On certain semigroups of full contraction maps of a finite chain

Goje Uba GARBA, Muhammad Jamilu IBRAHIM, Abdussamad Tanko IMAM*
Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria

Received: 14.02.2016 • Accepted/Published Online: 06.06.2016 • Final Version: 22.05.2017

Abstract: Let $X_n = \{1, 2, \ldots, n\}$ with its natural order and let $T_n$ be the full transformation semigroup on $X_n$. A map $\alpha \in T_n$ is said to be order-preserving if, for all $x, y \in X_n$, $x \leq y \Rightarrow x\alpha \leq y\alpha$. The map $\alpha \in T_n$ is said to be a contraction if, for all $x, y \in X_n$, $|x\alpha - y\alpha| \leq |x - y|$. Let $CT_n$ and $OCT_n$ denote, respectively, subsemigroups of all contraction maps and all order-preserving contraction maps in $T_n$. In this paper we present characterisations of Green’s relations on $CT_n$ and starred Green’s relations on both $CT_n$ and $OCT_n$.

Key words: Full transformation, order-preserving, contraction, Green’s relations, starred Green’s relations

1. Introduction

The full transformation semigroup on $X_n = \{1, 2, \ldots, n\}$, under its natural order, is denoted by $T_n$. The importance of the study of $T_n$, as a naturally occurring semigroup, is justified by its universal property in which every finite semigroup is embeddable in some $T_n$. This is analogous to Cayley’s theorem for symmetric group $S_n$, of all permutations of $X_n$, in group theory. Thus, just as the study of alternating and dihedral groups has made a significant contribution to group theory, there is some interest in identifying and studying certain special subsemigroups of $T_n$. The subsemigroups $O_n = \{\alpha \in T_n : x \leq y \Rightarrow x\alpha \leq y\alpha, \text{ for all } x, y \in X_n\}$, of order-preserving elements and $S_n^- = \{\alpha \in T_n : x\alpha \leq x, \text{ for all } x \in X_n\}$, of order-decreasing elements of $T_n$ have been studied. In [14], Howie showed that every element of $O_n$ is expressible as a product of idempotents and also obtained formulae for the number of elements and the number of idempotents in $O_n$. Umar in [22] showed that every element of $S_n^-$ is expressible as a product of idempotents. The rank and idempotent rank of $O_n$ were computed by Gomes and Howie [12] to be $n$ and $2(n-1)$, respectively. Maximal subsemigroups, maximal idempotent-generated/regular subsemigroups, and locally maximal idempotent-generated subsemigroups of $O_n$ were described and classified in [24–26]. The results of [26] were simplified in [28]. Maximal regular subsemibands of the two-sided ideals of $O_n$ were completely described by Zhao [27]. In [8], a description of the endomorphisms of $O_n$ was presented. Other algebraic properties in the semigroup $O_n$ and some of its notable subsemigroups and oversemigroups may be found in [3–7,9].

On a semigroup $S$ the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^* \text{ if and only if } a, b$ are related by the Green’s relation $\mathcal{L}$ in some over semigroup of $S$. The relation $\mathcal{R}^*$ is defined dually. These relations have played a fundamental role in the study of many important classes of semigroups; see for example the work by Fountain [10, 11]. Moreover, many papers have appeared describing the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ in certain

*Correspondence: atimam@abu.edu.ng
2010 AMS Mathematics Subject Classification: 20M20.
characterising contraction maps in \( O \)

On \( O \) and \( O \) contraction maps and of all order-preserving contraction maps in \( T \),

Proof  Since Pei and Zhou \cite{18} characterised \( \mathcal{L}^* \) and \( \mathcal{R}^* \) in the subsemigroup of \( T_n \), consisting of all transformations preserving an equivalence relation. Similar characterisations of \( \mathcal{L}^* \) and \( \mathcal{R}^* \) were presented in \cite{16–21}. In this current article we consider an algebra study for the so-called subsemigroups of contraction mappings of \( T_n \). In particular, we present characterisations of both Green’s and starred Green’s relations for these semigroups.

A map \( \alpha \) in \( T_n \) is said to be a contraction if \( |x\alpha - y\alpha| \leq |x - y| \), for all \( x, y \in X_n \). The sets of all contraction maps and of all order-preserving contraction maps in \( T_n \) are, respectively, denoted by \( CT_n \) and \( OCT_n \), which are subsemigroups of \( T_n \). The term contraction map first appeared in \cite{13} but algebraic and combinatorial studies of the semigroups \( CT_n \) and \( OCT_n \) were initiated by Dauda \cite{1}. Orders and regularity for both \( CT_n \) and \( OCT_n \) were investigated in \cite{1}. He also characterises Green’s relations on \( OCT_n \). Here we investigate Green’s relations on \( CT_n \) and starred Green’s relations on both \( CT_n \) and \( OCT_n \).

2. Preliminaries

Let \( O_n = \{ \alpha \in T_n \setminus S_n : (\forall x, y \in X_n) \; x \leq y \Rightarrow x\alpha \leq y\alpha \} \), \( CT_n = \{ \alpha \in T_n \setminus S_n : (\forall x, y \in X_n) |x\alpha - y\alpha| \leq |x - y| \} \), and \( OCT_n = CT_n \cap O_n \) be the subsemigroups of \( T_n \setminus S_n \) consisting of all order-preserving maps, all contraction maps, and all order-preserving contraction maps, respectively.

**Definition 2.1** Let \( A \) be a subset of \( X_n \) and let \( \{A_1, A_2, \ldots, A_r\} \) be a partition of \( X_n \). Then \( A \) is called convex if, for all \( x, y \in X_n \), \( (x, y \in A \; \text{and} \; x \leq z \leq y) \Rightarrow z \in A \). \( A \) is called a transversal of \( \{A_1, A_2, \ldots, A_r\} \) if \( |A| = r \) and each \( A_i \) \( (1 \leq i \leq r) \) contains exactly one point of \( A \). The partition \( \{A_1, A_2, \ldots, A_r\} \) is called a convex partition if it possesses a convex transversal.

From the definition of contraction maps, it is easy to notice (which is also noticed in \cite[Lemma 3.1.2]{1}) that if \( \alpha \in T_n \) is a contraction, then there exists \( s \in X_n \) such that

\[
\text{im}(\alpha) = \{s, s + 1, \ldots, t - 1, t\},
\]

in other words, \( \text{im}(\alpha) \) is convex.

Each map \( \alpha \in O_n \) can be written as

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},
\]

where \( \text{im}(\alpha) = \{a_1 < a_2 < \ldots < a_r\} \) and \( A_1, A_2, \ldots, A_r \) are equivalence classes under the equivalence \( \ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\} \). Thus, \( x\alpha = a_i \) for all \( x \in A_i \) \( (1 \leq i \leq r) \). It is then easy to see, from the order-preserving property, that the \( \ker(\alpha) \)-classes \( A_i \) \( (1 \leq i \leq r) \) are convex subsets of \( X_n \). We start by characterising contraction maps in \( O_n \).

**Lemma 2.1** \( \alpha \in O_n \) is a contraction if and only if \( \text{im}(\alpha) \) is convex.

**Proof** Since \( O_n \) is a subsemigroup of \( T_n \) it is clear, from our observation just after Definition 2.1, that \( \text{im}(\alpha) \) is convex whenever \( \alpha \in O_n \) is a contraction.

501
Conversely, suppose that \( \text{im}(\alpha) = \{a_1 < a_2 < \ldots < a_r\} \) is convex. Then \( a_{i+1} = a_i + 1 \) (1 \( \leq i \leq r - 1 \)). Let \( x, y \in X_n \) and suppose (without loss of generality) that \( x < y \). Then either \( x, y \in a_i\alpha^{-1} \) (for some \( i \)) or \( x \in a_i\alpha^{-1} \) and \( y \in a_j\alpha^{-1} \) (for some \( i < j \)). In the former, we have \( |x\alpha - ya| = |a_i - a_i| = 0 < |x - y| \). In the latter, assume that \( j = i + k \), where \( k \) is any positive integer, so that \( |x\alpha - ya| = |a_i + k - a_i| = |a_i + k - a_i| = k \leq |x - y| \) since \( \ker(\alpha) \)-classes \( a_i\alpha^{-1} \) (1 \( \leq i \leq r \)) are convex. Thus, \( |x\alpha - ya| \leq |x - y| \) for all \( j \geq i \) and so \( \alpha \) is a contraction. 

Next we characterise contraction maps in \( T_n \).

**Theorem 2.2** Let \( \alpha \) be an element of \( T_n \) of height \( r \), where \( r \leq n \). Then \( \alpha \) is contraction if and only if

(i) \( \text{im}(\alpha) \) is a convex subset of \( X_n \), and

(ii) for each \( i \in \text{im}(\alpha) \) and each \( x \in i\alpha^{-1} \), if \( x - 1 \in k\alpha^{-1} \) and \( x + 1 \in t\alpha^{-1} \), then \( k, t \in \Phi_i \), where

\[
\Phi_i = \begin{cases} 
\{i, i+1\} & \text{if } i = 1 \\
\{i-1, i, i+1\} & \text{if } 1 < i < r \\
\{i-1, i\} & \text{if } i = r.
\end{cases}
\]

**Proof** Suppose that \( \alpha \) in \( T_n \) is a contraction. Then, by [1, Lemma 3.1.2], part (i) holds, that is, \( \text{im}(\alpha) \) is convex. Now suppose that, for each \( i \in \text{im}(\alpha) \) and each \( x \in i\alpha^{-1} \), \( x - 1 \in k\alpha^{-1} \) and \( x + 1 \in t\alpha^{-1} \). We need to show that \( s, t \in \Phi_i \). Suppose that either \( s \notin \Phi_i \) or \( t \notin \Phi_i \). Then

\[
|x\alpha - (x-1)\alpha| = |i - s| > 1 = |x - (x - 1)|
\]
or

\[
|(x + 1)\alpha - x\alpha| = |t - i| > 1\left|(x + 1) - x\right|,
\]

so that, in both cases, \( \alpha \) cannot be a contraction. This is a contradiction to the choice of \( \alpha \). Thus both \( s \) and \( t \) must be in \( \Phi_i \).

Conversely, suppose that \( \alpha \in T_n \) satisfies the two conditions of the theorem and let \( x, y \in X_n \). If both \( x \) and \( y \) belong to the same block of \( \alpha \), then

\[
|x\alpha - ya| = 0 \leq |x - y|.
\]

On the other hand, if \( x \) and \( y \) belong to different blocks of \( \alpha \), say \( x \in s\alpha^{-1} \) and \( y \in t\alpha^{-1} \), where \( s, t \in \text{im}(\alpha) \) and \( s \neq t \), it is then not so hard to see that the two conditions of the theorem ensure that

\[
|x\alpha - ya| = |s - t| \leq |x - y|.
\]

Thus, \( \alpha \) is a contraction. \( \square \)

### 3. Green’s relations

For the definition of Green’s relations \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \) and \( \mathcal{J} \) on a semigroup see [15]. As in [23], we shall throughout this and the next sections write \( \mathcal{K}(S) \) to emphasise that \( \mathcal{K} \) is a relation on a semigroup \( S \). In this section we characterise the relations \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \) and \( \mathcal{J} \) on \( C\mathcal{T}_n \).

Let \( \text{Ker}(\alpha) \) be the set of all the equivalence classes of the equivalence relation \( \ker(\alpha) \) on \( X_n \), that is \( \text{Ker}(\alpha) = X_n/\ker(\alpha) \).
Theorem 3.1 Let $\alpha, \beta \in CT_n$. Then

(i) $(\alpha, \beta) \in L(CT_n)$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$, and both Ker$(\alpha)$ and Ker$(\beta)$ are convex partitions of $X_n$;

(ii) $(\alpha, \beta) \in R(CT_n)$ if and only if Ker$(\alpha) = \text{ker}(\beta)$;

(iii) $(\alpha, \beta) \in D(CT_n)$ if and only if $|\text{im}(\alpha)| = |\text{im}(\beta)|$, and both Ker$(\alpha)$ and Ker$(\beta)$ are convex partitions of $X_n$.

Proof (i) Suppose that $(\alpha, \beta) \in L(CT_n)$, then

$$\delta \beta = \alpha \quad \text{and} \quad \gamma \alpha = \beta \quad \text{for some} \quad \delta, \gamma \in CT_n^1.$$ 

This clearly implies that $\text{im}(\alpha) = \text{im}(\beta)$. Therefore, $\text{im}(\gamma)$ and $\text{im}(\delta)$ must be transversal of Ker$(\alpha)$ and Ker$(\beta)$, respectively. However, since $\delta, \gamma \in CT_n^1$ it follows, by Theorem 2.2(i), that $\text{im}(\delta)$ and $\text{im}(\gamma)$ are convex subsets of $X_n$. Thus, Ker$(\alpha)$ and Ker$(\beta)$ are convex partitions of $X_n$.

Conversely, suppose that $\text{im}(\alpha) = \text{im}(\beta) = \{c_1, c_2, \ldots, c_r\}$ and Ker$(\alpha)$, Ker$(\beta)$ are convex partitions of $X_n$. Let $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ be convex transversal of Ker$(\alpha)$ and Ker$(\beta)$, respectively, arranged in a way that $a_i \in c_i\alpha^{-1}$ and $b_i \in c_i\beta^{-1}$ for each $1 \leq i \leq r$. Define maps $\delta$ and $\gamma$ by $\text{ker}(\delta) = \text{ker}(\alpha)$, ker$(\gamma) = \text{ker}(\beta)$, $(c_i\alpha^{-1})\delta = b_i$, and $(c_i\beta^{-1})\gamma = a_i$, for each $1 \leq i \leq r$. Then $\delta, \gamma \in CT_n$ and $\delta \beta = \alpha, \gamma \alpha = \beta$ so that $(\alpha, \beta) \in L(CT_n)$.

(ii) Suppose that $(\alpha, \beta) \in R(CT_n)$; then

$$\beta \delta = \alpha \quad \text{and} \quad \alpha \gamma = \beta \quad \text{for some} \quad \delta, \gamma \in CT_n^1.$$ 

From this it follows that Ker$(\alpha) = \text{ker}(\beta)$.

Conversely, suppose that Ker$(\alpha) = \text{Ker}(\beta) = \{C_1, C_2, \ldots, C_r\}$. Then, since $\alpha, \beta \in CT_n$, we may (without loss of generality) write

$$\alpha = \begin{pmatrix} C_1 & C_2 & \cdots & C_r \\ i & i+1 & \cdots & i+r-1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} C_1 & C_2 & \cdots & C_r \\ j & j+1 & \cdots & j+r-1 \end{pmatrix}$$

for some $i, j \in X_n$. Then the maps

$$\delta = \begin{pmatrix} \{1,2,\ldots,j\} & j+1 & \cdots & j+r-2 & \{j+r-1,j+r,\ldots,n\} \\ i & i+1 & \cdots & i+r-2 & i+r-1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} \{1,2,\ldots,i\} & i+1 & \cdots & i+r-2 & \{i+r-1,i+r,\ldots,n\} \\ j & j+1 & \cdots & j+r-2 & j+r-1 \end{pmatrix}$$

are in $CT_n^1$ and satisfy $\beta \delta = \alpha, \alpha \gamma = \beta$ so that $(\alpha, \beta) \in R(CT_n)$.

(iii) Suppose that $(\alpha, \beta) \in D(CT_n)$; then $(\alpha, \gamma) \in L(CT_n)$ and $(\gamma, \beta) \in R(CT_n)$, for some $\gamma \in CT_n$. Using Theorem 3.1, we have that $\text{im}(\alpha) = \text{im}(\gamma), \text{Ker}(\gamma) = \text{ker}(\beta)$, and Ker$(\alpha)$, Ker$(\gamma)$ are convex partitions of $X_n$. This implies that $|\text{im}(\alpha)| = |\text{im}(\beta)|$ and Ker$(\alpha)$, Ker$(\beta)$ are convex partitions of $X_n$.

Conversely, suppose that $|\text{im}(\alpha)| = |\text{im}(\beta)|$, and both Ker$(\alpha)$ and Ker$(\beta)$ are convex partitions of $X_n$. Then we can choose $\gamma \in CT_n$ such that Ker$(\gamma) = \text{ker}(\beta)$ and $\text{im}(\gamma) = \text{im}(\alpha)$. It is then clear that $(\alpha, \gamma) \in L(CT_n)$ and $(\gamma, \beta) \in R(CT_n)$, so that $(\alpha, \beta) \in D(CT_n)$.  

□
4. Starred Green’s relations

Recall that on a semigroup \( S \) the relation \( L^* \) is defined by the rule that \((a, b) \in L^* \) if and only if \( a, b \) are related by the Green’s relation \( L \) in some oversemigroup of \( S \). The relation \( R^* \) is defined dually. These relations also have the following characterisations (see \[10\])

\[
L^*(S) = \{ (a, b) : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by \} \tag{2}
\]

and

\[
R^*(S) = \{ (a, b) : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb \}. \tag{3}
\]

The join of the relations \( L^* \) and \( R^* \) is denoted by \( D^* \) and their intersection by \( H^* \).

**Theorem 4.1** Let \( S \in \{CT_n, OCT_n\} \) and let \( \alpha, \beta \in S \). Then

(i) \((\alpha, \beta) \in L^*(S) \) if and only if \( \text{im}(\alpha) = \text{im}(\beta) \),

(ii) \((\alpha, \beta) \in R^*(S) \) if and only if \( \ker(\alpha) = \ker(\beta) \),

(iii) \((\alpha, \beta) \in H^*(S) \) if and only if \( \text{im}(\alpha) = \text{im}(\beta) \) and \( \ker(\alpha) = \ker(\beta) \),

(iv) \((\alpha, \beta) \in D^*(S) \) if and only if \( |\text{im}(\alpha)| = |\text{im}(\beta)| \).

**Proof**

(i) Suppose that \((\alpha, \beta) \in L^*(S) \). Let \( \text{im}(\alpha) = \{a_1, \ldots, a_r\} \), where (by \[1\], Lemma 3.1.2, or Lemma 2.1) \( a_{i+1} = a_i + 1 \) for each \( i = 1, \ldots, n-1 \). Then

\[
\alpha \cdot \begin{pmatrix} 1, \ldots, a_1 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ a_{r-1} \\ a_r \\ \vdots \\ a_r \\ \end{pmatrix} = \alpha \cdot 1_{X_n}
\]

and, by Equation (2), if and only if

\[
\beta \cdot \begin{pmatrix} 1, \ldots, a_1 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ a_{r-1} \\ a_r \\ \vdots \\ a_r \\ \end{pmatrix} = \beta \cdot 1_{X_n}
\]

which implies that \( \text{im}(\beta) \subseteq \{a_1, \ldots, a_r\} = \text{im}(\alpha) \). Similarly, we can show that \( \text{im}(\alpha) \subseteq \text{im}(\beta) \), and so \( \text{im}(\alpha) = \text{im}(\beta) \).

Conversely, suppose that \( \text{im}(\alpha) = \text{im}(\beta) \). Then \((\alpha, \beta) \in L(T_n) \) and, since \( T_n \) is an oversemigroup of \( S \), it follows from definition that \((\alpha, \beta) \in L^*(S) \).

(ii) Suppose that \((\alpha, \beta) \in R^*(S) \). Then

\[
(x, y) \in \ker(\alpha) \iff x\alpha = y\alpha \\
\Leftrightarrow \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \alpha = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \alpha \\
\Leftrightarrow \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \beta = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \beta \quad \text{(by Equation (3))} \\
\Leftrightarrow x\beta = y\beta \\
\Leftrightarrow (x, y) \in \ker(\beta).
\]

Hence \( \ker(\alpha) = \ker(\beta) \).
Similarly, the converse part is clear.

(iii) This follows from parts (i) and (ii).

(iv) Suppose \((\alpha, \beta) \in \mathcal{D}^*(S)\). Then, by [15, Proposition 1.5.11], for some \(n \in \mathbb{N}\), there exist elements \(\delta_1, \delta_2, \ldots, \delta_{2n-1} \in S\) such that

\[
(\alpha, \delta_1) \in \mathcal{L}^*(S), (\delta_1, \delta_2) \in \mathcal{R}^*(S), (\delta_2, \delta_3) \in \mathcal{L}^*(S), \ldots, (\delta_{2n-1}, \beta) \in \mathcal{R}^*(S).
\]

Now, by parts (i) and (ii) of the theorem, we have

\[
|\text{im}(\alpha)| = |\text{im}(\delta_1)| = |X_n/\ker(\delta_1)| = |X_n/\ker(\delta_2)| = |\text{im}(\delta_2)| = |\text{im}(\delta_3)| = \cdots = |X_n/\ker(\delta_{2n-1})| = |X_n/\ker(\beta)| = |\text{im}(\beta)|.
\]

Conversely, suppose that \(|\text{im}(\alpha)| = |\text{im}(\beta)|\) and let

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}
\]

where \(a_{i+1} = a_i + 1\), \(b_{i+1} = b_i + 1\) for each \(i = 1, 2, \ldots, r - 1\). Then the map

\[
\gamma = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}
\]

is in \(S\) and, by parts (i) and (ii), \((\alpha, \gamma) \in \mathcal{L}^*(S)\) and \((\gamma, \beta) \in \mathcal{R}^*(S)\) so that, by [15, Proposition 1.5.11], \((\alpha, \beta) \in \mathcal{D}^*(S)\) follows.

The \(\mathcal{L}^* - \text{class}\) containing an element \(a\) is denoted by \(L_a^*\) and corresponding notations are used for the remaining starred relations. We define a left(right) \(*-\text{ideal}\) of a semigroup \(S\) to be a left(right) ideal \(I\) of \(S\) for which \(L_a^* \subseteq I\) \((R_a^* \subseteq I)\) for all \(a \in I\). A subset \(I\) of \(S\) is a \(*-\text{ideal}\) if it is both left and right \(*-\text{ideals}\) of \(S\). The principal \(*-\text{ideal}\), \(J^*(a)\), generated by \(a \in S\) is the intersection of all \(*-\text{ideals}\) of \(S\) to which \(a\) belongs. The relation \(J^*\) is defined by the rule that: \(aJ^*b\) if and only if \(J^*(a) = J^*(b)\).

Now we are going to show that on the semigroup \(S \in \{\mathcal{C}T_n, \mathcal{OCT}_n\}\), \(\mathcal{D}^* = J^*\) but first we record the following lemma from [11].

**Lemma 4.2** Let \(a, b\) be elements of a semigroup \(S\). Then \(b \in J^*(a)\) if and only if there are elements \(a_0, a_1, \ldots, a_n \in S\), \(x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1\) such that \(a = a_0, b = a_n\) and \((a_i, x_ia_{i-1}y_i) \in \mathcal{D}^*(S)\) for \(i = 1, \ldots, n\).

Immediately we adopt the method used in [23] to have

**Lemma 4.3** Let \(S \in \{\mathcal{C}T_n, \mathcal{OCT}_n\}\). Then for each \(\alpha, \beta \in S\), \(\alpha \in J^*(\beta)\) implies \(|\text{im}(\alpha)| \leq |\text{im}(\beta)|\).

**Proof** Let \(\alpha \in J^*(\beta)\), then by Lemma 4.2, there exist \(\beta_0, \ldots, \beta_n \in S\), \(\delta_1, \ldots, \delta_n, \gamma_1, \ldots, \gamma_n \in S^1\) such that \(\beta = \beta_0, \alpha = \beta_n\) and \((\beta_i, \delta_i, \beta_{i-1} \gamma_i) \in \mathcal{D}^*(S)\), for \(i = 1, \ldots, n\). However, by Theorem 4.1(iv), this implies that

\[
|\text{im}(\beta_i)| = |\text{im}(\delta_i, \beta_{i-1} \gamma_i)| \leq |\text{im}(\beta_{i-1})|
\]

for all \(i = 1, \ldots, n\), which implies \(|\text{im}(\alpha)| \leq |\text{im}(\beta)|\) as required.

The fact that \(\mathcal{D}^* \subseteq J^*\) together with Lemma 4.3 gives the following result.

**Theorem 4.4** On the semigroup \(S \in \{\mathcal{C}T_n, \mathcal{OCT}_n\}\), \(\mathcal{D}^* = J^*\).
References


