times u_k^{x_{k,1}} \), i.e. a product of disjoint cycles since \( \text{supp}(u_k) = \Delta_k \) and \( x_{k+1,1} \) sends each \( 1 \leq i \leq p^{k+1} \) to \( i + p^k \mod (p^{k+1}) \). (Always \( c = a \times b \) means that \( a, b \) are disjoint permutations.) Put \( g_k = u_k \times \nu_k \) for every \( k \geq 1 \) and define \( G = \langle g_k : k \geq 1 \rangle \). Then \( G \) satisfies the cyclic-block property by [4, Theorem 1.1]. We see from the definitions that \( \{g_k : k \geq 1\} \) is an ascending set of homogeneous elements of \( G \). Furthermore, it follows from the definition that
\[
u_k = u_k^{x_{k,1}} \times \ldots \times u_k^{x_{k,1}} = \langle (x_{k+1,1}u_k)^{p^n} = x_{k+1,1}^{x_{k+1,1}} \cdots \times u_k^{x_{k,1}}u_k = u_k \times u_k^{x_{k,1}} \times \ldots \times u_k^{x_{k,1}} = g_k
\]
for every \( k \geq 1 \). Hence, it follows that the \( g_k \) satisfy (\( \ast \)) as can be seen from the proof of Corollary 1.2. It can also be shown easily that \( G \subseteq W' \). Indeed,
\[
g_1 = x_{1,1} \cdots x_{1,1} = x_{1,1}^{x_{1,1}} = x_{1,1}^{x_{1,1}}[x_{1,1}, x_{2,1}] \cdots [x_{1,1}, x_{2,1}^{-1}] \in W'
\]
since \( x_{1,1} = 1 \). Assume that \( g_k \in W' \) for a \( k \geq 1 \). Now
\[
g_{k+1} = u_ku_k^{x_{k,1}} \cdots u_k^{x_{k,1}} = u_k^{x_{k,1}}[u_k, x_{k+1,1}] \cdots [u_k, x_{2,1}^{-1}].
\]
Since \( u_k^{x_{k,1}} = g_{k-1} \in W' \) it follows that \( g_{k+1} \in W' \), which completes the induction, and so \( G \subseteq W' \).
As was indicated above, each \( u_k \) is a cycle of length \( p^k \) with \( \text{supp}(u_k) = \Delta_k \) by [4, Lemma 3.2(a)]. Hence, \( \text{supp}(u_k^{x_{k,1} + 1}) = x_{k,x_{k,1}}^{-1}(\Delta_k) \) for every \( i \geq 1 \) and hence
\[
\text{supp}(v_k) = \bigcup_{i=1}^{p-1} \text{supp}(u_k^{x_{k,1} + 1}) = \Delta_{k+1} \setminus \Delta_k.
\]
For $p = 2$
\[ u_1 = (1, 2); u_2 = (1, 4, 2, 3); u_3 = (1, 8, 4, 6, 2, 7, 3, 5) \]
and
\[ u_4 = (1, 16, 8, 12, 4, 14, 6, 10, 2, 15, 7, 11, 3, 13, 5, 9). \]

Hence,
\[ g_1 = (1, 2)(3, 4); g_2 = (1, 4, 2, 3)(5, 8, 6, 7); g_3 = (1, 8, 4, 6, 2, 7, 3, 5)(9, 16, 12, 14, 10, 15, 11, 13). \]

Finally, it follows from [4, Theorem 1.1, Lemmas 2.2 and 3.4] that $G$ satisfies the cyclic-block property, any two blocks for $G$ are either disjoint or one is contained in the other one, and the blocks for $G$ are the blocks for $W$. Thus, the set of the blocks of the same $p$-power size for $G$ form a block system for $G$ and hence also for $W$.

We end this section with a characterization of $W$.

**Proposition 2.1** $W$ is a transitive maximal $p$-subgroup of $FSym(N^*)$, $Z(W) = 1$, self-normalizing, and $W/W'$ is infinite elementary abelian.

**Proof** Put $W = \bigcup_{k=1}^{\infty} W_k$ and also $N^* = \bigcup_{k=1}^{\infty} \Delta_k$, where $\Delta_k = \{1, 2, \ldots, p^k\}$ for every $k \geq 1$. It is easy to see that each $W_k$ is transitive on $\Delta_k$, which implies that $W$ is transitive on $\Omega$, and then $Z(W) = 1$ by [10, Lemma 8.3C(ii)].

First we show that $W_k$ is a Sylow $p$-subgroup of $Sym(\Delta_k)$ for every $k \geq 1$. Note that $supp(W_k) = \Delta_k$. Put
\[ P_k = \langle (x_{1,1}, \ldots, x_{k,1}) \rangle. \]

Then $P_k$ is isomorphic to a Sylow $p$-subgroup of $Sym(\Delta_k)$ by [11, Proposition 19.10] since each $G_{k,i}$ has order $p$. It will suffice to show that $W_k \cong P_k$. For $k = 1$ the assertion holds since $\langle x_{1,1} \rangle = \langle (1, 2, \ldots, p) \rangle$ is a Sylow $p$-subgroup of $Sym\{1, 2, \ldots, p\}$. Suppose that the assertion holds for $k \geq 1$. Then $W_k \cong \langle x_{1,1} \rangle \langle x_{2,1} \rangle \cdots \langle x_{k,1} \rangle$, and by identifying these two groups, $W_k$ becomes a Sylow $p$-subgroup of $Sym(\Delta_k)$. Thus, we get $W_{k+1} \cong W_k \langle x_{k+1,1} \rangle$. Let $B_k$ be the base subgroup; that is, $B_k = \prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b$, where each $(W_k)_b$ is equal to $W_k$. Then $P_{k+1}$ is isomorphic to $B_k \langle x_{k+1,1} \rangle$ and so $|P_{k+1}| = p|B_k| = p|W_k|^p$. However, we have seen above that $x_{k+1,1}$ permutes the sets $\{1, 2, \ldots, p^k\}$, $\{1 + p^k, 2 + p^k, \ldots, 2p^k\}$, $\{1 + (p - 1)p^k, 2 + (p - 1)p^k, \ldots, p^k + 1\}$ among themselves and induces a cycle of length $p$ on them. Also, $W_k$ is a Sylow $p$-subgroup of $Sym(\Delta_k)$. Clearly then $x_{i,1}^{-1} W_k x_{i,1}^{x_{k+1,1}}$ are Sylow $p$-subgroups on the corresponding sets $x_{i,1}^{-1} \Delta_k$ for $i = 1, \ldots, p$. Thus, $x_{k+1,1}^{-1} W_k x_{k+1,1}^{x_{k+1,1}}$ and $x_{k+1,1}^{-1} W_k x_{k+1,1}^{x_{k+1,1}}$ have disjoint supports for $i \neq j$ and so they commute. Therefore, we get
\[ W_{k+1} = (W_k x_{k+1,1}^{-1} W_k x_{k+1,1} \cdots x_{k+1,1}^{-1}) x_{k+1,1} \]

This gives $|W_{k+1}| = |W_k|^p |x_{k+1,1}| = p|W_k|^p$ and hence $|W_{k+1}| = |P_{k+1}|$. This implies that $W_{k+1}$ is a Sylow $p$-subgroup of $Sym(\Delta_{k+1})$ since $W_{k+1} \leq Sym(\Delta_{k+1})$, which completes the induction. Clearly it follows from this that $W$ is a Sylow $p$-subgroup of $FSym(N^*)$ since $FSym(N) = \bigcup_{k=1}^{\infty} Sym(\Delta_k)$.

Next we show that $W$ is self-normalizing. Assume not. Then there exists a subgroup $Y$ of $FSym(N^*)$ with $W < Y$ and $Y/W$ is abelian. Also, $Y$ is transitive since $W$ is. Moreover, $Y' \leq W$ and so $Y'$ is a $p$-group, but then $Y'$ is a $p$-group by [20, Lemma 2.1], which is a contradiction.
Finally, we show that $W/W' = \prod_{k=1}^{\infty} \langle x_{k,1}W' \rangle$, as a direct product. This will be the case if we can show that $W_{k}/W'_{k} = \prod_{i=1}^{k} \langle x_{i,1}W'_{k} \rangle$, as a direct product, for every $k \geq 1$. For $k = 1$ this is trivial. Suppose that the assertion holds for $k \geq 1$. We have seen above that

$$W_{k+1} \cong W_k \triangleright \langle x_{k+1,1} \rangle = (\prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b) \langle x_{k+1,1} \rangle = B_k \langle x_{k+1,1} \rangle,$$

which implies that $W_{k+1}/W'_{k+1} \cong B_k \langle x_{k+1,1} \rangle/(B_k \langle x_{k+1,1} \rangle)'$. We can now apply [17, Corollary 4.5] to $B_k \langle x_{k+1,1} \rangle$. This gives

$$(B_k \langle x_{k+1,1} \rangle)' = M$$

where $M = \{ f \in B_k : \pi(f) \in W'_{k} \}$ and $\pi(f) = \prod_{b \in \langle x_{k+1,1} \rangle} f(b)$. Next define $x_{i,1}^*(1) = x_{i,1}$ and $x_{i,1}^*(b) = 1$ for $b \neq 1$ for $1 \leq i \leq k$. Each $x_{i,1}^* \in B_k = \prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b$. We claim that $x_{1,1}^* W \cdots x_{k,1}^* M$ are linearly independent over $\mathbb{Z}_p$, the field of $p$ elements. Assume if possible that there exists an $f = (x_{1,1}^*)^{s_1} \cdots (x_{r,1}^*)^{s_r}$, where $1 \leq r \leq k$ and $1 \leq s_i < p$ so that $f \in M$. Then $f = (x_{1,1}^{s_1} \cdots x_{r,1}^{s_r}, 1, \ldots, 1)$ and $\pi(f) = x_{1,1}^{s_1} \cdots x_{r,1}^{s_r} \in W'_{k}$, but since $W_k/W'_{k} = \langle x_{1,1}W'_{k} \rangle \cdots \langle x_{k,1}W'_{k} \rangle$ by the induction hypothesis it follows that $x_{1,1}^{s_1} W_k = \cdots = x_{r,1}^{s_r} W_k = 1$, which means that $x_{i,1}^{s_i} \in W_k$ and then $p | s_i$ since $|x_{i,1}| = p$, which is impossible since $1 \leq s_i < p$ for every $i \geq 1$. Consequently it follows that $x_{1,1}^* M, \ldots, x_{k,1}^* M$ are linearly independent in $B_k \langle x_{k+1,1} \rangle/M$. Then also $x_{1,1}^* M, \ldots, x_{k,1}^* M, x_{k+1,1}^* M$ are linearly independent in $B_k \langle x_{k+1,1} \rangle/M$ since $\langle x_{k+1,1} \rangle \cap B_k = 1$. Therefore,

$$B_k \langle x_{k+1,1} \rangle/M = \langle x_{1,1}^* M \rangle \cdots \langle x_{k+1,1}^* M \rangle.$$ 

Hence, using the above isomorphism, we get

$$W_{k+1}/W'_{k+1} = \langle x_{1,1}W'_{k+1} \rangle \cdots \langle x_{k+1,1}W'_{k+1} \rangle,$$

which completes the induction. Now since $W = \bigcup_{k=1}^{\infty} W_k$ it follows easily that

$$W/W' = \prod_{k=1}^{\infty} \langle x_{k,1}W' \rangle$$

as a direct product. Suppose that $\langle x_{t,1}W' \rangle \cap \langle x_{k,1}W' : k \geq 1, k \neq t \rangle \neq 1$ for a $t \geq 1$. Then $\langle x_{t,1}W' \rangle \leq \langle x_{t,1}W' : k \geq 1, k \neq t \rangle$ since $|x_{t,1}| = p$. Hence, $x_{t,1}$ is a finite product of elements of certain cosets of the right side. Also, $W' = \bigcup_{k=1}^{\infty} W'_k$. Clearly then there exists an $n > t$ so that $x_{t,1} \in \langle x_{k,1}W'_n : 1 \leq k \leq n, k \neq t \rangle$, but since $W_n/W'_n = \langle x_{1,1}W'_n \rangle \cdots \langle x_{n,1}W'_n \rangle$, as was shown above, this is impossible. Therefore, the assumption is false and so $W/W'$ is a direct product of the $\langle x_{k,1}W' \rangle$ as $k$ ranges over the positive integers. 

**Remark.** The commutator subgroup $W'$ of $W$ is perfect and transitive by [19, Theorem 1]. Also, $W$ does not satisfy the normalizer condition by [1, Theorem 1.2(b)] since $G \leq W$ and $G'$ is not an MNFC-group by Corollary 1.3. The reader may observe that $W$ is exactly the same group that is constructed in [12, 18.2.2 Example], where it is shown also that this group does not satisfy the normalizer condition.

### 3. Proof of Theorem 1.1

We begin with a known result on the cyclic-block property for the convenience of the reader. (See also [5, Proposition 1.7].)
Lemma 3.1 3.1 Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$ satisfying the cyclic-block property, where $\Omega$ is infinite. Then the following hold:

(a) Let $\Delta$ be a finite block for $G$ and let $\alpha \in \Delta$. Then for every $y \in G \setminus G_{\{\Delta\}}$, $\langle y^{s(y,\Delta)} \rangle(\alpha) = \Delta$.

(b) Let $\Delta$ be a finite block for $G$. Then

$$Ker_G(\Delta) = \{ g \in G : |g| \leq |\Delta| \}.$$

Furthermore, $\exp(G_{\{\Delta\}})$ is infinite.

(c) Any NFC-subgroup of $G$ is transitive on $\Omega$.

Proof (a) (See [4, Lemma 2.1].) Put $H = G_{\{\Delta\}}$ and let $y \in G \setminus H$. Put $t = s(y, \Delta)$. Then $t$ is the smallest number such that $y^t \in H$. Also, $t = p^r$ for an $r \geq 1$. Next put $\langle y \rangle(\alpha) = \Gamma$ and $\langle y^t \rangle(\alpha) = \Lambda$. Then $\Gamma$ and $\Lambda$ are blocks for $G$ by the cyclic-block property. Also, $\Delta \subseteq \Gamma$ and $\Lambda \subseteq \Delta$ by [4, Lemma 2.2] since $y \not\in H$ but $y^t \in H$. Clearly $|\Gamma| = p^r|\Delta|$. Assume if possible that there exists a $y^t(\alpha) \in \Delta \setminus \Lambda$. Then $j \nmid p^r$ and so $j < p^r$, but since $\alpha \in \Delta \cap y^{-j}(\Delta)$ and since $\Delta$ is a block, it follows that $y^t(\Delta) = \Delta$, which is a contradiction since $t = p^r$ is the smallest number with the property that $y^t(\Delta) = \Delta$. Therefore, the assumption is false and so $\langle y^t \rangle(\alpha) = \Delta$.

(b) Put $M = Ker_G(\Delta)$. Then $M < H$ since $H \neq G$ due to the fact that $\Omega$ is infinite and $G$ is transitive. Let $y \in G$ and put $|y| = t$. First suppose that $t \leq |\Delta|$. Then we claim that $y \in H$. This is trivial if $\supp(y) \cap \Delta = \emptyset$ since then $y(\Delta) = \Delta$. Suppose that $\langle y \rangle(\alpha) \neq \alpha$ for an $\alpha \in \Delta$. Put $\Gamma = \langle y \rangle(\alpha)$. Then $\Gamma$ is a block for $G$ by the hypothesis and $|\Gamma| \leq t \leq |\Delta|$. Also, since $\alpha \in \Gamma \cap \Delta$ applying [4, Lemma 2.2], we get $\Gamma \subseteq \Delta$, which implies that $y \in H$. Thus, $\{ g \in G : |g| \leq |\Delta| \} \subseteq M$. Next suppose that $t > |\Delta|$. There exists a $\beta \in \Omega$ so that $t = |\langle y \rangle(\beta)|$. Also, there exists a $g \in G$ so that $g(\beta) = \alpha$. Since $\langle y \rangle(\beta) = \{ \beta, y(\beta), \ldots, y^{t-1}(\beta) \}$, it follows that $\langle g y g^{-1} \rangle(\alpha) = \{ g y(\beta), \ldots, g y^{t-1}(\beta), g(\beta) \}$. Now if $y \in M$ then also $gg^{-1} \in M$, but since $\langle g y g^{-1} \rangle(\alpha)$ is a block containing $\alpha$ and has size greater than $|\Delta|$, this is a contradiction. Therefore, $M = \{ g \in G : |g| \leq |\Delta| \}$. In particular it follows that any subgroup of finite exponent of $G$ is contained in a kernel subgroup which is nilpotent of finite exponent. It is well-known that a transitive subgroup of $FSym(\Omega)$ has infinite exponent if $\Omega$ is infinite by [18, Lemma 3.1] or [10, Theorem 8.3A]). Let $\alpha \in \Omega$. We show that $G_{\alpha}$ contains a conjugate of every element of $G$. Let $g \in G$. There exists a $\beta \in \Omega$ so that $g(\beta) = \beta$ and so $g \in G_{\beta}$. Also, $\beta = x(\alpha)$ for an $x \in G$. Hence, $g \in G_{x(\alpha)} = x G_{\alpha} x^{-1}$ and so $g^x \in G_{\alpha}$, which completes the proof of (b).

(c) Let $X$ be a proper NFC-subgroup of $G$. Then $X$ cannot be contained in the set-wise stabilizer of a finite block for $G$ since $X$ is not an FC-group. However, if $\exp(X) \leq |\Delta|$ for a finite block $\Delta$, then $X \leq Ker(\Delta) \leq G_{\{\Delta\}}$ by (b), which is impossible. Therefore, $\exp(X) = \infty$. Let $\alpha, \beta \in \Omega$ and let $\Delta$ be a finite block for $G$ containing both of them. Then there exists a $g \in X \setminus X_{\{\Delta\}}$ so that $\langle g^p \rangle(\alpha) = \Delta$ by (a), which implies that $\beta = (g^p)^j(\alpha)$ for a $j \geq 1$, and so $X$ is transitive.

Lemma 3.2 Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$ and let $c, d$ be two cycles in $G$ such that $\supp(c), \supp(d)$ are blocks for $G$ and $\supp(c) \subseteq \supp(d)$. Let $|c| = p^a, |d| = p^b$ and put $t = s(d, \supp(c))$. 990
Then \( t = p^{b-a} \). If \( d^{b-a}|_{\text{supp}(c)} = e^k \), then \((p,k) = 1\) and

\[
d^{b-a} = e^k \times (e^k)^d \times \cdots \times (e^k)^{d^{b-a}-1}.
\]

**Proof** Put \( \Delta = \text{supp}(c) \), \( \Gamma = \text{supp}(d) \). Then \(|\Delta| = p^a\) and \(|\Gamma| = p^b\). Let \( \alpha \in \Delta \). Now \( \Delta \subseteq \Gamma \). Clearly \( \Gamma = \Delta \cup d(\Delta) \cup \cdots \cup d^{t-1}(\Delta) \) as a disjoint union since \( \Delta \) is a block and \( d \) is a cycle. Hence, \( p^b = tp^a \) and hence \( t = p^{b-a} \).

Put \( H = G_{(\Delta)} \). Then \( t \) is the smallest number with \( d^t \in H \). Hence, \( \langle d^{b-a} \rangle(\alpha) \subseteq \Delta \) and \( |d^{b-a}| = |\langle d^{b-a} \rangle(\alpha)| \) since \( d \) is a cycle, which implies that \( |\langle d^{b-a} \rangle(\alpha)| = p^a \). Now suppose that \( d^{b-a} \Delta = e^k \). Then \( p \nmid k \) since \( |c| \) is a cycle of length \( p^a \). Thus, \((p,k) = 1\) and \( e^k \) is a cycle.

Now \( d^{b-a}|_{d^t(\Delta)} = d^e e^k d^{-t} \) for every \( 1 \leq i \leq p^{b-a} \) and \( \Gamma = \Delta \cup d(\Delta) \cup \cdots \cup d^{b-a-1}(\Delta) \). Obviously then

\[
d^{b-a} \Gamma = e^k \times (e^k)^d \times \cdots \times (e^k)^{b-a-1}.
\]

\[\square\]

**Lemma 3.3** Let \( G \) be a totally imprimitive \( p \)-subgroup of \( \text{FSym}(\Omega) \). Let \( X \) be an ascending subset of \( G \) satisfying the cyclic-block property. Suppose also that \( X \) satisfies \((*)\). Then \( X \) contains an ascending subset \( Y = \{y_i : i \geq 1\} \) of \( G \) such that the following holds. Each \( y_i \) can be expressed as a direct product of cycles as

\[
y_i = c_{i,1} \times \cdots \times c_{i,r(i)}
\]

so that \( \text{supp}(y_i) \subseteq \text{supp}(c_{i+1,1}) \), \( m(y_i) \leq m(y_{i+1}) \), and the following hold. Let \( 1 \leq j \leq r(i) \) and \( k \geq i \). Put \( |c_{i,j}| = p^a \) and \( |c_{k,1}| = p^b \). Then

\[
[c_{k,1}^{b-a}|_{\text{supp}(c_{i,j})}, c_{i,j}] = 1.
\]

**Proof** Choose a \( y_1 \neq 1 \) in \( X \) so that \( m(y_1) \leq m(x) \) for every \( x \in X \) and let

\[
y_1 = c_{1,1} \times \cdots \times c_{1,r(1)}
\]

be the cycle decomposition of \( y_1 \). Let \( \Gamma_1 \) be the smallest block containing \( \text{supp}(y_1) \). Next choose a \( y_2 \) in \( X \setminus G_{(\Gamma_1)} \) so that \( m(y_2) \leq m(x) \) for every \( x \in X \setminus G_{(\Gamma_1)} \). Now \( \langle y_2 \rangle(\Gamma_1) \) is a block by the cyclic-block property and \( \Gamma_1 \supseteq \text{supp}(y_2)(\alpha) \) by [4, Lemma 2.2] since \( y_2 \notin G_{(\Gamma_1)} \). Also, \( m(y_1) \leq m(y_2) \). Put \( c_{2,1} = (\alpha, \ldots, y_{2}^{t-1}(\alpha)) \), where \( t_2 \) is the smallest number such that \( y_{2}^{t_2}(\alpha) = \alpha \). Thus, \( \Gamma_1 \supseteq \text{supp}(c_{2,1}) \). Continuing in this way we obtain an infinite subset \( Y = \{y_i : i \geq 1\} \) of \( X \) such that \( m(y_i) \leq m(y_{i+1}) \) and \( \text{supp}(y_i) \subseteq \text{supp}(c_{i+1,1}) \) for every \( i \geq 1 \), where

\[
y_i = c_{i,1} \times \cdots \times c_{i,r(i)}
\]

is the cycle decomposition of \( y_i \). Let \( 1 \leq i < k \) and let \( 1 \leq j \leq r(i) \). Then \( \text{supp}(c_{i,j}) \subseteq \text{supp}(c_{k,1}) \). Also, \( Y \) satisfies \((*)\) since \( Y \) is a subset of \( X \). Therefore,

\[
[c_{k,1}^{(c_{i,1}, \text{supp}(c_{i,j}))}|_{\text{supp}(c_{i,j})}, c_{i,j}] = 1.
\]
Let $|c_{i,j}| = p^a$ and $|c_{k,1}| = p^b$. Then since $s(c_{k,1}, \text{supp}(c_{i,j})) = p^{b-a}$ by Lemma 3.2, substituting this value above the desired equality is obtained. Furthermore, $Y$ is ascending since $\text{supp}(c_{i,1})$ is a block and $\text{supp}(c_{i,1}) \subseteq \text{supp}(c_{i+1,1})$ for every $i \geq 1$.

\textbf{Lemma 3.4} Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$. Let $X$ be an ascending subset of $G$ satisfying the cyclic-block property and ($\ast$). Let $Y = \{y_i : i \geq 1\}$ be the subset of $X$ obtained in Lemma 3.3. Thus, for each $i \geq 1$, the cycle decomposition of $y_i$ can be written as \[ y_i = c_{i,1} \times \cdots \times c_{i,r(i)} \] such that $\text{supp}(y_i) \subseteq \text{supp}(c_{i+1,1})$ and $m_1(y_i) \leq m_1(y_{i+1})$ for every $i \geq 1$. Moreover, if we put $|c_{i,j}| = p^{a_{i,j}}$, for every $i \geq 1$ and $1 \leq j \leq r(i)$, then for $1 \leq i \leq k$ the equality \[ c_{k,1}^{p^{a_{(k,1)-a_{(i,j)}}}}|_{\text{supp}(c_{i,j})} = c_{i,j}^{q_{(i,j)}} \] holds for a $q(i,j) \geq 1$ with $(p,q(i,j)) = 1$ by Lemma 3.3.

Now let $j, k, t \geq 1$ be integers with $j \leq k, t$ and suppose that $|y_j| \leq \min\{m(y_k), m(y_t)\}$. Let $m, n \geq 1$. Then \[ c_{j,1}^{y_j^m} y_j^n = c_{j,1}^{y_j^m} y_j^n \] for an $r \in \{k, t\}$ and $s \geq 1$.

\textbf{Proof} Put $c_i = c_{i,1}$ and let $|\text{supp}(c_i)| = p^{a(i)}$ for $i = j, k, t$. Then $c_i$ is a factor of the cycle decomposition of $y_i$ for $i = j, k, t$ and $\text{supp}(y_u) \subseteq \text{supp}(c_i)$ for every $1 \leq u < v$. We may suppose that $j < k, t$.

\textbf{Case 1} $j < k < t$. Now \[ c_{j,1}^{y_j^m} y_j^n = c_{j,1}^{y_j^m} y_j^n \] since $\text{supp}(c_j) \subseteq \text{supp}(c_k)$. On the other hand, \[ c_{i}^{a_{(i)-a(k)}} = c_{k}^{a_{(k,1)}} \times \cdots \times (c_{k}^{a_{(k,1)}} c_{i}^{a_{(i)-a(k)}} - 1) = c_{k}^{a_{(k,1)}} \times v_k \] by (1) and Lemma 3.2, where $\text{supp}(v_k) \cap \text{supp}(c_k) = \emptyset$. Also, $bq(k,1) \equiv 1 \mod (p^{a(k)})$ for an integer $b$ since $(q(k,1), p) = 1$ by Lemma 3.2. Using this above gives \[ c_{k}^{a_{(k,1)-a(k)}} = c_{k}^{a_{(k,1)}} \times c_{k}^{a_{(k,1)-a(k)} - 1} = c_{k} \times v_k. \]

Hence, $c_{k}^{m} = c_{i}^{mbp^{a_{(i)-a(k)}} v_k^{-b}}$. Substituting this in (2) gives \[ c_{j}^{v_k^{-b}} y_j^n = c_{j}^{v_k^{-b}} c_{j}^{bmbp^{a_{(i)-a(k)} - n}} y_j^n = c_{j}^{v_k^{-b}} c_{j}^{bmbp^{a_{(i)-a(k)} + n}} y_j^n = c_{j}^{v_k^{-b}} c_{j}^{bmbp^{a_{(i)-a(k)} + n}} y_j^n \] since $\text{supp}(c_j) \subseteq \text{supp}(c_k)$.

\textbf{Case 2} $k > t > j$. We may suppose that $\text{supp}(c_{j,1}^m) \cap \text{supp}(y_t^n) \neq \emptyset$; otherwise, $c_{j,1}^{m} y_t^n = c_{j,1}^{m}$ and we are done. Then there exists a cycle $c_{t,1}$ in the cycle decomposition of $y_t$ so that $\text{supp}(c_{j,1}^m) \subseteq \text{supp}(c_{t,1})$ by...
the cyclic-block property since \(|y_j| < m(y_1)| by the hypothesis. For simplicity, put \(u_t = c_{t,r}\) and \(q(t) = q(t,r)\). Clearly now \(c_j^{c_k y_t^p} = c_j^{y_t^p u_t^q}\). Let \(|u_t| = p^z\). Then
\[
e^p_{(k)} = u_t^{q(t)} \times \cdots \times (u_t^{q(t)})^k = u_t^{q(t)} \times v_t
\]
where \((q(t), p) = 1\) by (1). Then, as in Case 1, there exists an integer \(b\) so that
\[
\epsilon^p_{(k)} = u_t \times v_t^b
\]
where \(\text{supp}(u_t) \cap \text{supp}(v_t) = \emptyset\). Hence \(u_t^n = v_t^{-bn} \epsilon^p_{(k)} u_t^b\). Substituting this above gives
\[
\epsilon^m_{(k)} u_t^n = c_j^{c_k y_t^p u_t^n} = c_j^{c_k y_t^p u_t^b} = c_j^{y_t^p u_t^b} c_j^{c_k y_t^p b^+(k) - z}
\]
where \(\text{supp}(c_j^m) \subseteq \text{supp}(u_t)\) and \(\text{supp}(u_t) \cap \text{supp}(v_t) = \emptyset\), which completes the proof of the lemma.

\textbf{Lemma 3.5} Let \(G\) be a totally imprimitive \(p\)-subgroup of \(FSym(\Omega)\). Let \(X\) be an ascending subset of \(G\) satisfying the cyclic-block property and (\(\ast\)). Let \(Y = \{y_i : i \geq 1\}\) be the subset of \(X\) obtained in Lemma 3.3 and suppose that \(|y_i| < m(y_{i+1})\) for every \(i \geq 1\). Let \(j \geq 1\) and put \(Y^*_j = \langle y_i : i \geq j \rangle\). Let \(y = y_{k_1} m_1 \cdots y_{k_r} m_r \in Y^*_j\), where \(k_i \geq j\), \(r \geq 1\), \(m_i \geq 1\), and \(k_u \neq k_{u+1}\). Then \(c_{j,1}^y = c_{j,1}^y a \) for every \(a \in \{k_1, \ldots, k_r\}\) and \(s \geq 1\).

\textbf{Proof} We may use induction on \(r \geq 1\). For \(r = 1\) the assertion is obvious. Suppose that \(r > 1\) and the assertion holds for numbers less than \(r\). Note that \(|y_j| < m(y_{k_1})\) for \(i = 1, \ldots, q\) by the hypothesis. Hence, applying Lemma 3.4 we obtain a \(k \in \{k_1, k_2\}\) and an \(m \geq 1\) so that \(c_{j,1}^{y_{k_1} m_1 m_2 \cdots y_{k_r} m_r} = c_{j,1}^{y_{k_1} m_1 m_2 \cdots y_{k_r} m_r}\). Then the induction hypothesis applies to the right side of the preceding equality. Therefore, there exist a \(u \in \{k, k_3, \ldots, k_r\}\) and an \(s \geq 1\) so that \(c_{j,1}^{y_{k_1} m_1 m_2 \cdots y_{k_r} m_r} = c_{j,1}^{y_{k_1} m_1 m_2 \cdots y_{k_r} m_r}\). Then since \(c_{j,1}^y = c_{j,1}^y a \) the induction and the proof of the lemma are complete.

\textbf{Lemma 3.6} Let the hypothesis and the notation be as in Lemma 3.5. Let \(j \geq 1\). Then \([c_{j,1}^y, c_{j,1}] = 1\) for every \(y \in Y^*_j\).

\textbf{Proof} Put \(c_j = c_{j,1}\). Let \(y \in Y^*_j\). We have \(c_j^y = c_j^{y_k^1}\) for a \(k \geq j\) and an \(s \geq 1\) by Lemma 3.5. Let \(\text{supp}(c_j) = \Gamma_j\) and put \(H = G(\Gamma_j)\). If \(y_k \not\in H\), then \(y_k(G_j) \cap \Gamma_j = \emptyset\) and since \(\text{supp}(c_j^{y_k}) = y_k^s(\Gamma_j)\) it follows that \([c_j^{y_k}, c_j] = 1\), and the assertion holds in this case since \(c_j^y = c_j^{y_k}\).

Next suppose that \(y_k \in H\). Let \(c_{k,1} = c_k\), \(\text{supp}(c_k) = \Gamma_k\), \(|\Gamma_j| = p^a(\Gamma_j)\), and \(|\Gamma_k| = p^b(\Gamma_k)\). Then \(p^a(\Gamma_k - a(\Gamma_j))s\) by Lemma 3.2 and hence \(s = p^a(\Gamma_k - a(\Gamma_j))t\) for \(a \geq 1\). Also, \(\text{supp}(y_j) \subseteq \Gamma_k\) and \(c_j^{p^a(\Gamma_k - a(\Gamma_j))} = c_j^{y_j} c_j^{v_k}\) for a \(v_k \in FSym(\Omega)\) and \(q(j) \geq 1\) by (1) in Lemma 3.4. Hence, \(c_j^y = c_j^{y_j} v_k^t\), but also \(y_k = c_k \times z_k\) for a
$z_k \in FSym(\Omega)$. Combining these values we get $y^i_k = c_j^{\nu(j)(v_k^i z_k^i)}$, where $supp(v_k z_k) \cap \Gamma_k = \emptyset$. Using this last equality we get

$$c_j y^i_k = c_j^{\nu(j)(v_k^i z_k^i)} = c_j z_k^i = c_j$$

and hence

$$[c_j y^i_k, c_j] = [c_j, c_j] = 1,$$

which was to be shown.

\[\square\]

**Lemma 3.7** Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$, where $\Omega$ is infinite. Let $y \in G$ and let $j > 1$ so that $\alpha \in supp(y) \subset \langle c_j^p \rangle(\alpha)$ and $|c_j, 1| = p^t$ for $t \geq 4$. Then $[c_j^{p^t}, c_j, 1] \neq 1$.

**Proof** Put $c = c_{j, 1}$. Then $supp(y) \subseteq \{\alpha, c^{p^t} \alpha, \ldots, (c^{p^t})^{p^{t-1}} \alpha\}$ by the hypothesis. This means that if $y$ moves an element of $\Omega$, then it must be of the form $(c^{p^t})^k(\alpha)$ for $0 \leq k \leq p^{t-2} - 1$.

Assume if possible that $c^y c = c e^y$. Then

$$y c y^{-1} c(\alpha) = c y c y^{-1}(\alpha). \quad (1)$$

Now

$$y c y^{-1} c(\alpha) = yc(\alpha) = y(c^2(\alpha))$$

since $y$ cannot move $c(\alpha)$ and

$$c y c y^{-1}(\alpha) = c y (c^{kp^2}(\alpha)) = c y(c^{kp^2+1}(\alpha)) = c^{kp^2+2}(\alpha)$$

since $y$ cannot move $c^{kp^2+1}(\alpha)$, where $y^{-1}(\alpha) = (c^{p^t})^k(\alpha)$ and $1 \leq k \leq p^{t-2} - 1$ since $y(\alpha) \neq \alpha$. Thus the equality (1) takes the form

$$y(c^2(\alpha)) = c^{kp^2+2}(\alpha).$$

Now if $p > 2$, then $y(c^2(\alpha)) = c^2(\alpha)$ since $c^2(\alpha)$ is not of the form $(c^{p^t})^k(\alpha)$. Indeed, if $c^2(\alpha) = (c^{p^t})^k(\alpha)$, then $c^{kp^2-2}(\alpha) = \alpha$, which implies that $p^t k p^2 - 2$ since $|c| = p^t$, which is impossible. Therefore, $c^2(\alpha) = c^{kp^2+2}(\alpha)$ and hence $\alpha = c^{kp^2}(\alpha)$, which is a contradiction since $1 \leq k \leq p^{t-2} - 1$, $c$ is a cycle, $|c| = p^t$, and $t \geq 4$. Next suppose that $p = 2$. Again since $y$ can move only elements of the form $(c^{p^t})^k(\alpha) = c^{k}(\alpha)$ and since $c^2(\alpha)$ is not of this form, we get $y(c^2(\alpha)) = c^2(\alpha)$ and hence $c^2(\alpha) = c^{k+2}(\alpha)$. Hence, $c^{k}(\alpha) = \alpha$, which is another contradiction since $|c| = 2^t$, $t \geq 4$, and $1 \leq k \leq 2^{t-2} - 1$. \[\square\]

**Lemma 3.8** Let $G$ be a totally imprimitive $p$-subgroup of $FSym(\Omega)$, where $\Omega$ is infinite. Let $X$ be the ascending subset of $G$ satisfying the cyclic-block property such that for every $x \in X$ there exists a $y \in X$ such that $m(x) < m(y)$. Then there exists an ascending subset $Z = \{z_i : i \geq 1\}$ of $G$ so that $m(z_i) < m(z_{i+1})$ for every $i \geq 1$. 994
Proof  By the hypothesis we can obtain easily an infinite subset \( X^* = \{ x_i : i \geq 1 \} \) of \( X \) so that \( m(x_i) < m(x_{i+1}) \) for every \( i \geq 1 \). We may suppose that \( x_1 \neq 1 \). Let \( d_i \) be a cycle of \( x_i \) of the smallest length, that is, of length \( m(x_i) \) for every \( i \geq 1 \). Then \( |d_i| < |d_{i+1}| \) for every \( i \geq 1 \). Choose an \( \alpha \in \text{supp}(d_1) \). By the transitivity of \( G \) for every \( i \geq 1 \) there exists an \( a_i \in G \) so that \( \alpha \in \text{supp}(d_i^a) \). Then \( \text{supp}(d_i^a) \subseteq \text{supp}(d_{i+1}^{a_i}) \) by [4, Lemma 2.2] since \( \alpha \in \text{supp}(d_i^a) \cap \text{supp}(d_{i+1}^{a_i}) \) for every \( i \geq 1 \). Put \( z_i = x_i^{a_i} \) for every \( i \geq 1 \) and define \( Z = \{ z_i : i \geq 1 \} \). Since \( \text{supp}(d_i^a) \subseteq \text{supp}(d_{i+1}^{a_i}) \) and since each \( \text{supp}(d_i^a) \) is a block for \( G \) it follows that \( \bigcup_{i=1}^{\infty} \text{supp}(d_i^a) = \Omega \) due to the fact that every proper block is finite by the transitivity of \( G \) on \( \Omega \). Hence, it follows that \( Z \) cannot be contained in the set stabilizer of a finite subset of \( G \) and also \( \exp(Z) \) is infinite. Therefore, \( Z \) is an ascending subset of \( G \). □

Proof of Theorem 1.1 Let \( G \) be a totally imprimitive \( p \)-subgroup of \( \text{FSym}(\Omega) \), where \( \Omega \) is infinite. Let \( X \) be an ascending subset of \( G \) satisfying the cyclic-block property so that conditions (a) and (b) are satisfied. Then applying Lemma 3.8 we obtain an ascending subset \( Z = \{ z_i : i \geq 1 \} \) of \( G \) so that \( m(z_i) < m(z_{i+1}) \) for every \( i \geq 1 \). Next we can choose an infinite subset \( U = \{ u_i : i \geq 1 \} \) of \( Z \) so that \( |u_i| < m(u_{i+1}) \) for every \( i \geq 1 \) since the numbers \( m(z_i) \) are increasing without bound. We now substitute \( U \) in place of \( X \) in Lemma 3.3. This gives an ascending subset \( Y = \{ y_i : i \geq 1 \} \) of \( G \) so that the following hold. The cycle decomposition of each \( y_i \) can be expressed as

\[
y_i = c_{i,1} \times \cdots \times c_{i,r(i)}
\]

so that \( \text{supp}(y_i) \subseteq \text{supp}(c_{i+1,1}) \) and if \( 1 \leq j \leq r(i) \), \( k \geq i \), \( |c_{i,j}| = p^a \), \( |c_{k,1}| = p^b \), then

\[
c_{k,1}^{b-a} | \text{supp}(c_{i,j}) = c_{i,j}^{q(i,j)}
\]

for a \( q(i,j) \geq 1 \). Furthermore, for each \( i \geq 1 \), the inequality \( |y_i| < m(y_{i+1}) \) is satisfied by definition of \( U \). Thus, Lemmas 3.4, 3.5, and 3.6 can be applied to \( Y \).

Next we may suppose that \( y_1 \neq 1 \). Choose an \( \alpha \in \text{supp}(y_1) \). Let \( \Delta \) be the smallest block such that \( \text{supp}(y_1) \subseteq \Delta \) and let \( |\Delta| \leq p^t \), for a \( t \geq 4 \). There exists a \( j > 1 \) so that \( |c_{j,1}| \geq p^{2t} \) and \( \Delta \subseteq \text{supp}(c_{j,1}) \). Put \( c_j = c_{j,1} \). Then \( c_j = (c_{j,1}(\alpha), \ldots, c_{j,r(j)-1}(\alpha)) \). Now \( c_j^2 \) is a product of \( p^2 \) cycles each of length \( \geq p^{2(t-2)} = p^{2(t-1)} \geq p^t \) since \( t \geq 4 \). Then it is easy to see that \( \Delta \subseteq \langle c_j^2 \rangle(\alpha) \) by the cyclic-block property since \( \langle c_j^2 \rangle(\alpha) \) is a block and \( \alpha \in \Delta \cap \langle c_j^2 \rangle(\alpha) \).

Put \( Y_j^* = \langle y_i : i \geq j \rangle \). Then the application of Lemmas 3.4, 3.5, and 3.6 gives \( [c_{j,1}^p, c_{j,1}] = 1 \) for every \( y \in Y^* \), but application of Lemma 3.7 gives \( [c_{j,1}^{y_1}, c_j] \neq 1 \), which implies that \( y_1 \notin Y^* \) and so \( Y^* \neq G \). However, since \( \{ y_i : i \geq j \} \) is ascending by definition of \( Y \), the subgroup \( Y^* \) cannot be an \( FC \)-subgroup of \( G \). Therefore, \( G \) cannot be an \( MNFC \)-group and so the proof of the theorem is complete. □

Proof of Corollary 1.2 Let \( G \) be a totally imprimitive \( p \)-subgroup of \( \text{FSym}(\Omega) \), where \( \Omega \) is infinite. Let \( X \) be an ascending subset of homogeneous elements of \( G \) satisfying the cyclic-block property so that \( X \) satisfies the (a) condition. Then condition (a) of Theorem 1.1 is satisfied. Therefore, we need only show that condition (b) of Theorem 1.1 is satisfied. Since \( X \) is ascending by the hypothesis, \( \exp(X) \) is infinite and \( \langle X \rangle \) is a non-\( FC \)-subgroup of \( G \). Also, since \( G \) is locally finite, it follows that for every \( x \in X \) there exists a \( y \in X \) so that
Proof of Theorem 1.1

Let $G$ be a subgroup of $FSym(N^*)$ satisfying the cyclic-block property. Let $p$ be the smallest prime dividing the order of $G$. Then $G$ contains a subgroup $G'$ isomorphic to $FSym(N^*)$, which is also cyclic-block, and $G'$ contains a subgroup isomorphic to $FSym(N^*)$. By the classification of finitary permutation groups, $G'$ is isomorphic to $FSym(N^*)$. Thus, $G'$ is also cyclic-block.

Proof of Corollary 1.3

Let $G$ be the $p$-subgroup of $FSym(N^*)$ described in Section 2. Then $G$ satisfies the cyclic-block property by [4, Theorem 1.1]. We have $G = \langle g_k : k \geq 1 \rangle$, where $g_k = u_k \times v_k$, $u_k = (a_1, \ldots, a_{p^k})$, $v_k = u_k \times \cdots \times u_k^{p^{k+1}}$, $supp(u_k) = \Delta_k$, and $supp(v_k) = \Delta_{k+1} \setminus \Delta_k$. Hence, it follows that each $g_k$ is homogeneous; that is, $|g_k| = m(g_k) = p^k$ for every $k \geq 1$. Furthermore,

$$g_k^{p^k} \Delta_k = g_k$$

since $u_k^{p^k} = g_k$ as was shown in Section 2. Thus, $G$ satisfies the hypothesis of Corollary 1.2 and therefore $G$ cannot be $MNFC$. 

Next we show that $G'$ cannot be $MNFC$. For each $s \geq 2$ let $Y_s = \{g_k^{-1} g_k^{p^s} : 1 \leq k < s\}$ and put $Y = \bigcup_{s \geq 2} Y_s$. Then $Y$ is an ascending subset of homogeneous elements of $G'$. To see this let $1 \leq k < s$. Then $g_k^{-1} g_k^{p^s} = g_k^{-1} g_k^{p^s}$ since $supp(g_k) = \Delta_{k+1} = supp(u_{k+1}) \subseteq supp(u_s)$. Also $u_{k+1}^p = g_k$ (see Section 2). Hence $g_k^{-1} g_k^{p^s} = g_k^{-1} g_k^{p^{k+1}} = 1$. So suppose that $s > k + 1$. Then $u_s(\Delta_{k+1}) \cap \Delta_{k+1} = \emptyset$. Also,

$$supp(g_k^{p^s}) = u_s^{-1}(supp(g_k)) = u_s^{-1}(\Delta_{k+1}).$$

Clearly it follows from this that $g_k^{-1} g_k^{p^s} = g_k^{-1} \times g_k^{p^s}$ and so $g_k^{-1} g_k^{p^s}$ is homogeneous since $g_k$ is homogeneous. Furthermore, $g_k \notin G_{\Delta_{k-1}}$ since $g_k = u_k \times v_k$, $supp(u_k) = \Delta_k$ and $\Delta_{k-1} \subseteq \Delta_k$. Now suppose that $s > k + 1$. Then also $g_k^{-1} g_k^{p^s} \notin G_{\Delta_{k-1}}$ since $\Delta_{k-1} \subset supp(g_k)$ and $g_k^{p^s} \in G_{\Delta_{k-1}}$. In particular, (b) of Theorem 1.1 is satisfied.

Finally, let $1 \leq k + 1 < s$. Then

$$(g_k^{-1} g_k^{p^s})^p |_{\Delta_k} = g_k^{-p^s} |_{\Delta_k} = u_k^{-p^s} |_{\Delta_k} = u_k^{-p^s} |_{\Delta_k} = g_k^{-p^s} \times g_k^{p^s} |_{\Delta_k}$$

and so (a) of Theorem 1.1 is satisfied. Therefore, $G'$ cannot be $MNFC$ by Theorem 1.1. (A different proof of this result is given in [5, Theorem 1.6].)

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References


