Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications

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Abstract: Let $a, b$ be two commutative generalized Drazin invertible elements in a Banach algebra; the expressions for the generalized Drazin inverse of the product $ab$ and the sum $a + b$ were studied in some current literature on this subject. In this paper, we generalize these results under the weaker conditions $a^2b = aba$ and $b^2a = bab$. As an application of our results, we obtain some new representations for the generalized Drazin inverse of a block matrix with the generalized Schur complement being generalized Drazin invertible in a Banach algebra, extending some recent works.

Key words: Generalized Drazin inverse, Banach algebra, additive result, block matrix

1. Introduction

The generalized Drazin inverse in a Banach algebra was introduced in [10]. The expressions for the generalized Drazin inverse of the product and the sum were studied by many authors. For instance, in [10], for two commutative generalized Drazin invertible elements $a, b$ in a Banach algebra, Koliha gave the expression of $(ab)^d$. Meanwhile, the representation of $(a + b)^d$ was obtained under the conditions $ab = ba = 0$ in a Banach algebra. Later, Djordjević and Wei [8] gave the expression of $(a+b)^d$ under the assumption $ab = 0$ in the context of the Banach algebra of all bounded linear operators on an arbitrary complex Banach space. In [1], Castro-González and Koliha obtained a formula for $(a + b)^d$ under the conditions $a\sigma b = b, b\sigma a = a, b^*aba^* = 0$, which are weaker than $ab = 0$ in Banach algebras. In [6], Deng and Wei derived necessary and sufficient conditions for the existence of $(P + Q)^d$ under the condition $PQ = QP$, where $P, Q$ are bounded linear operators. Moreover, the expression of $(P + Q)^d$ was given. In [3], Cvetković-Ilić et al. extended the result of [6] to Banach algebras. More results on generalized Drazin inverse can be found in [2, 4, 7, 8, 12, 14].

In [13], Liu et al. deduced the explicit expressions for the Drazin inverses of the product $ab$ and the sum $a + b$ under the conditions $a^2b = aba$ and $b^2a = bab$, where $a$ and $b$ are complex matrices. In [18], the corresponding results of [13] were studied for the pseudo Drazin inverse (which is a special case of generalized Drazin inverse [17]) in a Banach algebra. In this paper, we will further consider the results of [13] and [18] for the generalized Drazin inverse, which extend [10, Theorem 5.5] and [3, Theorem 2.1].

Another relevant topic is to establish a representation for the generalized Drazin inverse of a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of its blocks under certain conditions. The generalized Schur complement $S = D - CA^d B$
plays an important role in the representation for $M^d$. Here we list partially some conditions as follows:

1. $S$ is invertible, $A^*BC = 0$, $CA^*B = 0$, and $AA^*B = A^*BD$ (see [5]);
2. $S$ is invertible, $BCA^* = 0$, $CA^*B = 0$, and $CAA^* = DCA^*$ (see [5]);
3. $S$ is generalized Drazin invertible, $BCA^* = 0$, $DCA^* = 0$, $S^*CA = 0$, and $ABS^* = 0$ (see [15]);
4. $S$ is generalized Drazin invertible, $A^*B = 0$, and $S^*CA = 0$ (see [15]).

In this paper, we will extend the above results under weaker conditions as applications of our additive result.

2. Preliminaries

Throughout this paper, $\mathcal{A}$ denotes a complex Banach algebra with unity 1. For $a \in \mathcal{A}$, denote the spectrum and the spectral radius of $a$ by $\sigma(a)$ and $r(a)$, respectively. $\mathcal{A}^{-1}$ and $\mathcal{A}^{qnil}$ stand for the sets of all invertible and quasinilpotent elements ($\sigma(a) = \{0\}$) in $\mathcal{A}$, respectively. The commutant of an element $a \in \mathcal{A}$ is defined by $\text{comm}(a) = \{b \in \mathcal{A} : ab = ba\}$. In addition, denote by $C^k_n$ the binomial coefficient $\frac{n!}{k!(n-k)!}$ ($0 \leq k \leq n$).

For the readers’ convenience, we first recall the definitions of some generalized inverses. The generalized Drazin inverse [10] of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of $a$) is the element $x \in \mathcal{A}$ that satisfies

$$xax = x, \ ax = xa \quad \text{and} \quad a - a^2 x \in \mathcal{A}^{qnil}.$$  

Such $x$, if it exists, is unique and will be denoted by $a^d$. It is well known that $a \in \mathcal{A}$ has a generalized Drazin inverse if and only if 0 is not an accumulation point of $\sigma(a)$. Let $\mathcal{A}^d$ denote the set of all generalized Drazin invertible elements in $\mathcal{A}$. If $a \in \mathcal{A}^d$, the spectral idempotent $a^\pi$ of $a$ corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. In this case, the resolvent $R(\lambda, a) = (\lambda 1 - a)^{-1}$ has a Laurent series

$$R(\lambda, a) = \sum_{n=1}^{\infty} \lambda^{-n} a^{n-1} a^\pi - \sum_{n=0}^{\infty} \lambda^n (a^d)^{n+1},$$

on some punctured disc $\{\lambda : 0 < |\lambda| < r\}, r > 0$ (see [10, Theorem 5.1]).

The group inverse of $a \in \mathcal{A}$ is the element $x \in \mathcal{A}$ that satisfies

$$axa = a, \ xax = x \quad \text{and} \quad ax = xa.$$  

If the group inverse of $a$ exists, it is unique and denoted by $a^\#$.

Let $p \in \mathcal{A}$ be an idempotent ($p^2 = p$). Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_1 & a_3 \\ a_4 & a_2 \end{bmatrix}_p,$$

where $a_1 = pap, \ a_2 = (1-p)a(1-p), \ a_3 = pa(1-p), \ \text{and} \ a_4 = (1-p)ap$.

It is well known that if $a \in \mathcal{A}^d$, then we have the following matrix representations:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathcal{A}p)^{-1}$, and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$.

Now we present two useful lemmas, which play an important role in the sequel.
Lemma 2.1 [1, Theorem 2.3] Let \( p^2 = p \), \( x, y \in \mathcal{A} \) and let \( x \) and \( y \) have the representations

\[
x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}.
\]

(1) If \( a \in (p\mathcal{A}p)^d \) and \( b \in ((1-p)\mathcal{A}(1-p))^d \), then \( x, y \in \mathcal{A}^d \) and

\[
x^d = \begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_p, \quad y^d = \begin{bmatrix} b^d & 0 \\ u & a^d \end{bmatrix}_{1-p},
\]

where

\[
u = \sum_{n=0}^{\infty} (a^d)^{n+2}cb^n + \sum_{n=0}^{\infty} a^n a^{(b^d)^{n+2} - a^d} c b^d.\]

(2) If \( x \in \mathcal{A}^d \) [resp. \( y \in \mathcal{A}^d \)] and \( a \in (p\mathcal{A}p)^d \), then \( b \in ((1-p)\mathcal{A}(1-p))^d \), and \( x^d \) [resp. \( y^d \)] is given by (2) and (3).

Lemma 2.2 [10, Theorem 5.5] Let \( a, b \in \mathcal{A} \) be such that \( ab=ba \). Then \( ab \in \mathcal{A}^d \) and \( (ab)^d = a^d b^d \).

Next, the commuting property for the generalized Drazin inverse is investigated in a Banach algebra.

Theorem 2.3 Let \( a, b \in \mathcal{A} \) and \( c \in \mathcal{A} \). If \( ca = bc \), then \( ca^d = b^d c \).

Proof Suppose that \( a, b \in \mathcal{A} \) and \( ca = bc \), for any \( n \in \mathbb{N} \), we have the following equations:

\[
bb^d c - bb^d ca^d = bb^d c(1 - aa^d) = (bb^d)^n c(1 - aa^d) = (bb^d)^n (b^n c)(1 - aa^d) = (bb^d)^n (b^n c)(1 - aa^d),
\]

which imply

\[
\|(bb^d c - bb^d ca^d)\|^{\frac{1}{n}} = \|(bb^d)^n c(1 - aa^d)\|^{\frac{1}{n}} \leq \|b^d\|\|c\|^{\frac{1}{n}}\|a^n (1 - aa^d)\|^{\frac{1}{n}} \xrightarrow{n \to \infty} 0.
\]

Thus, \( bb^d c = bb^d ca^d \), i.e. \( b^d c = b^d ca^d \).

On the other hand, we have that

\[
ca^d a - b^d caa^d a = ca^d a - b^d bca^d a = (1 - bb^d) ca^d a = (1 - bb^d) c(a^d a)^n = (1 - bb^d) (c a^n) (a^d)^n = (1 - bb^d) (b^n c)(a^d)^n.
\]

Then we obtain

\[
\|ca^d - b^d caa^d a\|^{\frac{1}{n}} = \|(1 - bb^d) b c(a^d)^n\|^{\frac{1}{n}} \leq \|b^n (1 - bb^d)\|^{\frac{1}{n}}\|c\|^{\frac{1}{n}}\|a^d\| \xrightarrow{n \to \infty} 0.
\]

Thus, \( ca^d = b^d caa^d a \), i.e. \( ca^d = b^d caa^d \). Therefore, we deduce that \( ca^d = b^d c \). □

Corollary 2.4 [10, Theorem 4.4] Let \( a \in \mathcal{A} \) and \( c \in \mathcal{A} \). If \( ca = ac \), then \( ca^d = a^d c \).

The following lemmas will also be useful.

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Lemma 2.5 Let \(a, b \in \mathcal{A}^d\) be such that \(a^2b = aba\) and \(b^2a = bab\). Then

(i) \(\{ab, a^db, abd, a^dbd\} \subseteq \text{comm}(a) \cap \text{comm}(a^d)\).

(ii) \(\{ba, b^da, bab, b^dab\} \subseteq \text{comm}(b) \cap \text{comm}(b^d)\).

Proof (i) By Corollary 2.4, it suffices to prove \(\{ab, a^db, abd, a^dbd\} \subseteq \text{comm}(a)\).

Since \(a^2b = aba\), then \((a^d)ab = (a^d)(aba) = (a^d)a^2b = a(a^db)\).

Note that \(bab^d = b^dab\), and we get \(a(ab^d) = a^2b(b^d)^2 = aba(b^d)^2 = a(b^d)(b^d)2ba = (ab^d)a\), which implies \(a(ab^d) = (a^d)^2a(ab^d) = (a^d)^2(ab^d)a = (a^d)abcd\).

(ii) It is analogous to the proof of (i). \(\square\)

Remark 2.6 In Lemma 2.5, the conditions \(a^2b = aba\) and \(b^2a = bab\) are weaker than \(ab = ba\). Indeed, it is clear that \(ab = ba\) can imply \(a^2b = aba\) and \(b^2a = bab\). However, in general, the converse is false. The following example can illustrate this fact.

Example 2.7 Let \(\mathcal{A}\) be the Banach algebra of all complex \(3 \times 3\) matrices, and take

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

Clearly, \(a^2b = aba\) and \(b^2a = bab\). However, \(ab \neq ba\).

Remark 2.8 We have seen that if \(a \in \mathcal{A}^d\), \(b \in \mathcal{A}\), and \(ab = ba\), then \(a^db = ba^d\). However, under the conditions of Lemma 2.5, \(a^db = ba^d\) may not be true, which can also be illustrated by the previous Example 2.7. Note that \(a^3 = a\) and \(b^3 = b\); then \(a^d = a\) and \(b^d = b\). However, \(a^db \neq ba^d\).

The next result was proved for complex matrices (see [13, Lemma 2.3]). Indeed, it is true in a Banach algebra.

Lemma 2.9 Let \(a, b \in \mathcal{A}\) be such that \(a^2b = aba\) and \(b^2a = bab\). Then

\[
(a + b)^n = \sum_{i=0}^{n-1} C_{n-1}^i (a^nib^i + b^ni^a), \quad \text{where} \quad n \in \mathbb{N}.
\]

Next, we establish two crucial auxiliary results.

Lemma 2.10 Let \(a, b \in \mathcal{A}\) be such that \(a^2b = aba\) and \(b^2a = bab\). Then

(i) \(r(a + b) \leq r(a) + r(b)\).

(ii) If both \(a\) and \(b\) are quasinilpotent, then \(a + b\) is quasinilpotent.

Proof (i) Take any \(\alpha > r(a)\) and \(\beta > r(b)\). Let \(a_1 = \frac{1}{\alpha}a\) and \(b_1 = \frac{1}{\beta}b\). Then \(r(a_1) < 1\) and \(r(b_1) < 1\). From
Lemma 2.9, we have that

\[
\|(a + b)^{n+1}\| = \left\| \sum_{i=0}^{n} C_n^i (a^{n+1-i} b^i + b^{n+1-i} a^i) \right\|
\]

\[
= \left\| a \sum_{i=0}^{n} C_n^i a^{n-i} b^i + b \sum_{i=0}^{n} C_n^i b^{n-i} a^i \right\|
\]

\[
\leq \left\| a \right\| \sum_{i=0}^{n} C_n^i ||a^{n-i}|| ||b^i|| + \left\| b \right\| \sum_{i=0}^{n} C_n^i ||b^{n-i}|| ||a^i||
\]

\[
= (\left\| a \right\| + \left\| b \right\|) \sum_{i=0}^{n} C_n^i ||a^i|| ||b^{n-i}||
\]

\[
= (\left\| a \right\| + \left\| b \right\|) \sum_{i=0}^{n} C_n^i \alpha \beta^{n-i} ||a^i|| ||b^{n-i}||.
\]

For each \(n\), choose \(n', n'' \in \mathbb{N}\) such that \(n' + n'' = n\) and \(\|a''\| ||b''|| = \max_{0 \leq i \leq n} ||a_i|| ||b_i''||\), then we have

\[
\|(a + b)^{n+1}\| \leq (\left\| a \right\| + \left\| b \right\|)(\alpha + \beta)^n ||a''\| ||b''||,
\]

which implies

\[
r(a + b) = \lim_{n \to \infty} (\|(a + b)^{n+1}\|^{1/(n+1)})^{n+1} = \lim_{n \to \infty} (\|(a + b)^n\|^{1/n})^{n+1}
\]

\[
\leq (\alpha + \beta) \lim_{n \to \infty} (\left\| a \right\| + \left\| b \right\|)^{2n} \lim_{n \to \infty} \inf ||a''\| \|b''\|^{1/2}
\]

\[
= (\alpha + \beta) \lim_{n \to \infty} \inf ||a''\| \|b''\|^{1/2}.
\]

According to the proof of [9, Lemma 1.2.13], we obtain \(r(a + b) \leq \alpha + \beta\), which yields \(r(a + b) \leq r(a) + r(b)\).

(ii) This can be obtained by (i).

\[\square\]

**Lemma 2.11** Let \(a, b \in \mathcal{A}\) be such that \(a^2 b = aba\) or \(b^2 a = bab\). Then

(i) \(r(ab) \leq r(a)r(b)\).

(ii) If either \(a\) or \(b\) is quasinilpotent, then \(ab\) is quasinilpotent.

**Proof** (i) Note the symmetry of \(a^2 b = aba\) and \(b^2 a = bab\), it suffices to prove the case \(a^2 b = aba\).

Assume \(a^2 b = aba\); then \((ab)^n = a^n b^n\) for \(n \in \mathbb{N}\) by induction. Therefore,

\[
\|(ab)^n\|^{1/n} = \|a^n b^n\|^{1/n} \leq \|a^n\|^{1/n} \|b^n\|^{1/n}.
\]

Let \(n \to \infty\); then we obtain that \(r(ab) \leq r(a)r(b)\).

(ii) This follows from (i) directly.

\[\square\]

### 3. Main results

In this section, for \(a, b \in \mathcal{A}\), we will investigate the representations of \((ab)^d\) and \((a + b)^d\) under the new conditions \(a^2 b = aba\) and \(b^2 a = bab\).

We start with a theorem that is an extension of [10, Theorem 5.5].

**Theorem 3.1** Let \(a, b \in \mathcal{A}\) be such that \(a^2 b = aba\) and \(b^2 a = bab\). Then \(ab \in \mathcal{A}\) and \((ab)^d = a^d b^d\).
Thus, we have $ab = aba$ expressed in matrix form yields

$$\begin{bmatrix} a_1 b_1 & a_2 b_3 \\ a_1 b_4 & a_2 b_2 \end{bmatrix}_p = a^2 b = aba = \begin{bmatrix} a_1 b_1 a_1 & a_1 b_3 a_2 \\ a_2 b_4 a_1 & a_2 b_2 a_2 \end{bmatrix}_p.$$  

Thus, we have $a_1 b_3 = a_1 b_3 a_2$, i.e. $b_3 = a_1^{-1} b_3 a_2$, which implies $b_3 = a_1^{-n} b_3 a_2^n$ for any $n \in \mathbb{N}$. Since $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$, then

$$\|b_3\|_{\mathcal{A}} = \|a_1^{-n} b_3 a_2^n\|_{\mathcal{A}} \leq \|a_1^{-1}\| \|b_3\|_{\mathcal{A}} \|a_2^n\|_{\mathcal{A}} \xrightarrow{n \to \infty} 0.$$  

Hence, $b_3 = 0$. Similarly, from $a_2 b_4 = a_2^2 b_4 a_1^{-1}$, it follows that $a_2 b_4 = 0$. In addition, we can get $a_1 b_1 = b_1 a_1$ and $a_2^2 b_2 = a_2 b_2 a_2$. Then we have

$$b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p \text{ and } ab = \begin{bmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{bmatrix}_p.$$  

Next, we prove that $b_1 \in (p\mathcal{A}p)^d$ and $b_1^d = aa^d b^d a a^d$ by the definition of generalized Drazin inverse. Note that $b_1 = aa^d b a a^d = aa^d b$ and $aa^d b a a^d = aa^d b$ by Lemma 2.5(i). Therefore, we need to prove $b_1^d = aa^d b^d$.

Let $v = aa^d b$. Then we have

1. $b_1 v = aa^d b aa^d b^d = aa^d b b v = v b_1.$
2. $v b_1 v = aa^d b^d a a^d b = aa^d b a a^d b^d = aa^d b a a^d b^d = aa^d b a a^d b^d = v.$

3. Note that $b_1 - b_1^2 v = aa^d b (1 - bb^d)$. By induction and Lemma 2.5, we have that $(aa^d b (1 - bb^d))^n = aa^d b^n (1 - bb^d)$ for any $n \in \mathbb{N}$. Since $b (1 - bb^d) \in \mathcal{A}^{qnil}$, then

$$\|(b_1 - b_1^2 v)^n\|_{\mathcal{A}} = \|aa^d b^n (1 - bb^d)\|_{\mathcal{A}} \leq \|aa^d\|_{\mathcal{A}} \|b^n (1 - bb^d)\|_{\mathcal{A}} \xrightarrow{n \to \infty} 0.$$  

Thus $b_1 - b_1^2 v \in (p\mathcal{A}p)^{qnil}$. Hence, $b_1^d = v$. Similarly, we have that $b_2^d = b^d (1 - aa^d)$.

According to the equation $a_1 b_1 = b_1 a_1$ and Lemma 2.2, we have that $a_1 b_1 \in (p\mathcal{A}p)^d$ and $(a_1 b_1)^d = a_1^{-1} b_1^d$. Observe that $a_2 b_2 = a_2 b_2 a_2$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$; applying Lemma 2.11(ii) to the elements $a_2, b_2$, we get $a_2 b_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$, i.e. $(a_2 b_2)^d = 0$.

Finally, applying Lemma 2.1(i), we have $ab \in \mathcal{A}^d$ and

$$(ab)^d = \begin{bmatrix} (a_1 b_1)^d & 0 \\ 0 & (a_2 b_2)^d \end{bmatrix}_p = \begin{bmatrix} a_1^{-1} b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p = a^d b^d.$$

$\square$
Remark 3.2 (1) From Lemma 2.2 and Corollary 2.4, we can see that \((ab)^d = a^{d}b^{d} = b^{d}a^{d}\) for commutative generalized Drazin invertible elements \(a, b \in \mathcal{A}\). However, in general, \((ab)^d \neq b^{d}a^{d}\) under the conditions of Theorem 3.1. For example, let \(a, b\) be the same as the elements in Example 2.7. Clearly,

\[
ab = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} = (ab)^d.
\]

However, \((ab)^d \neq b^{d}a^{d}\).

(2) In Theorem 3.1, if we replace \(b^{2}a = aba\) with \(ba^{2} = aba\), then we can conclude that \((ab)^d = a^{d}b^{d} = b^{d}a^{d}\). The proof of the previous result is similar to the proof of Theorem 3.1 and so we omit the proof. The following example shows that the conditions \(a^{2}b = aba\) and \(ba^{2} = aba\) are weaker than \(ab = ba\). Let \(\mathcal{A} = M_2(\mathbb{C})\) and take

\[
a = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix}.
\]

Then we can get that \(a^{2}b = aba\) and \(ba^{2} = aba\). However, \(ab \neq ba\).

Next, we present our main result, which recovers [3, Theorem 2.1].

Theorem 3.3 Let \(a, b \in \mathcal{A}^d\) be such that \(a^{2}b = aba\) and \(b^{2}a = bab\). Then the following conditions are equivalent:

(i) \(a + b \in \mathcal{A}^d\).

(ii) \(1 + a^{d}b \in \mathcal{A}^d\).

(iii) \(c = aa^{d}(a + b)bb^{d} \in \mathcal{A}^d\).

In this case,

\[
(a + b)^d = a^d(1 + a^{d}b)^d + a^{d}b(a^d)^2((1 + a^{d}b)^d)^2 + \sum_{n=0}^{\infty} (b^d)^{n+1}(-a)^n a^d
\]

\[+ \sum_{n=0}^{\infty} (n+1)b^{n}a(b^d)^{n+2}(-a)^n a^d, \tag{4}\]

\[
(a + b)^d = c^d + \sum_{n=0}^{\infty} (a^d)^{n+1}(-b)^n b^d + a^{d}b(c^d)^2 + \sum_{n=0}^{\infty} a^{d}bc^d(a^d)^{n+1}(-b)^n b^d
\]

\[+ \sum_{n=0}^{\infty} a^{d}b(a^d)^{n+1}(-b)^n b^d c^d + \sum_{n=0}^{\infty} (n+1)a^{d}b(a^d)^{n+2}(-b)^n b^d
\]

\[+ \sum_{n=0}^{\infty} (b^d)^{n+1}(-a)^n a^d + \sum_{n=0}^{\infty} (n+1)b^{n}a(b^d)^{n+2}(-a)^n a^d, \tag{5}\]

\[
(1 + a^{d}b)^d = a^d + a^{2}a^{d}(a + b)^d \quad \text{and} \quad (aa^{d}(a + b)bb^{d})^d = aa^{d}(a + b)^d bb^{d}. \tag{6}\]
**Proof**  As in the proof of Theorem 3.1, we have that

\[
a = \begin{bmatrix}
a_1 & 0 \\
0 & a_2
\end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix}
b_1 & 0 \\
b_4 & b_2
\end{bmatrix}_p,
\]

where \( p = aa^d, \ a_1 \in (p, \mathcal{A} p)^{-1}, \) and \( a_2 \in ((1-p), \mathcal{A}(1-p))^{qni}. \) Moreover, we have \( a_1 b_1 = b_1 a_1, \ a_2 b_4 = 0, \ a_2^2 b_2 = a_2 b_2 a_2, \ b_i^d = aa^d b^d, \) and \( b_i^d = b^d(1-aa^d). \) From the condition \( b^2 a = bab, \) it follows that \( b_2^d a_2 = b_2^d a_2 b_2 \) and \( b_2 b_4 = 0. \)

Let \( p_1 = b_1 b_1^d \) and \( p_2 = b_2 b_2^d. \) Then \( p_1 p_2 = p p_1 = p_1 \) and \( p_2 (1-p) = (1-p) p_2 = p_2 \) by Lemma 2.5. We now consider the matrix representations of \( b_1 \) and \( b_2 \) relative to idempotents \( p_1 \) and \( p_2, \) respectively. We have that

\[
b_1 = \begin{bmatrix}
b_1' & 0 \\
0 & b_2'
\end{bmatrix}_{p_1} \quad \text{and} \quad b_2 = \begin{bmatrix}
b_2' & 0 \\
0 & b_2'
\end{bmatrix}_{p_2},
\]

where \( b_1' \in (p_1, \mathcal{A} p_1)^{-1}, \ b_2' \in (p_2, \mathcal{A} p_2)^{-1}, \) and \( b_2' \in ((p_1, \mathcal{A} (p_1), p_1))^{qni}. \)

Note that \( p_1 a_1 (p - p_1) = b_1 b_1' a_1 (p - b_1 b_1') = b_1 a_1 b_1' (p - b_1 b_1') = b_1 a_1 (b_2^d - b_2 b_2') = 0. \) Similarly, \( (p - p_1) a_1 p_1 = 0 \) and \( p_2 a_2 (1-p - p_2) = 0. \) Thus, we get the following matrix representations:

\[
a_1 = \begin{bmatrix}
a_1' & 0 \\
0 & a_2'
\end{bmatrix}_{p_1} \quad \text{and} \quad a_2 = \begin{bmatrix}
a_2'' & 0 \\
a_4' & a_2''
\end{bmatrix}_{p_2}.
\]

Note that \( a_2^2 b_2 = a_2 b_2 a_2 \) and \( b_2^d a_2 = b_2 a_2 b_2; \) as in the proof of Theorem 3.1, we have that \( b_i'' a_i'' = a_i b_i'' , \) \( (b_2')^2 a_i'' = b_i'' a_i'' b_i'' \) and \( (a_i')^d b_i'' = a_i b_i'' a_i''. \) Moreover, \( (a_i'')^d = p_2 a_i'' = 0 \) and \( (a_i'')^d = a_i'' (1-p - p_2) = 0, \) which imply \( a_i'' \) and \( a_i'' \) are quasinilpotent. Besides these, \( b_2^d a_4'' = a_2 a_4'' = 0. \)

Next, we will prove that \( a_2 + b_2 \in ((1-p) \mathcal{A} (1-p))^d. \) Observe that

\[
a_2 + b_2 = \begin{bmatrix}
a_i'' + b_i'' \\
a_i'' + b_i''
\end{bmatrix}_{p_2}.
\]

Since \( a_i'' + b_i'' = b_i'' (p_2 + (b_i'')^{-1} a_i'') \) and \( a_i'' \) is quasinilpotent, we have that \( a_i'' + b_i'' \) is invertible in subalgebra \( p_2 \mathcal{A} p_2 \) and

\[
(a_i'' + b_i'')^{-1} = (b_i'')^{-1} (p_2 + (b_i'')^{-1} a_i'')^{-1} = (b_i'')^{-1} (p_2 + \sum_{n=1}^{\infty} (b_i'')^{-n} (-a_i'')^n).
\]

Note that \( (b_i'')^{-1} = b_2 = b^d (1 - aa^d). \) By induction, we can obtain that \( (b_i'')^{-n} = (b_i'')^{-n} (1 - aa^d) \) for any \( n \in \mathbb{N}. \) In addition, we verify that

\[
a_i'' = p_2 a_2 p_2 = b_2 b_2^d a_2 b_2^d = b_2 b_2^d a_2 = (ba^d) b^d a^d (a^d a) = b^d a^d a,
\]

which implies \( (-a_i'')^n = b^d (-a)^n a^d \) for any \( n \in \mathbb{N} \) by induction. Note that \( a^d b^d a^d = b^d a^d \) and \( p_2 = b_2 b_2^d = 555.
\[ ba^\pi b^d a^\pi = bb^d a^\pi; \text{ then we get} \]

\[
(a''_1 + b''_1)^{-1} = b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty} (b^d)^n a^\pi (bb^d(-a)^n a^\pi)) \\
= b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty} (b^d)^n bb^d(-a)^n a^\pi) \\
= b^d a^\pi bb^d a^\pi + b^d a^\pi \sum_{n=1}^{\infty} (b^d)^n (-a)^n a^\pi \\
= b^d a^\pi + \sum_{n=1}^{\infty} (b^d)^{n+1} (-a)^n a^\pi \\
= \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi.
\]

Applying Lemma 2.10(ii) to the element \( a''_2, b''_2 \), we have that \( a''_2 + b''_2 \) is quasinilpotent, i.e. \( (a''_2 + b''_2)^d = 0 \).

Lemma 2.1(i) ensures that \( a_2 + b_2 \in ((1-p)\mathcal{A}(1-p))^d \) and

\[
(a_2 + b_2)^d = \begin{bmatrix} (a''_1 + b''_1)^{-1} & 0 \\ x & 0 \end{bmatrix}_{p_2},
\]

where \( x = a''_4(a'' + b''_4)^{-2} \). Note that

\[
a''_4 = (1-p-p_2)a_2p_2 = (b^\pi a^\pi)(a^\pi a)(bb^d a^\pi) = b^\pi a^\pi bb^d a^\pi = b^\pi abb^d a^\pi.
\]

Because \( a^\pi (b^d)^n a^\pi = (b^d)^n a^\pi \) for any \( n \in \mathbb{N} \), then

\[
x = b^\pi abb^d a^\pi \left( \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi \right)^2 \\
= b^\pi abb^d a^\pi \left( \sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi \right) \\
= b^\pi a \sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi.
\]

Therefore, we can obtain

\[
(a_2 + b_2)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi + b^\pi a \sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi.
\]

Since

\[
a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_4 & a_2 + b_2 \end{bmatrix}_p,
\]

by Lemma 2.1, we have that \( a + b \in \mathcal{A}^d \) if and only if \( a_1 + b_1 \in (p\mathcal{A}_p)^d \). In this case, we have

\[
(a + b)^d = \begin{bmatrix} (a_1 + b_1)^d & 0 \\ y & (a_2 + b_2)^d \end{bmatrix}_p,
\]

where \( y = b_4((a_1 + b_1)^d)^2 \).

(i) \( \iff \) (ii) From

\[
1 + a^d b = \begin{bmatrix} p + a_1 b_1 & 0 \\ 0 & 1 - p \end{bmatrix}_p,
\]

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it follows that $1 + a^d b \in \mathcal{A}^d$ if and only if $p + a_1^{-1} b_1 \in (p\mathcal{A} p)^d$. By Lemma 2.2, we have that $a_1 + b_1 = a_1 (p + a_1^{-1} b_1) \in (p\mathcal{A} p)^d$ if and only if $p + a_1^{-1} b_1 \in (p\mathcal{A} p)^d$. Hence, $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$. In this case, we have

$$(1 + a^d b)^d = \left[ \begin{array}{cc} (p + a_1^{-1} b_1)^d & 0 \\ 0 & 1 - p \end{array} \right].$$

Moreover, we deduce that

$$(a_1 + b_1)^d = a_1^{-1} (p + a_1^{-1} b_1)^d = a^d ((1 + a^d b)^d - (1 - p)) = a^d (1 + a^d b)^d.$$  

By a straightforward computation, we obtain that the equation (4) holds.

(i) $\iff$ (iii) From $a_1 \in (p\mathcal{A} p)^{-1}$, we have $a_1' \in (p_1 \mathcal{A} p_1)^{-1}$ and $a_2' \in (\mathcal{A} (p - p_1) \mathcal{A} (p - p_1))^{-1}$. Note that $a_2' b_2 = b_2 a_2$ and $a_1'$ is quasi-nilpotent; then $a_1' + b_1' = a_2' ((p - p_1) + (a_2' - 1)b_2')$ is invertible in subalgebra $(p - p_1) \mathcal{A} (p - p_1)$ and $(a_2' + b_2')^{-1} = \sum_{n=0}^{\infty} (a_2')^{n+1} (-b)^n b^\sigma$, which is similar to the proof of the expression for $(a_1' + b_1')^{-1}$. Since

$$a_1 + b_1 = \left[ \begin{array}{cc} a_1' + b_1' & 0 \\ 0 & a_2' + b_2' \end{array} \right]_{p_1},$$

we have $a_1 + b_1 \in (p\mathcal{A} p)^d$ if and only if $a_1' + b_1' \in (p_1 \mathcal{A} p_1)^d$. In this case,

$$(a_1 + b_1)^d = (a_1' + b_1')^d + (a_2' + b_2')^{-1}.$$  

The following matrix representations

$$c = aa^d (a + b)bb^d = \left[ \begin{array}{cc} (a_1 + b_1) b_1 b_1' & 0 \\ 0 & 0 \end{array} \right]_p$$

and

$$(a_1 + b_1) b_1 b_1' = \left[ \begin{array}{cc} a_1' + b_1' & 0 \\ 0 & 0 \end{array} \right]_{p_1}$$

yield the equality $c = a_1' + b_1'$. Therefore, we conclude that $a + b \in \mathcal{A}^d$ if and only if $c \in \mathcal{A}^d$. In this case, we have

$$y = a^\sigma b ((c^d)^2 + \sum_{n=0}^{\infty} c^d (a_2')^{n+1} (-b)^n b^\sigma + \sum_{n=0}^{\infty} (a_2')^{n+1} (-b)^n b^\sigma c^d + \sum_{n=0}^{\infty} (n + 1)(a_2')^{n+2} (-b)^n b^\sigma),$$

and the equation (5) holds. Finally, the equation (6) can be obtained by an elemental computation.

Next, we consider some specializations of our main result.

**Corollary 3.4** [3, Theorem 2.1] Let $a, b \in \mathcal{A}^d$ be such that $ab = ba$. Then $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$. In this case,

$$(a + b)^d = a^d (1 + a^d b)bb^d + \sum_{n=0}^{\infty} b^\sigma (-b)^n (a_2')^{n+1} + \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\sigma.$$  

(7)

**Proof** Only the expression for $(a + b)^d$ needs a proof. It follows directly from (6) that $(aa^d (a + b)bb^d)^d = a^d (1 + a^d b)bb^d$. Note that $a^\sigma a^d = 0$ and $b^\sigma b^d = 0$; then the equation (7) holds by (5).  

$\square$
Corollary 3.5 Let \( a, b \in \mathcal{A}^d \) be such that \( a^2b = aba \) and \( b^2a = bab \).

(i) If \( 1 \notin \sigma(-a^d b) \) (or \( \sigma(a^d b) = \{0\} \)), then \( a + b \in \mathcal{A}^d \),
\[
(a + b)^d = a^d(1 + a^d b)^{-1} + a^\pi b(a^d)^2(1 + a^d b)^{-2} + \sum_{n=0}^{\infty} (b^d)^{n+1}(-a)^n a^\pi + \sum_{n=0}^{\infty} (n+1)b^\pi a(b^d)^{n+2}(-a)^n a^\pi,
\]
and
\[
(1 + a^d b)^{-1} = a^\pi + a^2 a^d (a + b)^d.
\]

(ii) If \( \sigma(b) = \{0\} \), then \( a + b \in \mathcal{A}^d \) and
\[
(a + b)^d = a^d(1 + a^d b)^{-1} + a^\pi b(a^d)^2(1 + a^d b)^{-2} = \sum_{n=0}^{\infty} (a^d)^{n+1}(-b)^n + \sum_{n=0}^{\infty} (n+1)a^\pi b(a^d)^{n+2}(-b)^n.
\]

Proof (i) This follows from Theorem 3.3 directly.

(ii) Since \( \sigma(b) = \{0\} \), then \( b \in \mathcal{A}^q_{\text{nil}} \), i.e. \( b^d = 0 \), which implies \( ab^d(a + b)bb^d = 0 \). Thus, we have that \( a + b \in \mathcal{A}^d \) by Theorem 3.3. To show that \( 1 + a^d b \in \mathcal{A}^{-1} \), it suffices to prove that \( a^d b \in \mathcal{A}^q_{\text{nil}} \). From Lemma 2.5(i), it follows that \( (a^d)^2 b = a^d ba^d \), which yields \( a^d b \in \mathcal{A}^q_{\text{nil}} \) by Lemma 2.11(ii). The expressions of \( (a + b)^d \) can be obtained by the equations (4) and (5).

4. Applications to block matrices

Let
\[
x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}
\]
relative to idempotent \( p \in \mathcal{A} \), \( a \in (p\mathcal{A}p)^d \), and let \( s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d \) be the generalized Schur complement of \( a \) in \( x \).

In this section, we get some representations for the generalized Drazin inverse of a block matrix \( x \) with applications of our previous result.

For future reference we state two known results.

Lemma 4.1 [1, Example 4.5] Let \( a, b \in \mathcal{A}^d \). If \( ab = 0 \), then \( a + b \in \mathcal{A}^d \) and
\[
(a + b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1}a^n a^\pi + \sum_{n=0}^{\infty} b^n b^\pi (a^d)^{n+1}.
\]

Lemma 4.2 [11, Lemma 2.1] Let \( x \) be defined as in (8). Then the following statements are equivalent:

(i) \( x \in \mathcal{A}^d \) and \( x^d = r \), where
\[
r = \begin{bmatrix} a^d + a^d b s^d c a^d & -a^d b s^d \\ -s^d c a^d & s^d \end{bmatrix}
\]

(ii) \( a^\pi b s^d = a^d b s^\pi \), \( s^\pi c a^d = s^d c a^\pi \), and \( y = \begin{bmatrix} a a^\pi & a^\pi b \\ s^\pi a^\pi & s s^\pi \end{bmatrix} \in \mathcal{A}^q_{\text{nil}} \).
Note that, in Lemma 4.2, if \( y = 0 \), then we can check that \( xrx = x \), and so that we have the following corollary.

**Corollary 4.3** Let \( x \) be defined as in (8). If \( a^\pi bs^d = a^dbs^\pi \), \( s^\pi ca^d = s^dca^\pi \), and \( y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} = 0 \), then \( x \in \mathcal{A}^\# \) and

\[
x^\# = \begin{bmatrix} a^\# + a^\# b s^\# c a^\# & -a^\# b s^\# \\ -s^\# c a^\# & s^\# \end{bmatrix}.
\]

**Remark 4.4** For item (ii) of Lemma 4.2, we can see that \( a^\pi bs^d = a^dbs^\pi \) is equivalent to \( a^\pi bs^d = a^dbs^\pi = 0 \). Moreover, \( s^\pi ca^d = s^dca^\pi \) is equivalent to \( s^\pi ca^d = s^dca^\pi = 0 \). Now, we drop any one of the four equations \( a^\pi bs^d = 0 \), \( a^dbs^\pi = 0 \), \( s^\pi ca^d = 0 \), \( s^dca^\pi = 0 \) and replace the quasinilpotency by the generalized Drazin invertibility of \( y \). Here, we only give the one of the four cases. Similarly, we can prove the others.

**Theorem 4.5** Let \( x \) be defined as in (8). If \( a^\pi bs^d = 0 \), \( s^\pi ca^d = 0 \), \( s^dca^\pi = 0 \), and \( y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} \in \mathcal{A}^d \), then \( x \in \mathcal{A}^d \) and

\[
x^d = \begin{bmatrix} a^\pi & -a^\pi b s^\pi \\ 0 & s^\pi \end{bmatrix} y^d + \sum_{n=0}^{\infty} y^{n+1} \begin{bmatrix} p & a^dbs^\pi \\ 0 & 1-p \end{bmatrix} y^n y^\pi,
\]

where \( r \) is defined as in (9).

**Proof** From the condition \( s^\pi ca^d = 0 \), we have \( s^\pi ca^\pi + ss^d c = c \) and \( s^\pi s + ss^d d = d \). Then we can write

\[
x = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^db \\ ss^d c & ss^d d \end{bmatrix} := y + z.
\]

The equations \( a^\pi a^d = 0 \) and \( a^\pi bs^d = 0 \) imply \( yz = 0 \).

To show that \( z \in \mathcal{A}^d \), we consider the following decomposition:

\[
z = \begin{bmatrix} 0 & aa^db s^\pi \\ 0 & ss^d d s^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^db s^d \\ ss^d c & ss^d d s^d \end{bmatrix} := z_1 + z_2.
\]

Clearly, \( z_1 z_2 = 0 \) and \( z_1^2 = 0 \).

Next, we will prove that \( z_2 \in \mathcal{A}^d \). Let \( z_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \), where \( a_2 = a^2a^d \), \( b_2 = aa^db s^d \), \( c_2 = ss^d c \), and \( d_2 = ss^d d s^d \). It is clear that \( a_2 \) is group invertible, \( a_2^\# = a^d \), and \( a_2^\pi = a^\pi \). Note that \( s_2 := d_2 - c_2 a_2^\# b_2 = ss^d d s^d - ss^d c a_2^\# s^d d s^d = s^2 s^d \), which gives \( s_2 \) is group invertible, \( s_2^\# = s^d \), and \( s_2^\pi = s^\pi \). Furthermore, we can deduce that \( a_2^\pi b s_2^\# = 0 \), \( a_2^\pi b_2 s_2^\pi = 0 \), \( s_2^\pi c_2 a_2^\# = 0 \), \( s_2^\pi c_2 a_2^\pi = s^d c a^\pi = 0 \), and

\[
y_2 := \begin{bmatrix} a_2^\pi a_2^\pi & a_2^\pi b_2 \\ s_2^\pi c_2 a_2^\pi & s_2^\pi s_2^\pi \end{bmatrix} = 0.
\]

By Corollary 4.3, we obtain that \( z_2 \) is group invertible and \( z_2^\# = r \), where \( r \) is defined as in (9).
It follows directly from Lemma 4.1 that \( z \in \mathcal{S}^d \) and \( z^d = r + r^2 z_1 \). By a direct computation, we have

\[
z^\pi = \begin{bmatrix} a^\pi & -a^d b s^\pi \\ 0 & s^\pi \end{bmatrix}
\]
and \( zz^\pi = 0 \). Thus, \( z \) is group invertible.

Finally, we deduce that \( x \in \mathcal{S}^d \) by Lemma 4.1 again. In addition, the equation (10) holds. \( \square \)

In the following result, we give a new representation for the generalized Drazin inverse of block matrix \( x \) in (8) in terms of \( a^d \) and \( s^d \).

**Theorem 4.6** Let \( x \) be defined as in (8). If \( a a^\pi b c = 0, c a^\pi b = 0, a^\pi b c a^\pi = 0, a^2 a^\pi b + b c a^\pi b = a a^\pi b d, \) and \( c a a^\pi b + d c a^\pi b = c a^\pi b d, \) then \( x \in \mathcal{S}^d \) and

\[
x^d = w + \sum_{n=1}^{\infty} w^{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ a^n a^{-1} a^\pi & 0 \end{bmatrix} - 2 \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & a^n a^{-1} a^\pi b \end{bmatrix}
\]

\[
\quad + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} \left( w^2 + \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} a^n a^\pi & 0 \\ a^n a^{-1} a^\pi & 0 \end{bmatrix} \right),
\]

where

\[
w^k = r^k \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1 - p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+k} \begin{bmatrix} 0 & 0 \\ 0 & s^n s^\pi \end{bmatrix}, \quad k \in \mathbb{N},
\]

and \( r \) is defined as in (9).

**Proof** Since \( a a^d b + a^\pi b = b \), then

\[
x = \begin{bmatrix} a & a a^d b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} := x_1 + x_2.
\]

By a computation, the hypotheses imply \( x_1^2 x_2 = x_1 x_2 x_1 \) and \( x_2^2 x_1 = x_2 x_1 x_2 \).

We must show that \( x_1 \in \mathcal{S}^d \). Let

\[
x_1 = \begin{bmatrix} a a^\pi & 0 \\ c a^\pi & 0 \end{bmatrix} + \begin{bmatrix} a^2 a^d & a a^d b \\ c a^d & d \end{bmatrix} := x'_1 + x''_1,
\]

then \( x'_1 x''_1 = 0 \).

In order to prove that \( x'_1 \in \mathcal{S}^d \), we can write \( x''_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \), where \( a_1 = a^2 a^d, b_1 = a a^d b, c_1 = c a a^d \) and \( d_1 = d \). Obviously, \( a_1 \) is group invertible, \( a_1^\pi = a_1^\pi \), and \( a_1^\pi = \pi \). Besides these, we can obtain that \( s_1 := d_1 - c_1 a_1^\pi b_1 = d - c a^d b = s \in \mathcal{S}^d \). Moreover, we clearly have that \( a_1^\pi b_1 s_1^d = a^\pi a a^d b s^d = 0, s_1^d c_1 a_1^\pi = s^\pi c a^d = 0, \) and \( s_1^d c_1 a_1^\pi = s^d c a^d a^\pi = 0 \). Let \( y_1 := \begin{bmatrix} a_1 & a_1^\pi \\ s_1^d c_1 a_1^\pi & s_1^d s_1^\pi \end{bmatrix} \), then \( y_1 = \begin{bmatrix} 0 & 0 \\ 0 & s^\pi s^\pi \end{bmatrix} \in \mathcal{S}^d \).

Therefore, according to Theorem 4.5, we have that \( x''_1 \in \mathcal{S}^d \) and \( (x''_1)^d = w \), where

\[
w = r \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1 - p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ 0 & s^n s^\pi \end{bmatrix}.
\]

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Observe that \( \sigma(x'_1) \subseteq \sigma(aa^\pi) \cup \{0\} \) and \( aa^\pi \in \mathcal{A}^{q_{n_{11}}} \); then \( x'_1 \in \mathcal{A}^{q_{n_{41}}} \), i.e. \( (x'_1)^d = 0 \). Applying Lemma 4.1, we deduce that \( x_1 \in \mathcal{A}^d \) and

\[
x_1^d = w + \sum_{n=1}^{\infty} w_{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ ca^n a^\pi & 0 \end{bmatrix}.
\]

From the equality \( x_2^d = 0 \), it follows that \( x_1^d = 0 \), which yields \( x_1 x_1^d(x_1 + x_2) x_2 x_2^d = 0 = \mathcal{A}^d \). Applying Theorem 3.3, we obtain that \( x \in \mathcal{A}^d \) and

\[
x^d = x_1^d - (x_1^d)^2 x_2 + x_1^d x_2 (x_1^d)^2 - 2x_1^d x_2 (x_1^d)^3 x_2.
\]

Note that \( x_2 (x_1^d)^3 x_2 = x_2^3 (x_1^d) = 0 \) by Lemma 2.5. Then

\[
x^d = x_1^d - (x_1^d)^2 x_2 + x_1^d x_2 (x_1^d)^2 = x_1^d - 2(x_1^d)^2 x_2 + x_2 (x_1^d)^2.
\]

Next, we prove the expression of \( x^d \). Note that, for \( n \in \mathbb{N} \),

\[
\begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix} r = 0 \quad \text{and} \quad \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1 - p \end{bmatrix} r = r;
\]

then the equation (12) holds. By substituting the expression of \( x_1^d \) into the equation (13) and using the following equalities

\[
\begin{bmatrix} a^n a^\pi & 0 \\ ca^n a^\pi & 0 \end{bmatrix} r = 0 \quad \text{and} \quad w \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} = 0,
\]

we can get the equation (11). \( \square \)

From Theorem 4.6, we can obtain the following corollary, which recovers [5, Theorem 8] for a \( 2 \times 2 \) operator matrix.

**Corollary 4.7** Let \( x \) be defined as in (8). If \( a^\pi b c = 0 \), \( c a^\pi b = 0 \), \( a a^\pi b = a^\pi b d \), and \( s = d - ca^\pi b \) is invertible, then \( x \in \mathcal{A}^d \) and

\[
x^d = \left( r - \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r^2 \right) \left( 1 + \sum_{n=0}^{\infty} r_{n+1} \begin{bmatrix} 0 & a^\pi \end{bmatrix} c a^n a^n \right),
\]

where \( r \) is defined as in (9) with \( s^d = s^{-1} \).

**Proof** As in the proof of Theorem 4.6. Note that \( x_1 x_2 = x_2 x_1 \); then \( x_1^d x_2 = x_2 x_1^d \). Thus \( x^d = x_1^d - (x_1^d)^2 x_2 \). By a computation, we can get the equation (14). \( \square \)

**Remark 4.8** Theorem 4.6 generalizes [15, Theorem 2.3], where an expression for \( x^d \) is given under the conditions \( a^\pi b = 0 \) and \( s^\pi ca = 0 \). Indeed, \( a^\pi b = 0 \) and \( s^\pi ca = 0 \) can imply the conditions of Theorem 4.6. However, in general, the converse is false. The following example can illustrate this fact.
Example 4.9 Let $A$ be the Banach algebra of all complex $3 \times 3$ matrices, and take

$$x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

Then

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } c = d = 0.$$

Obviously,

$$a^d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } a^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$  

We can see that the conditions of Theorem 4.6 hold. However, $a^\pi b \neq 0$.

Following the same strategy as in the proof of Theorem 4.6, we derive another formula for $x^d$. Here we omit the proof.

Theorem 4.10 Let $x$ be defined as in (8). If $bca^\pi = 0, dca^\pi = 0, ca^\pi bca^\pi = 0, sa^\pi ca = 0, d^2ca^\pi + cbca^\pi = dcaa^\pi$, and $abca^\pi + bdca^\pi = bcaa^\pi$, then $x \in A^d$ and

$$x^d = w + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & 0 \\ 0 & a^{n-1} a^\pi b \end{bmatrix} w^{n+1} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & ca^{n-1} a^\pi b \end{bmatrix} w^{n+2}$$

$$- 2 \left( w^2 + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & 0 \\ 0 & a^{n-1} a^\pi b \end{bmatrix} w^{n+2} \right) \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix},$$

where $w^k$ is defined as in (12) for $k \in \mathbb{N}$, and $r$ is defined as in (9).

Now, we state a special case of Theorem 4.10, which also generalizes [5, Theorem 9] for a $2 \times 2$ operator matrix.

Corollary 4.11 Let $x$ be defined as in (8). If $bca^\pi = 0, ca^\pi b = 0, caa^\pi = dca^\pi$, and $s = d - ca^d b$ is invertible, then $x \in A^d$ and

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi b & 0 \end{bmatrix} r^{n+3},$$

where $r$ is defined as in (9) with $s^d = s^{-1}$.

Remark 4.12 Theorem 4.10 extends [16, Theorem 3.2], where the generalized Drazin inverse of $x$ is considered in the case that $bca^\pi = 0, dca^\pi = 0, s^\pi ca = 0$, and $abs^\pi = 0$. In fact, Example 4.9 can also illustrate that the conditions of Theorem 4.10 are weaker than those of [16, Theorem 3.2].
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