Ranges and kernels of derivations

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Abstract: In this paper we establish some properties concerning the class of operators $A \in \mathcal{L}(\mathcal{H})$ that satisfy $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$, where $\overline{\mathcal{R}(\delta_A)}$ is the norm closure of the range of the inner derivation $\delta_A$, defined on $\mathcal{L}(\mathcal{H})$ by $\delta_A(X) = AX -XA$. Here $\mathcal{H}$ stands for a Hilbert space; as a consequence, we show that the set $\{A \in \mathcal{L}(\mathcal{H}) \mid \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}\}$ is norm-dense. We also describe some classes of operators $A$, $B$ for which we have $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$, and many of their problems remain also open (see [2, 5, 8, 9, 15, 27]).

Key words: Generalized derivation, p-hyponormal operator, log-hyponormal operator, range and kernel

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators acting on a complexe infinite dimensional Hilbert space $\mathcal{H}$. For $A$, $B \in \mathcal{L}(\mathcal{H})$ we define the generalized derivation $\delta_{A,B}$ associated with $(A, B)$ by $\delta_{A,B}(X) = AX -XB$ for $X \in \mathcal{L}(\mathcal{H})$. If $A = B$, then $\delta_{A,A} = \delta_A$ is called the inner derivation implemented by $A \in \mathcal{L}(\mathcal{H})$. These concrete operators on $\mathcal{L}(\mathcal{H})$ occur in many settings in mathematical analysis and application, their properties, spectrum (see [7, 13, 20]), norm (see [23]), ranges, and kernels (see [4, 5, 8, 9, 15, 27]) have been much studied, and many of their problems remain also open (see [3, 18, 26]).

Let $\mathcal{N} = \bigcup_{A \in \mathcal{L}(\mathcal{H})} \mathcal{R}(\delta_A) \cap \{A\}'$, where $\mathcal{R}(\delta_A)$ denotes the range of $\delta_A$ and $\{A\}'$ is the commutant of $A$. In finite dimension, it is known that $\mathcal{N}$ is exactly the set of nilpotent operators. In infinite dimension the theorem of Kleinecke–Shirokov [19] confirms that any operator in $\mathcal{N}$ is quasinilpotent. However, an operator in $\overline{\mathcal{R}(\delta_A)} \cap \{A\}'$ is not necessarily quasinilpotent (Anderson [1] proved that there exists an operator $A$ in $\mathcal{L}(\mathcal{H})$ such that $I \in \overline{\mathcal{R}(\delta_A)}$, where $\overline{\mathcal{R}(\delta_A)}$ is the normal closure of $\mathcal{R}(\delta_A)$).

In [2] Anderson proved the remarkable result that $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$ if $A$ is normal or isometric. In the same direction, it should be noted that Bouali and Bouhafsi [6] showed that if $A$ is a cyclic subnormal operator then $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$.

The purpose of the first section is to establish some properties of the class of operators $A \in \mathcal{L}(\mathcal{H})$ that satisfy $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$. As a consequence, we give a large class of operators $A \oplus B$ verifying $\overline{\mathcal{R}(\delta_{A,B})} \cap \{A \oplus B\}' = \{0\}$, and we prove that the set $\{A \in \mathcal{L}(\mathcal{H}) \mid \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}\}$ is norm-dense in $\mathcal{L}(\mathcal{H})$.

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If $H$ is a finite dimensional Hilbert space $<X,Y> = tr(XY^*)$ is an inner product on $L(H)$ and we have the orthogonal direct sum decomposition $L(H) = R(\delta_A) \bigoplus \{A^*\}'$. However, when $H$ is infinite dimensional we do not have $\overline{R}(\delta_A) \cap \{A^*\}' = \{0\}$ in general. The class of operators $A$ that have the property $\overline{R}(\delta_A) \cap \{A^*\}' = \{0\}$ includes the normal operators [2], isometries [25], the cyclic subnormal operators [16], the class of operators $A$ such that $P(A)$ is normal for some quadratic polynomial $P$ [16], and Jordan operators [22].

In [12] Elalami proved that $\overline{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*}) = \{0\}$ if $A^*$ and $B$ are hyponormal operators, where $\ker(\delta_{A^*,B^*})$ denotes the kernel of $\delta_{A^*,B^*}$. In the second section we consider this problem; we show that $\overline{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*}) = \{0\}$ if $\langle P(A), P(B) \rangle$ and $(P(B), P(A))$ has the $(F - P)_{L(H)}$ property for some quadratic polynomial $P$. Consequently, we extend the result of [16] to $\delta_{A,B}$. Using the $(F - P)_{L(H)}$ property we prove that $\overline{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*}) = \{0\}$ in each of the following cases:

(a) $B$ is normal and $A^*$ is p-hyponormal or log-hyponormal, ($0 < p \leq 1$).

(b) $A$ is normal and $B$ is p-hyponormal or log-hyponormal, ($0 < p \leq 1$).

An operator $A \in L(H)$ is p-hyponormal, $0 < p \leq 1$, if $|A|^2p \leq |A|^{2p}$ (a 1-hyponormal operator is hyponormal and a $\frac{1}{2}$-hyponormal operator is semihyponormal). It is an immediate consequence of the Lowner–Heinz inequality that a p-hyponormal operator is q-hyponormal for all $0 < q \leq p$. An invertible operator $A \in L(H)$ is log-hyponormal if $\log|A|^2p \leq \log|A|^{2p}$. An invertible p-hyponormal operator is log-hyponormal, but the converse is false; see [17, p. 169] for a reference. Log-hyponormal and p-hyponormal operators, which share a number of properties with hyponormal operators, have been considered by a number of authors in the recent past; see [11, 17, 24] for further references.

2. Commutants and derivation ranges

**Definition 2.1** A vector $x \in H$ is cyclic for $A \in L(H)$ if $H$ is the smallest invariant subspace for $A$ that contains $x$. The operator $A$ is said to be cyclic if it has a cyclic vector.

**Definition 2.2** Let $A \in L(H)$. The operator $A$ is said to be subnormal if there exists a normal operator $N$ on a Hilbert space $K$ such that $H$ is a subspace of $K$, invariant under the operator $N$, and the restriction of $N$ to $H$ coincides with $A$.

Consider the set $\mathcal{M}_C(H) = \{A \in L(H) / \overline{R}(\delta_A) \cap \{A\}' = \{0\}\}$.

**Theorem 2.3** Let $A$ and $B$ be in $\mathcal{M}_C(H)$, such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then $A \oplus B \in \mathcal{M}_C(H \oplus H)$.

**Proof** Assume that $A, B \in \mathcal{M}_C(H)$, and $\sigma(A) \cap \sigma(B) = \emptyset$. Let $C = A \oplus B \in L(H \oplus H)$, and $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \in \overline{R}(\delta_C) \cap \{C\}'$. Then there exists a net $(X_n)_n \subset L(H \oplus H)$ such that $X_n = \begin{pmatrix} X^1_n & X^2_n \\ X^3_n & X^4_n \end{pmatrix}$,

$CX_n - X_nC \xrightarrow{\|\|} D$ and $CD = DC$.

A simple calculation shows that

$AX^1_n - X^1_nA \xrightarrow{\|\|} D_1$ and $AD_1 = D_1A$,
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\[ BX_4 - X_4B \parallel D_4 \quad \text{and} \quad BD_4 = D_4B, \]
\[ BX_3 - X_3A \parallel D_3 \quad \text{and} \quad BD_3 = D_3A, \]
\[ AX_2 - X_2B \parallel D_2 \quad \text{and} \quad AD_2 = D_2B. \]

Hence \( D_1 \in \mathcal{R}(\delta_A) \cap \{A\}' = \{0\}, \quad D_4 \in \mathcal{R}(\delta_B) \cap \{B\}' = \{0\}, \quad D_3 \in \mathcal{R}(\delta_{B,A}) \cap \ker(\delta_{B,A}), \) and \( D_2 \in \mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A,B}). \) Since \( \sigma(A) \cap \sigma(B) = \emptyset, \) it follows from Rosemblem’s theorem [21] that \( D_2 = D_3 = 0. \) Thus \( A \oplus B \in \mathcal{M}_C(\mathcal{H} \oplus \mathcal{H}). \)

**Theorem 2.4** Let \( A, B \in \mathcal{L}(\mathcal{H}), \) with \( B \) similar to \( A \) and \( A \in \mathcal{M}_C(\mathcal{H}). \) Then \( B \in \mathcal{M}_C(\mathcal{H}). \)

**Proof** Let \( A, B \in \mathcal{L}(\mathcal{H}), \) such that \( A \in \mathcal{M}_C(\mathcal{H}) \) and there exists an invertible operator \( S \in \mathcal{L}(\mathcal{H}) \) verifying \( B = S^{-1}AS. \) Then for all \( X \in \mathcal{L}(\mathcal{H}), \)

\[ S^{-1}(AX - XA)S = B(S^{-1}XS) - (S^{-1}XS)B. \]

Thus \( S^{-1}\mathcal{R}(\delta_A)S = \mathcal{R}(\delta_B). \) Hence

\[
\mathcal{R}(\delta_B) \cap \{B\}' = S^{-1}\left[\mathcal{R}(\delta_A) \cap \{A\}'\right]S = S^{-1}\left[\mathcal{R}(\delta_A) \cap \{A\}'\right]S = \{0\}.
\]

This completes the proof.

**Corollary 2.5** Let \( A \in \mathcal{L}(\mathcal{H}). \) If \( A \) is similar to a normal, isometric, or cyclic subnormal operator then

\[ \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}. \]

**Proof** Anderson proved that \( \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\} \) if \( A \) is normal or isometric [2], and in [6] Bouali and Bouhafsi showed that if \( A \) is cyclic subnormal then \( \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}. \)

**Corollary 2.6** Let \( A, B \in \mathcal{L}(\mathcal{H}), \) with \( \sigma(A) \cap \sigma(B) = \emptyset. \) If \( A \) and \( B \) are similar to normal, isometric, or cyclic subnormal operators, all combinations are allowed; then

\[ \mathcal{R}(\delta_{A \oplus B}) \cap \{A \oplus B\}' = \{0\}. \]

**Definition 2.7** [14] we shall say that a certain property \( (P) \) of operators acting on a Hilbert space \( \mathcal{H} \) is a bad-property, or \( b \)-property, if:

(i) Whenever \( A \) satisfies \( (P), \) then for \( \alpha \in \mathcal{C}, \) with \( \alpha \neq 0, \) and \( \beta \in \mathcal{C}, \) the operator \( \alpha A + \beta \) satisfies \( (P); \)

(ii) If \( B \) is similar to \( A, \) and \( A \) satisfies \( (P), \) then \( B \) also satisfies \( (P); \)
(iii) If $A$ and $B$ satisfy (P), such that $\sigma(A) \cap \sigma(B) = \emptyset$, then $A \oplus B$ satisfies (P).

**Theorem 2.8** $\mathcal{M}_C(\mathcal{H})$ is norm-dense in $\mathcal{L}(\mathcal{H})$.

**Proof** Using [14], theorem 3.5.1, it is sufficient to establish that the property $A \in \mathcal{M}_C(\mathcal{H})$ is a b-property.

(i) If $A \in \mathcal{M}_C(\mathcal{H})$, then for $\alpha \in \mathcal{G}$, with $\alpha \neq 0$, and $\beta \in \mathcal{G}$,
\[ \mathcal{R}(\delta_{\alpha A + \beta}) \cap \{\alpha A + \beta\}' = \mathcal{R}(\delta_A) \cap \{A\}' = \{0\}. \]

Thus $\alpha A + \beta \in \mathcal{M}_C(\mathcal{H})$. This proves the first condition.

(ii) By theorem 2.4, $A \in \mathcal{M}_C(\mathcal{H})$ is invariant for similarity. The second condition is then verified.

(iii) By theorem 2.3, the third condition of the b-property is fulfilled. This completes the proof.

\[ \square \]

**Remark 2.9** In [16], theorem 2, Ho shows that $N = \{A \in \mathcal{L}(\mathcal{H}) \mid I \notin \mathcal{R}(\delta_A)\}$ is norm-dense in $\mathcal{L}(\mathcal{H})$. Clearly $\mathcal{M}_C(\mathcal{H}) \subset N$. Theorem 2.8 generalizes Ho’s result.

3. Ranges and kernels of generalized derivations

**Definition 3.1** Let $A$, $B$ be in $\mathcal{L}(\mathcal{H})$. The pair $(A, B)$ is said to possess the Fuglede–Putnam property $(F - P)_{\mathcal{L}(\mathcal{H})}$ if $AT = TB$ and $T \in \mathcal{L}(\mathcal{H})$ implies $A^*T = TB^*$.

**Lemma 3.2** Let $A, X \in \mathcal{L}(\mathcal{H})$ such that $P \geq 0$ and $PX + XP = 0$. Then $PX = XP = 0$.

**Proof** Assume that $PX + XP = 0$. Then $P^2X = XP^2$, and since $P \in \{P^2\}''$ ($\{P^2\}''$ is the bicommutant of $P^2$), it follows that $PX = XP$. Thus $PX = XP = 0$.

**Lemma 3.3** Let $A, B \in \mathcal{L}(\mathcal{H})$. If $(A, B)$ has the $(F - P)_{\mathcal{L}(\mathcal{H})}$ property, then
\[ \mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A,B}) = \{0\}. \]

**Proof** In the proof of theorem 1 [27], Yusun shows that $\|\delta_{A,B}(X) + T\| \geq \|T\|$ for all $X \in \mathcal{L}(\mathcal{H})$ and $T \in \ker(\delta_{A,B})$, if $(A, B)$ has the $(F - P)_{\mathcal{L}(\mathcal{H})}$ property.

**Theorem 3.4** Let $A, B$ be in $\mathcal{L}(\mathcal{H})$. If $(P(A), P(B))$ and $(P(B), P(A))$ have the $(F - P)_{\mathcal{L}(\mathcal{H})}$ property for some quadratic polynomial $P$ then
\[ \mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A,B}) = \{0\}. \]

**Proof** Since for all $(\alpha, \beta) \in \mathcal{G}^2$, with $\alpha \neq 0$,
\[ \mathcal{R}(\delta_{\alpha A + \beta, aB + \beta}) = \mathcal{R}(\delta_{A,B}) \quad \text{and} \quad \ker(\delta_{\alpha A + \beta, aB + \beta}) = \ker(\delta_{A,B}) \]
we may assume without loss of generality that $(A^2, B^2)$ and $(B^2, A^2)$ have the $(F - P)_{L(H)}$ property. Let $T^* \in \overline{R}(\delta_{A, B}) \cap \ker(\delta_{A^*, B^*})$. Then there exists a sequence $(X_n)_n$ in $L(H)$ such that:

$$AX_n - X_n B \xrightarrow{\| \cdot \|} T^* \quad \text{and} \quad TA = BT.$$ 

This implies that

$$A^2 X_n - X_n B^2 \xrightarrow{\| \cdot \|} AT^* + T^* B \quad \text{and} \quad TA^2 = B^2 T.$$ 

Since $(B^2, A^2)$ has the $(F - P)_{L(H)}$ property, it follows that $A^2 T^* = T^* B^2$. Hence $A^2 (AT^* + T^* B) = (AT^* + T^* B) B^2$. Consequently,

$$AT^* + T^* B \in \overline{R}(\delta_{A^2, B^2}) \cap \ker(\delta_{A^2, B^2}).$$ 

Using lemma 3.3 we have $AT^* + T^* B = 0$. By multiplication right by $T$, and using $BT = TA$, we obtain $AP + PA = 0$ with $P = T^* T$. It follows from lemma 3.2 that $AP = PA = 0$. On the other hand, $A(X_n T) - (X_n T) A \xrightarrow{\| \cdot \|} T^* T = P$; and by multiplication of right and left by $P$, we get $P^3 = 0$. Since $P$ is self-adjoint, then $P = 0$, and this necessarily implies $T = 0$. Thus $\overline{R}(\delta_{A, B}) \cap \ker(\delta_{A^*, B^*}) = \{0\}$. \hfill \Box

**Corollary 3.5** [16] Let $A \in L(H)$. If $P(A)$ is normal for some quadratic polynomial $P$, then $\overline{R}(\delta A) \cap \{A^*\}' = \{0\}$.

**Corollary 3.6** Let $A, B \in L(H)$. If $P(A)$ and $P(B)$ are normal operators for some quadratic polynomial $P$, then $\overline{R}(\delta_{A, B}) \cap \ker(\delta_{A^*, B^*}) = \{0\}$.

**Proposition 3.7** Let $A, B$ be in $L(H)$, such that $(B, A)$ has the $(F - P)_{L(H)}$ property. If $T \in \overline{R}(\delta_{A, B}) \cap \ker(\delta_{A^*, B^*})$, then $T^* T \in \overline{R}(\delta_B) \cap \{B\}'$ and $TT^* \in \overline{R}(\delta_A) \cap \{A\}'$.

**Proof** Assume that $T \in \overline{R}(\delta_{A, B}) \cap \ker(\delta_{A^*, B^*})$. Then there exists a sequence $(X_n)_n$ of elements of $L(H)$ such that

$$AX_n - X_n B \xrightarrow{\| \cdot \|} T \quad \text{and} \quad BT^* = T^* A.$$ 

Since right and left multiplication are continuous with respect to the norm topology, it follows that

$$B(T^* X_n) - (T^* X_n) B = T^* (AX_n - X_n B) \xrightarrow{\| \cdot \|} T^* T,$$

and

$$A(X_n T^*) - (X_n T^*) A = (AX_n - X_n B) T^* \xrightarrow{\| \cdot \|} TT^*.$$ 

Hence $T^* T \in \overline{R}(\delta_B)$ and $TT^* \in \overline{R}(\delta_A)$. On the other hand, $(B, A)$ has the $(F - P)_{L(H)}$ property; then $TB = AT$. Consequently we get $T^* T \in \overline{R}(\delta_B) \cap \{B\}'$ and $TT^* \in \overline{R}(\delta_A) \cap \{A\}'$. \hfill \Box
Corollary 3.8 Let $A$, $B$ be in $\mathcal{L}(\mathcal{H})$, such that $(B, A)$ has the \((F-P)\) property. If $A \in \mathcal{M}_\mathcal{L}(\mathcal{H})$ or $B \in \mathcal{M}_\mathcal{L}(\mathcal{H})$, then $\overline{\mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*})} = \{0\}$.

Corollary 3.9 Let $A$, $B$ in $\mathcal{L}(\mathcal{H})$, then $\overline{\mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*})} = \{0\}$ in one of the following conditions:

1. $B$ is normal and $A^*$ is $p$-hyponormal or log-hyponormal, $(0 < p \leq 1)$.
2. $A$ is normal and $B$ is $p$-hyponormal or log-hyponormal, $(0 < p \leq 1)$.

Proof (1). Assume that $B$ is normal and $A^*$ is $p$-hyponormal or log-hyponormal. Then $B$ is $p$-hyponormal and $A^*$ is $p$-hyponormal or log-hyponormal. It follows from lemma 2.1 [10] that $(B, A)$ has the \((F-P)\) property. Since $B$ is normal, $B \in \mathcal{M}_\mathcal{L}(\mathcal{H})$ [2]. Using the corollary 3.8 we obtain $\overline{\mathcal{R}(\delta_{A,B}) \cap \ker(\delta_{A^*,B^*})} = \{0\}$. We obtain (2) in the same way. \hfill \Box

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**References**


