Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 655 – 680
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doi:10.3906/mat-1601-19

Stability of nonmonotone critical traveling waves for spatially discrete reaction-diffusion equations with time delay

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Received: 07.01.2016 • **Accepted/Published Online:** 27.06.2016 • **Final Version:** 22.05.2017

Abstract: This paper is concerned with the existence and stability of critical traveling waves (waves with minimal speed $c = c_*$) for a nonmonotone spatially discrete reaction-diffusion equation with time delay. We first show the existence of critical traveling waves by a limiting argument. Then, using the technical weighted energy method with some new variations, we prove that the critical traveling waves $\phi(x + c_*t)$ (monotone or nonmonotone) are time-asymptotically stable when the initial perturbations are small in a certain weighted Sobolev norm.

Key words: Spatially discrete reaction-diffusion equations, nonmonotone critical traveling waves, stability, weighted energy

1. Introduction

This article is devoted to the stability of critical traveling waves for the following spatially discrete reaction-diffusion equation with time delay

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = \Delta_1 v(t,x) - v(t,x) + g(v(t-\tau,x)), \quad t>0, x\in\mathbb{R}, \\ v(s,x) = v_0(s,x), \quad (s,x)\in[-\tau,0]\times\mathbb{R}, \end{cases} \tag{1.1}$$

where $\tau > 0$ and

$$\Delta_1 v(t,x) = v(t,x+1) - 2v(t,x) + v(t,x-1).$$

Equation (1.1) describes the spatiotemporal evolution of a single-species population [8, 13], where $v(t,x)$ represents the mature population at time t and location x, and τ is the maturation delay. The function $g : [0,\infty) \to (0,\infty)$ is called the birth rate function, which is assumed to satisfy the following assumptions:

(H1) $g(0) = g(K) - K = 0$, $g'(0) > 1$ and $g'(K) < 1$.

(H2) There exists $v_* \in (0,K)$ such that $g(\cdot)$ is increasing in $[0,v_*]$ and decreasing in $[v_*,+\infty)$, which also implies $g'(0) > 0$ and $g'(K) < 0$.

(H3) $g \in C^2[0,\infty)$, $|g'(v)| < g'(0)$ for $v \in [0,\infty)$.

A specific function $g(v) = pve^{-av}$ with $p > 0$ and $a > 0$, which has been widely used in the mathematical biology literature, satisfies the above conditions for a wide range of parameters p and a. The assumption (H1)

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2010 *AMS Mathematics Subject Classification:* 34K12, 35K57, 92D25.

shows that 0 and K are two equilibria of (1.1), and 0 is unstable and K is stable. Hence, (1.1) is a monostable system. (H2) is a unimodality condition and implies that $g(v)$ is not monotone for $v \in [0, K]$.

For the spatially discrete reaction-diffusion equations like (1.1), there has been growing interest in the last decade; see Chen and Guo [2], Chen et al. [1], Guo and Zimmer [5, 6], Ma and Zou [13], and Wu and Zou [19] for the traveling wavefronts and Guo and Morita [4] and Guo [7] for the entire solutions. As we know, the study of traveling waves is important in many applications, since it can describe certain dynamical behavior of the studied problem. A traveling wave of (1.1) connecting with two steady states 0 and K is a special solution $u(t,x) = \phi(x+ct)$ satisfying

$$\begin{cases} c\phi'(\xi) - \Delta_1\phi(\xi) + \phi(\xi) = g(\phi(\xi - c\tau)), \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K, \end{cases} \tag{1.2}$$

where $\xi = x + ct$, $' = \frac{d}{d\xi}$, and c is the wave speed. Moreover, if $\phi(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wavefront. The existence of traveling wavefronts of (1.1) was first proved by Ma and Zou [13] under the assumption that $g(u)$ is increasing on the interval $[0, K]$. The methods they used are the upper-lower solutions and Schauder's fixed-point theorem. Recently, Yang et al. [20] studied the traveling waves of equation (1.1) when $g(u)$ is not increasing on the interval $[0, K]$. They obtained the existence of traveling wave solutions with speeds $c > c_*$ by constructing two auxiliary discrete reaction-diffusion equations with quasi-monotonicity. We need to point out that the existence of traveling waves with speed $c = c_*$ is unknown so far. Hence, in the first part of this paper, we prove the existence of traveling waves with speed $c = c_*$. In addition, the precise asymptotic behavior of traveling waves at positive infinity is obtained.

Our main goal in this paper is to show the stability of critical traveling waves of (1.1). The stability of traveling waves for the following delayed reaction-diffusion equation,

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} - v(t,x) + g(v(t-\tau, x)), \tag{1.3}$$

has been widely studied by many authors [12, 14–17]. Recently, Huang et al. [9], Huang et al. [10], and Zhang and Ma [21] respectively studied the stability of traveling waves of (1.3) with $\frac{\partial^2 v(t,x)}{\partial x^2}$ replaced by $J * v - v$, and nonlocal nonlinearity. To the best of our knowledge, previous works on the stability of traveling waves to discrete reaction-diffusion equations like (1.1) treated only the monotone case, which allowed the use of the comparison principle, e.g., see [2, 6]. For our equation (1.1), the comparison principle does not hold. More recently, Yang et al. [20] proved that all noncritical traveling waves $\phi(x+ct)$ with the wave speed $c > c_*$, including monotone or nonmonotone ones, are time-exponentially stable, by the technical weighted-energy method. However, the stability of the critical traveling waves $\phi(x+c_*t)$ still remains open. In the second part of this paper, we present a solution for this open problem. We should point out that using the method in [20] directly cannot obtain the stability of critical traveling waves. One crucial step for the stability proof in [20] is to get an energy estimate for the perturbed equation in a weighted L^2_w-space, that is

$$\begin{aligned} &\|v(t)\|^2_{L^2_w} + \int_0^t \int_{\mathbb{R}} \mathcal{A}_{\eta,w}(\xi) w(\xi) v^2(s,\xi)\, \mathrm{d}\xi\, \mathrm{d}s \\ \le & C\|v_0(0)\|^2_{L^2_w} + O(1)\int_0^t \int_{\mathbb{R}} w(\xi)|v(s,\xi)||v(s-\tau, \xi - c_*\tau)|^2 \mathrm{d}\xi\, \mathrm{d}s. \end{aligned} \tag{1.4}$$

In (4.10) of [20], the authors estimated the function $\mathcal{A}_{\eta,w}(\xi)$:

$$\mathcal{A}_{\eta,w}(\xi) \geq C_1 > 0 \qquad \text{for} \quad \xi \in \mathbb{R}, \quad c > c_*,$$

where C_1 is some positive constant. This estimate allows us to control the nonlinear term on the left-hand side of inequality (1.4) when the initial perturbation is small enough and makes us derive the exponential decay for the perturbed solution. However, when $c = c_*$, we can only obtain

$$\mathcal{A}_{\eta,w}(\xi) \geq C_1 = 0 \quad \text{for} \quad c = c_*,$$

which seems not enough to control the nonlinear term.

We remark that the stability of traveling waves for nonmonotone delayed reaction-diffusion equation (1.3) has been studied recently; see, e.g., [3, 11, 18]. In particular, Chern et al. [3] proved the stability of nonmonotone critical traveling waves of (1.3) by the weighted energy method with some development. It is natural to ask if the method in [3] can be extended to the discrete delayed reaction-diffusion equation (1.1). We shall give an affirmative answer. More precisely, by using the technical weighted energy method with some new variations, we shall prove the asymptotic stability of critical traveling waves of (1.1), including monotone and nonmonotone ones. It is remarked that there are some differences between equations (1.1) and (1.3) in obtaining the stability of critical traveling waves. First, $\frac{\partial^2 v(t,x)}{\partial x^2}$ is more convenient than the discrete diffusion operator $\Delta_1 v(t,x) = v(t,x+1) - 2v(t,x) + v(t,x-1)$ for obtaining the energy estimates in weighted space. We shall take Young's inequality to overcome the difficulties caused by the discrete diffusion operator Δ_1. Second, in order to obtain that the solution of the perturbed equation for (1.3) belongs to $X_{loc}(0,\tau)$, the fundamental solution of the perturbed equation plays an important role; see [3]. However, (1.1) and its perturbed equation (3.1) do not have fundamental solutions. Motivated by the classical transport equation, we give new forms of solutions of (3.1). It can help us to get the estimate in C- norm and the uniform limit of solutions.

Our paper is organized as follows. In Section 2, we establish the existence of critical traveling waves of (1.1). Section 3 is devoted to proving the stability of the critical traveling waves of (1.1), which is divided into three subsections. In Subsection 3.1, we obtain the global existence and uniqueness of the solution for the perturbed equation, where the initial perturbation can be allowed to be arbitrarily large. In Subsection 3.2, when the initial perturbation is suitably small, the solution of the perturbed equation can be proved to be uniformly bounded by the antiweighted energy method. Based on the uniform boundedness, we shall further prove asymptotic stability in Subsection 3.3.

2. Existence of critical traveling waves

This section is devoted to the existence of critical traveling waves. The characteristic function for (1.2) with respect to the trivial equilibrium 0 can be represented by

$$\mathcal{P}(c,\lambda) = c\lambda - (e^{\lambda} + e^{-\lambda} - 2) + 1 - g'(0)e^{-\lambda c\tau}. \tag{2.1}$$

Then the following result holds:

Lemma 2.1 *Assume that $g'(0) > 1$. Then there exist $\lambda_* > 0$ and $c_* > 0$ such that*

$$\mathcal{P}(c_*,\lambda_*) = 0, \quad \frac{\partial}{\partial\lambda}\mathcal{P}(c_*,\lambda)|_{\lambda=\lambda_*} = 0. \tag{2.2}$$

Furthermore, if $c > c_*$, *then* $\mathcal{P}(c,\lambda) = 0$ *has two distinct positive real roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_* < \lambda_2(c)$, *and* $\mathcal{P}(c,\lambda) > 0$ *for* $\lambda \in (\lambda_1(c), \lambda_2(c))$.

Proof Denote

$$G_c(\lambda) = c\lambda - \left(e^{\lambda} + e^{-\lambda} - 2\right) + 1, \quad H_c(\lambda) = g'(0)e^{-\lambda c\tau}.$$

It is easy to see that

$$G_c(0) = 1, \quad H_c(0) = g'(0) > 1.$$

Furthermore, we have

$$G''_c(\lambda) = -\left(e^{\lambda} + e^{-\lambda}\right) < 0,$$
$$H''_c(\lambda) = (c\tau)^2 g'(0)e^{-\lambda c\tau} > 0.$$

Hence, there exists a unique $c_* = c_*(\tau) > 0$ such that the two graphs of G_c and H_c are tangent at λ_*,

$$G_{c_*}(\lambda_*) = H_{c_*}(\lambda_*), \quad G'_{c_*}(\lambda_*) = H'_{c_*}(\lambda_*),$$

namely,

$$\mathcal{P}(c_*, \lambda_*) = c_*\lambda_* - \left(e^{\lambda_*} + e^{-\lambda_*}\right) + 3 - g'(0)e^{-\lambda_* c_* \tau} = 0,$$
$$\frac{\partial}{\partial\lambda}\mathcal{P}(c_*, \lambda)|_{\lambda=\lambda_*} = c_* - (e^{\lambda_*} - e^{-\lambda_*}) + c_*\tau g'(0)e^{-\lambda_* c_* \tau} = 0.$$

Then we have that for $c > c_*$, there exist two numbers $0 < \lambda_1(c) < \lambda_2(c)$ satisfying

$$c\lambda_i(c) - \left(e^{\lambda_i(c)} + e^{-\lambda_i(c)}\right) + 3 = g'(0)e^{-\lambda_i(c)c\tau} \text{ for } i = 1, 2$$

and

$$c\lambda(c) - \left(e^{\lambda(c)} + e^{-\lambda(c)}\right) + 3 > g'(0)e^{-\lambda(c)c\tau} \text{ for } \lambda(c) \in (\lambda_1(c), \lambda_2(c)).$$

The proof is completed. □

We assume that there exists $K^* \geq K$ such that $K^* \geq \max\{g(u)|0 \leq u \leq K^*\}$. Let

$$K_* := \inf\left\{u|u = \inf_{s\in(0,K^*]}\{g(s)|g(s) \leq s\}\right\} > 0.$$

Clearly, K_* is well defined and $g(u) > u$ for all $u \in (0, K_*)$.

By constructing two auxiliary discrete reaction-diffusion equations with monotonicity, Yang et al. [20] proved the existence of noncritical traveling waves (waves with speeds $c > c_*$) when g is a nonmonotone function. That is,

Lemma 2.2 *Assume that* (H1) − (H3) *hold. Let* $c_* > 0$ *be defined as in Lemma 2.1. Then for every* $c > c_*$, (1.1) *admits a traveling wave solution* $v(t,x) = \phi(x + ct)$ *satisfying* $\phi(-\infty) = 0$ *and*

$$0 < K_* \leq \liminf_{\xi\to+\infty} \phi(\xi) \leq \limsup_{\xi\to+\infty} \phi(\xi) \leq K^*.$$

Proof Although the lemma was proved in [20], for the completeness of this paper, here we still give the proof. Since $K_* > 0$, then there eixsts a small $\varepsilon_0 \in (0, K_*)$ such that $K_* - \varepsilon > 0$ for every $\varepsilon \in [0, \varepsilon_0]$.

For any $\varepsilon \in (0, \varepsilon_0)$, define two monotone continuous functions as follows:

$$g^*(u) = \begin{cases} \min\{g'(0)u, K^*\}, & \text{for } u \in [0, K^*], \\ \max\{K^*, g(u)\}, & \text{for } u > K^*, \end{cases}$$

and

$$g_*(u) = \begin{cases} \inf_{\eta \in [u, K^*]}\{g(\eta), K_* - \varepsilon\}, & \text{for } u \in [0, K^*], \\ \min\{g(u), K_* - \varepsilon\}, & \text{for } u > K^*. \end{cases}$$

Consider the following two auxiliary wave profile equations:

$$c\phi'(\xi) - \Delta_1\phi(\xi) + \phi(\xi) = g^*(\phi(\xi - c\tau)) \tag{2.3}$$

and

$$c\phi'(\xi) - \Delta_1\phi(\xi) + \phi(\xi) = g_*(\phi(\xi - c\tau)). \tag{2.4}$$

It is easy to obtain that for each $c \geq c_*$, (2.3) and (2.4) have strictly increasing traveling waves $\phi^*(x+ct)$ and $\phi_*(x+ct)$, respectively, satisfying

$$\phi^*(-\infty) = \phi_*(-\infty) = 0, \quad \phi^*(+\infty) = K^*, \quad \phi_*(+\infty) = K_* - \varepsilon$$

and

$$\lim_{\xi \to -\infty} \phi^*(\xi)e^{-\lambda_1(c)\xi} = \lim_{\xi \to -\infty} \phi_*(\xi)e^{-\lambda_1(c)\xi} = 1. \tag{2.5}$$

Let $a_1 > 0$ be such that $e^{\lambda_1(c)a_1} \geq 3$. Then

$$\lim_{\xi \to -\infty} \phi^*(\xi + a_1)e^{-\lambda_1(c)\xi} = e^{\lambda_1(c)a_1} \geq 3.$$

Therefore, there exists $M_1 > 0$ such that

$$\phi^*(\xi + a_1)e^{-\lambda_1(c)\xi} > 2 > \phi_*(\xi)e^{-\lambda_1(c)\xi} \quad \text{for all } \xi \leq -M_1. \tag{2.6}$$

Since $\phi^*(+\infty) = K^* > K_* - \varepsilon = \phi_*(+\infty)$, we choose $a_2 > 0$ sufficiently large so that

$$\phi^*(x + a_2) > \phi_*(\xi) \quad \text{for all } \xi \geq -M_1. \tag{2.7}$$

Let $a_0 = \max\{a_1, a_2\}$. Since $\phi^*(\cdot)$ is nondecreasing, it follows from (2.6) and (2.7) that

$$\phi^*(\xi + a_0) > \phi_*(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

Define

$$H^*[\phi](\xi) := \Delta_1\phi(\xi) - \phi(\xi) + g^*(\phi(\xi - c\tau)) + c\gamma\phi(\xi), \quad \xi \in \mathbb{R},$$

and

$$H_*[\phi](\xi) := \Delta_1\phi(\xi) - \phi(\xi) + g_*(\phi(\xi - c\tau)) + c\gamma\phi(\xi), \quad \xi \in \mathbb{R},$$

where $\gamma > \frac{3+g'(0)}{c}$. Then for any $\phi, \psi \in C(\mathbb{R}, [0, K^*])$ with $\phi(\xi) \geq \psi(\xi)$, $\xi \in \mathbb{R}$, we have

$$H^*[\phi](\xi) \geq H^*[\psi](\xi) \quad \text{and} \quad H_*[\phi](\xi) \geq H_*[\psi](\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{2.8}$$

For any $\lambda \in (0, \min\{\lambda_1(c), \lambda_2(c)\})$, let

$$X_\lambda = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) \middle| \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-\lambda \xi} < +\infty \right\}, \qquad \|\phi(\xi)\|_\lambda = \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-\lambda \xi}.$$

Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space. Since $\phi_*(\xi) \leq \phi^*(\xi + a_0)$ for all $\xi \in \mathbb{R}$, it is easy to see that the set

$$\Gamma := \left\{ \phi \in C(\mathbb{R}, [0, K^*]) \middle| \begin{array}{l} \text{(i)} \phi_*(\xi) \leq \phi(\xi) \leq \phi^*(\xi + a_0) \quad \text{for all } \xi \in \mathbb{R}; \\ \text{(ii)} |\phi(\xi_1) - \phi(\xi_2)| \leq 2\gamma K^* |\xi_1 - \xi_2| \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R} \end{array} \right\}$$

is nonempty, convex, and compact in X_λ.

Define $F : \Gamma \to C(\mathbb{R}, [0, K^*])$ by

$$F(\phi)(\xi) = \frac{1}{c} \int_\xi^{+\infty} e^{-\gamma(y-\xi)} H[\phi](y) dy,$$

where

$$H[\phi](\xi) = \Delta_1 \phi(\xi) - \phi(\xi) + g(\phi(\xi - c\tau)) + c\gamma\phi(\xi), \quad \xi \in \mathbb{R}. \tag{2.9}$$

Clearly, for any $\phi \in \Gamma \subset C(\mathbb{R}, [0, K^*])$, it follows from (2.8) that

$$0 \leq H_*[\phi](\xi) \leq H[\phi](\xi) \leq H^*[\phi](\xi) \leq -K^* + g^*(K^*) + c\gamma K^* = c\gamma K^* \tag{2.10}$$

for all $\xi \in \mathbb{R}$. Then we further obtain

$$0 \leq F(\phi)(\xi) \leq \frac{c\gamma K^*}{c} \int_\xi^{+\infty} e^{-\gamma(y-\xi)} dy = K^*,$$

and hence, $F : \Gamma \to C(\mathbb{R}, [0, K^*])$ is well defined. Furthermore, it is easily seen that a fixed point of F is a solution of the first equation of (1.2).

For any $\phi, \psi \in \Gamma$, we have

$$\begin{aligned}
&|H[\phi] - H[\psi]| e^{-\lambda\xi} \\
=& |\Delta_1(\phi - \psi) - (\phi - \psi) + (g(\phi)(\xi - c\tau) - g(\psi)(\xi - c\tau)) + c\gamma(\phi - \psi)| e^{-\lambda\xi} \\
=& |(\phi(\xi+1) - \psi(\xi+1)) + (\phi(\xi-1) - \psi(\xi-1)) + (c\gamma - 3)(\phi(\xi) - \psi(\xi)) \\
&+ (g(\phi)(\xi - c\tau) - g(\psi)(\xi - c\tau))| e^{-\lambda\xi} \\
\leq& |(\phi(\xi+1) - \psi(\xi+1))| e^{-\lambda\xi} + |(\phi(\xi-1) - \psi(\xi-1))| e^{-\lambda\xi} \\
&+ (c\gamma - 3)|\phi(\xi) - \psi(\xi)| e^{-\lambda\xi} + g'(0)|\phi(\xi - c\tau) - \psi(\xi - c\tau)| e^{-\lambda\xi} \\
\leq& [(e^\lambda + e^{-\lambda} - 2) - 1 + g'(0) e^{-\lambda c\tau} + c\gamma] \sup_{\xi \in \mathbb{R}} |\phi(\xi) - \psi(\xi)| e^{-\lambda\xi} \\
\leq& L \|\phi - \psi\|_\lambda,
\end{aligned}$$

where $L := c(\lambda + \gamma)$. Therefore, we have

$$
\begin{aligned}
&|F(\phi)(\xi) - F(\psi)(\xi)|e^{-\lambda\xi}\\
=&\left|\frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}(H[\phi](y) - H[\psi](y))dy\right| e^{-\lambda\xi},\\
\leq&\frac{e^{-\lambda\xi}}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}|H[\phi](y) - H[\psi](y)|\,dy\\
\leq&\frac{L}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}\,dy||\phi-\psi||_{\lambda}\\
=&\frac{L}{c\gamma}||\phi-\psi||_{\lambda},
\end{aligned}
$$

which implies that $F : \Gamma \to C(\mathbb{R}, [0, K^*])$ is continuous.

Next, we shall show that $F(\Gamma) \subseteq \Gamma$. Since $\phi_*(\xi)$ is the solution of (2.4), we have

$$
\phi_*(\xi) = \frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}H_*[\phi_*](y)dy. \tag{2.11}
$$

For any $\phi \in \Gamma$, we have $0 \leq \phi_*(\xi) \leq \phi(\xi) \leq \phi^*(\xi + a_0) \leq K^*$ for all $\xi \in \mathbb{R}$. Therefore, it follows from (2.8)–(2.11) that

$$
\begin{aligned}
F(\phi)(\xi) &= \frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}H[\phi](y)\,dy\\
&\geq \frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}H_*[\phi](y)\,dy\\
&\geq \frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)}H_*[\phi_*](y)\,dy\\
&= \phi_*(\xi).
\end{aligned}
$$

Since $\phi^*(\xi + a_0)$ is a solution of (2.3), by using a similar argument, we can show that $F(\phi)(\xi) \leq \phi^*(\xi + a_0)$ for all $\xi \in \mathbb{R}$. For any $\phi \in \Gamma$ and $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 < \xi_2$, it follows from (2.10) that

$$
\begin{aligned}
&|F(\phi)(\xi_1) - F(\phi)(\xi_2)|\\
=&\frac{1}{c}\left|\int_{\xi_1}^{+\infty} e^{-\gamma(y-\xi_1)}H[\phi](y)\,dy - \int_{\xi_2}^{+\infty} e^{-\gamma(y-\xi_2)}H[\phi](y)\,dy\right|\\
\leq&\frac{1}{c}\left\{e^{\gamma\xi_1}\left|\int_{\xi_1}^{\xi_2} e^{-\gamma y}H[\phi](y)\,dy\right| + (e^{\gamma\xi_2} - e^{\gamma\xi_1})\int_{\xi_2}^{+\infty} e^{-\gamma y}H[\phi](y)\,dy\right\}\\
\leq&\frac{1}{c}\sup_{\xi\in\mathbb{R}} H[\phi](\xi)\left\{e^{\gamma\xi_1}\left|\int_{\xi_1}^{\xi_2} e^{-\gamma y}\,dy\right| + e^{\gamma\xi_2}(1 - e^{\gamma(\xi_1-\xi_2)})\int_{\xi_2}^{+\infty} e^{-\gamma y}\,dy\right\}\\
\leq&\frac{2}{c\gamma}(c\gamma K^*)\left|e^{-\gamma(\xi_2-\xi_1)} - 1\right|\\
\leq&2\gamma K^*|\xi_1 - \xi_2|.
\end{aligned}
$$

Therefore, we conclude that $F(\phi) \in \Gamma$ for all $\phi \in \Gamma$. By virtue of Schauder's fixed point theorem, it follows that F has a fixed point ϕ in $\Gamma \subset X_\lambda$, which satisfies

$$\phi(\xi) = \frac{1}{c}\int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H[\phi](y)dy$$

and

$$\phi_*(\xi) \le \phi(\xi) \le \phi^*(\xi + a_0) \quad \text{for all } \xi \in \mathbb{R}. \tag{2.12}$$

Taking the limit $\xi \to -\infty$ and $\xi \to +\infty$ in (2.12), respectively, we get $\phi(-\infty) = 0$ and

$$K_* - \varepsilon \le \liminf_{\xi\to+\infty} \phi(\xi) \le \limsup_{\xi\to+\infty} \phi(\xi) \le K^*.$$

Since $\phi(\xi)$ is independent of ε, taking the limit as $\varepsilon \to 0^+$ in the last inequality, we get

$$K_* \le \liminf_{\xi\to+\infty} \phi(\xi) \le \limsup_{\xi\to+\infty} \phi(\xi) \le K^*.$$

The proof is completed. □

Now we further show that the traveling wave with critical speed also exists.

Theorem 2.3 *Assume that* (H1) – (H3) *hold. Then* (1.1) *admits a traveling wave solution* $v(t,x) = \phi(x + c_* t)$ *such that*

$$0 < K_* \le \liminf_{\xi\to+\infty} \phi(\xi) \le \limsup_{\xi\to+\infty} \phi(\xi) \le K^*.$$

Proof Choose a sequence $\{c_i\}_{i\ge1} \subset (c_*, +\infty)$ such that $\lim_{i\to+\infty} c_i = c_*$. By Lemma 2.2, it follows that (1.1) has a traveling wave $v(t,x) = \phi_i(x + c_i t)$ such that $\phi_i(-\infty) = 0$ and

$$0 < K_* \le \liminf_{\xi\to+\infty} \phi_i(\xi) \le \limsup_{\xi\to+\infty} \phi_i(\xi) \le K^*.$$

Without loss of generality, we assume that $\phi_i(0) = \frac{1}{2}K_* > 0,\ \forall i \ge 1$. It is easy to see that ϕ satisfies (1.2) if and only if ϕ satisfies

$$\phi(\xi) = \frac{1}{c}\int_0^{\infty} e^{-\gamma x} H[\phi](\xi + x)dx,$$

where $H[\phi](\xi)$ is defined in (2.9). Note that

$$\phi_i(\xi) = \frac{1}{c}\int_0^{\infty} e^{-\epsilon x} H[\phi_i](\xi + x)dx, \quad \forall \xi \in \mathbb{R}, i \ge 1. \tag{2.13}$$

Note that $0 < H(\phi)(\xi) \le c\gamma K^*$ for $\xi \in \mathbb{R}$. It follows that for any $\xi_1, \xi_2 \in \mathbb{R}$, and $i \ge 1$,

$$\begin{aligned}
&|\phi_i(\xi_1) - \phi_i(\xi_2)| \\
&= \left| \frac{1}{c} \int_0^\infty e^{-\gamma x} H[\phi_i](\xi_1 + x)dx - \frac{1}{c} \int_0^\infty e^{-\gamma x} H[\phi_i](\xi_2 + x)dx \right| \\
&= \left| \frac{1}{c} \int_{\xi_1}^\infty e^{-\gamma(x-\xi_1)} H[\phi_i](x)dx - \frac{1}{c} \int_{\xi_2}^\infty e^{-\gamma(x-\xi_2)} H[\phi_i](x)dx \right| \\
&= \frac{1}{c} \left| \int_{\xi_1}^{\xi_2} e^{-\gamma(x-\xi_1)} H[\phi_i](x)dx + \int_{\xi_2}^\infty e^{-\gamma(x-\xi_1)} H[\phi_i](x)dx - \int_{\xi_2}^\infty e^{-\gamma(x-\xi_2)} H[\phi_i](x)dx \right| \\
&= \frac{1}{c} \left| \int_{\xi_1}^{\xi_2} e^{-\gamma(x-\xi_1)} H[\phi_i](x)dx + \int_{\xi_2}^\infty \left(e^{-\gamma(x-\xi_1)} - e^{-\gamma(x-\xi_2)} \right) H[\phi_i](x)dx \right| \\
&\le \frac{1}{c} \left\{ \left| \int_{\xi_1}^{\xi_2} e^{-\gamma(x-\xi_1)} H[\phi_i](x)dx \right| + \int_{\xi_2}^\infty \left| e^{-\gamma(x-\xi_1)} - e^{-\gamma(x-\xi_2)} \right| H[\phi_i](x)dx \right\} \\
&\le \frac{1}{c} \left\{ \frac{1}{\gamma} \left| e^{-\gamma(\xi_2-\xi_1)} - 1 \right| + \left| e^{\gamma\xi_1} - e^{\gamma\xi_2} \right| \int_{\xi_2}^\infty e^{-\gamma x}dx \right\} \sup_{x\in\mathbb{R}} H[\phi_i](x) \\
&\le 2K^* \left| e^{-\gamma(\xi_2-\xi_1)} - 1 \right|.
\end{aligned}$$

Hence, the family of function $\{\phi_i : i \ge 1\}$ is uniformly bounded and equicontinuous in $\xi \in \mathbb{R}$. Thus, there exists $i_k \to +\infty$ and $\phi \in C(\mathbb{R}, \mathbb{R})$ such that $\lim_{k\to+\infty} \phi_{i_k}(\xi) = \phi(\xi)$ uniformly for ξ in any compact subset of $\mathbb{R}$. Clearly, $\phi_i(0) = \frac{1}{2}K_* > 0$. Let $i = i_k \to +\infty$ in (2.13). Then using the dominated convergence theorem, we obtain

$$\phi(\xi) = \frac{1}{c_*} \int_0^\infty e^{-\gamma x} H[\phi](\xi + x)dx, \quad \forall \xi \in \mathbb{R},$$

and hence $\phi(\xi + c_* t)$ is a traveling wave of (1.1). The proof is completed. □

Theorem 2.4 *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold. For any* $c \ge c_*$*, if equation* (1.2) *has no other solution* $\mathcal{W}$ *with* $0 < K_* \le \mathcal{W} \le K^*$ *and* $\mathcal{W} \not\equiv K$*, then* $\lim_{\xi\to+\infty} \phi(\xi) = K$.

Proof For any $c \ge c_*$, let ϕ be a traveling wave of (1.2) satisfying

$$0 < K_* \le \liminf_{\xi\to+\infty} \phi(\xi) \le \limsup_{\xi\to+\infty} \phi(\xi) \le K^*.$$

Choose a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n > 0$ and $\lim_{n\to\infty} \alpha_n = \infty$. Let $W_n = \phi(\xi + \alpha_n)$. By the translation invariance of the solutions of (1.2), it follows that

$$cW_n'(\xi) = \Delta_1 W_n(\xi) - W_n(\xi) + g(W_n(\xi - c\tau)). \tag{2.14}$$

Since $0 < K_* \le \liminf_{\xi\to+\infty} \phi(\xi) \le \limsup_{\xi\to+\infty} \phi(\xi) \le K^*$, W_n is uniformly bounded by K^*. From equation (2.14), we see that there exists a $M_0 > 0$ such that $|W_n'(\xi)| \le M_0, \forall n \in \mathbb{N}$. Thus, W_n is uniformly bounded and equicontinuous. We now prove that $W_n'(\xi)$ is equicontinuous. For any $\xi_1,\xi_2\in\mathbb{R}$, $n\in\mathbb{N}$, we have

$$\begin{aligned}
&|W_n'(\xi_1)-W_n'(\xi_2)|\\
=&c^{-1}|\Delta_1 W_n(\xi_1)-\Delta_1 W_n(\xi_2)-(W_n(\xi_1)-W_n(\xi_2))\\
&+g(W_n(\xi_1-c\tau))-g(W_n(\xi_2-c\tau))|\\
=&c^{-1}|\Delta_1(W_n(\xi_1)-W_n(\xi_2))-(W_n(\xi_1)-W_n(\xi_2))\\
&+g'(\phi)(W_n(\xi_1-c\tau)-W_n(\xi_2-c\tau))|\\
\le&c^{-1}\big[|W_n(\xi_1+1)-W_n(\xi_2+1)|+|W_n(\xi_1-1)-W_n(\xi_2-1)|\\
&+3|W_n(\xi_1)-W_n(\xi_2)|+g'(0)|W_n(\xi_1-c\tau)-W_n(\xi_2-c\tau)|\big].
\end{aligned}$$

Thus, $W_n'(\xi)$ is equicontinuous since $W_n(\xi)$ is equicontinuous. Then by the Arzelà–Ascoli theorem, we can see that there exist subsequences of $W_n(\xi)$ that converge pointwise to $\tilde{W}(\xi)$ as $n\to\infty$ in $\mathbb{R}$, which satisfies

$$c\tilde{W}'(\xi)=\Delta_1\tilde{W}(\xi)-\tilde{W}(\xi)+g(\tilde{W}(\xi-c\tau)).$$

Since $0<K_*\le\liminf_{\xi\to+\infty}\phi(\xi)\le\limsup_{\xi\to+\infty}\phi(\xi)\le K^*$, it follows that $0<K_*\le\liminf_{\xi\to+\infty}\tilde{W}(\xi)\le\limsup_{\xi\to+\infty}\tilde{W}\xi)\le K^*$. By the hypothesis, $\tilde{W}\equiv K$. Thus, we obtain that $\lim_{n\to\infty}W_n(\xi)=\tilde{W}(\xi)=K$ for any $\xi\in\mathbb{R}$. Hence, $\lim_{\xi\to+\infty}\phi(\xi)=K$. The proof is completed. □

3. Stability of critical traveling waves

In order to prove the stability of critical traveling waves, we reformulate (1.1) to a perturbed equation around the critical wave.

Let $\phi(x+c_*t)=\phi(\xi)$, $\xi=x+c_*t$ be a given traveling wave, and define

$$u(t,\xi)=v(t,\xi-c_*t)-\phi(\xi),\quad u_0(s,\xi)=v_0(s,\xi)-\phi(x+c_*s).$$

Then it is easy to see that $u(t,\xi)$ satisfies

$$\begin{cases}\frac{\partial u}{\partial t}+c_*\frac{\partial u}{\partial\xi}-\Delta_1 u(t,\xi)+u(t,\xi)\\ \qquad\qquad =P(u(t-\tau,\xi-c_*\tau)),\quad (t,\xi)\in\mathbb{R}^+\times\mathbb{R},\\ u(s,\xi)=u_0(s,\xi),\quad s\in[-\tau,0],\quad \xi\in\mathbb{R},\end{cases}\tag{3.1}$$

where

$$P(u(t-\tau,\xi-c_*\tau)):=g(\phi+u)-g(\phi),\tag{3.2}$$

with $u=u(t-\tau,\xi-c_*\tau)$ and $\phi=\phi(\xi-c_*\tau)$. Furthermore, by linearizing the delay term, we obtain

$$\begin{cases}\frac{\partial u}{\partial t}+c_*\frac{\partial u}{\partial\xi}-\Delta_1 u(t,\xi)+u(t,\xi)-g'(\phi(\xi-c_*\tau))u(t-\tau,\xi-c_*\tau)\\ \qquad\qquad =Q(u(t-\tau,\xi-c_*\tau)),\quad (t,\xi)\in\mathbb{R}^+\times\mathbb{R},\\ u(s,\xi)=u_0(s,\xi),\quad s\in[-\tau,0],\xi\in\mathbb{R},\end{cases}\tag{3.3}$$

where

$$Q(u(t-\tau,\xi-c_*\tau)) := g(\phi+u) - g(\phi) - g'(\phi)u, \tag{3.4}$$

and satisfies, by Taylor's formula,

$$Q(u) = O(1)|u|^2. \tag{3.5}$$

Before stating our stability results, we introduce some notations. Throughout this paper, $C > 0$ always denotes a generic constant, and $C_i > 0$ $(i = 1, 2, \cdots)$ represents a specific positive constant. Let $L^2(\mathbb{R})$ denote the space of the square integrable functions, $H^k(\mathbb{R})$ the Sobolev space, and $C(\mathbb{R})$ the space of the bounded continuous functions equipped with the sup norm. Let $T > 0$ be a number and $\mathfrak{B}$ be a Banach space.

We define a weight function

$$w(\xi) := e^{-2\lambda_*\xi}, \quad \xi \in \mathbb{R}. \tag{3.6}$$

Since $\lambda_* > 0$, we can see that $\lim_{\xi\to-\infty} w(\xi) = +\infty, \lim_{\xi\to+\infty} w(\xi) = 0$. Define

$$\begin{aligned} C_{unif}[-\tau,T] :=&\{v(t,x) \in C([-\tau,T]\times\mathbb{R}) \text{ such that} \\ &\lim_{x\to+\infty} v(t,x) \quad \text{exists uniformly in } t\in[-\tau,T]\ \}. \end{aligned} \tag{3.7}$$

Denote

$$\begin{aligned} X_0(-\tau,0) :=&\{u(t,\xi) \in C([-\tau,0]\times\mathbb{R}) \cap C_{unif}[-\tau,0], \\ &\sqrt{w}u \in C([-\tau,0];H^1(\mathbb{R})), \quad \sqrt{w}u \in L^2([-\tau,0];H^1(\mathbb{R}))\}, \end{aligned} \tag{3.8}$$

with

$$\mathcal{N}_0^2 := \sup_{t\in[-\tau,0]} \left(||u(t)||^2_{C(\mathbb{R})} + ||\sqrt{w}u(t)||^2_{H^1(\mathbb{R})} \right) + \int_{-\tau}^0 ||(\sqrt{w}u)(s)||^2_{H^1(\mathbb{R})} ds, \tag{3.9}$$

and

$$\begin{aligned} X_{loc}(0,\infty) :=&\{u(t,\xi) \in C_{loc}([0,\infty)\times\mathbb{R}) \cap C_{unif}[0,\infty), \\ &\sqrt{w}u \in C_{loc}([0,\infty);H^1(\mathbb{R})), \quad \sqrt{w}u \in L^2_{loc}([0,\infty);H^1(\mathbb{R}))\}. \end{aligned} \tag{3.10}$$

We further define

$$\begin{aligned} X(0,\infty) :=&\{u(t,\xi) \in C([0,\infty)\times\mathbb{R}) \cap C_{unif}[0,\infty), \\ &\sqrt{w}u \in C([0,\infty);H^1(\mathbb{R})), \\ &\sqrt{\phi w}u \in L^2([0,\infty);L^2(\mathbb{R}))\}, \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} \mathcal{N}_\infty^2 :=& \sup_{t\in[0,\infty)} \left(||u(t)||^2_{C(\mathbb{R})} + ||\sqrt{w}u(t)||^2_{H^1(\mathbb{R})} \right) \\ &+ \int_0^\infty ||\sqrt{\phi w}u(s)||^2_{L^2(\mathbb{R})} ds + \int_0^\infty ||\partial_\xi(\sqrt{w}u)(s)||^2_{L^2(\mathbb{R})} ds. \end{aligned} \tag{3.12}$$

Now we state the global existence, uniqueness, uniform boundedness, and stability for the solution to (1.1).

Theorem 3.1 (Global existence and uniqueness) *Assume that* (H1) − (H3) *hold. Let* $\phi(x + c_*t) = \phi(\xi)$ *be any given critical traveling wave and the initial perturbation* $u_0(s, \xi) := v_0(s, \xi) - \phi(\xi) \in X_0(-\tau, 0)$ *be arbitrary; then the solution* $u(t, \xi)$ *of the perturbed equation* (3.3) *globally and uniquely exists in* $X_{loc}(0, \infty)$.

Theorem 3.2 (Uniform boundedness) *Under the condition of Theorem 3.1, if the initial perturbation* $u_0 \in X_0(-\tau, 0)$ *is small enough, namely there exists a constant* $\delta_0 > 0$ *such that* $\mathcal{N}_0 \le \delta_0$, *then the solution* $u(t, \xi)$ *of the perturbed equation* (3.3) *satisfies* $u \in X(0, \infty)$, *and* $u(t, \xi)$ *is uniformly bounded in* $X(0, \infty)$,

$$\mathcal{N}_\infty^2 \le C\mathcal{N}_0^2. \tag{3.13}$$

Theorem 3.3 (Stability) *Under the condition of Theorem 3.2, it holds that*

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |u(t, \xi)| = 0. \tag{3.14}$$

3.1. Global existence and uniqueness

In this subsection, we shall prove Theorem 3.1, namely the global existence and uniqueness of the solution for the Cauchy problem (3.1).

Proof [Proof of Theorem 3.1] In order to establish the energy estimate, sufficient regularity of the solution to (3.1) is required. We thus mollify the initial data as

$$u_{0\varepsilon}(s, \xi) = (J_\varepsilon * u_0)(s, \xi) \in C\left([-\tau, 0]; H_w^2(\mathbb{R}) \cap H^2(\mathbb{R})\right),$$

where $J_\varepsilon(\xi)$ is the mollifier. Let $u_\varepsilon(t, \xi)$ be the solution to (3.1) with the initial data $u_{0\varepsilon}(s, \xi)$. We consider this mollification, with solution

$$u_\varepsilon(t, \xi) \in C\left([0, \infty); H_w^2(\mathbb{R}) \cap H^2(\mathbb{R})\right), \tag{3.15}$$

and then take the limit $\varepsilon \to 0$ to obtain the corresponding energy estimate for the original solution $u(t, \xi)$. For the sake of simplicity, we use $u(t, \xi)$ to establish the desired energy estimates.

We first consider $t \in [0, \tau]$. If $u_0 \in X_0(-\tau, 0)$, then we shall prove $u \in X_{loc}(0, \tau)$. Multiplying the equation (3.1) by $w(\xi)u(t, \xi)$ yields

$$\begin{aligned} &w(\xi)u(t,\xi)\frac{\partial u}{\partial t} + w(\xi)u(t,\xi)c_*\frac{\partial u}{\partial \xi} - w(\xi)u(t,\xi)\Delta_1 u(t,\xi) + w(\xi)u^2(t,\xi) \\ &= w(\xi)u(t,\xi)P(u_0(t-\tau, \xi - c_*\tau)). \end{aligned}$$

Integrating the equation above both sides with respect to ξ over $\mathbb{R}$, and noting that

$$\left\{\frac{c_*}{2}wu^2\right\}\Big|_{\xi=-\infty}^{\infty} = 0,$$

due to (3.15), we obtain

$$
\begin{aligned}
&\frac{d}{dt}||\sqrt{w}u||_{L^2}^2 - c_* \int_{\mathbb{R}} \frac{w'}{w} w(\xi)u^2(t,\xi)d\xi - 2\int_{\mathbb{R}} w(\xi)u(t,\xi)\Delta_1 u(t,\xi)d\xi \\
&+ 2\int_{\mathbb{R}} w(\xi)u^2(t,\xi)d\xi \\
=&\frac{d}{dt}||\sqrt{w}u||_{L^2}^2 - c_* \int_{\mathbb{R}} \frac{w'}{w} w(\xi)u^2(t,\xi)d\xi - 2\int_{\mathbb{R}} w(\xi)u(t,\xi)u(t,\xi+1)d\xi \\
&- 2\int_{\mathbb{R}} w(\xi)u(t,\xi)u(t,\xi-1)d\xi + 6\int_{\mathbb{R}} w(\xi)u^2(t,\xi)d\xi \\
=&2\int_{\mathbb{R}} w(\xi)u(t,\xi)P(u_0(t-\tau,\xi-c_*\tau))d\xi.
\end{aligned}
\tag{3.16}
$$

Using Young's inequality:

$$
2ab \le \eta a^2 + \frac{1}{\eta}b^2, \quad \text{for any } \eta > 0,
\tag{3.17}
$$

and choosing $\eta = e^{\lambda_*}$, we can obtain

$$
\begin{aligned}
&\left|2\int_{\mathbb{R}} w(\xi)u(t,\xi)u(t,\xi+1)d\xi\right| \\
\le&\int_{\mathbb{R}} w(\xi)\left[\eta u^2(t,\xi) + \frac{1}{\eta}u^2(t,\xi+1)\right]d\xi \\
=&\int_{\mathbb{R}} \eta w(\xi)u^2(t,\xi)d\xi + \int_{\mathbb{R}} \frac{1}{\eta}\frac{w(\xi-1)}{w(\xi)} w(\xi)u^2(t,\xi)d\xi \\
=&2e^{\lambda_*}\int_{\mathbb{R}} w(\xi)u^2(t,\xi)d\xi.
\end{aligned}
$$

Similarly, choosing $\eta = e^{-\lambda_*}$, we get

$$
\begin{aligned}
&\left|2\int_{\mathbb{R}} w(\xi)u(t,\xi)u(t,\xi-1)d\xi\right| \\
\le&\int_{\mathbb{R}} w(\xi)\left[\eta u^2(t,\xi) + \frac{1}{\eta}u^2(t,\xi-1)\right]d\xi \\
=&\int_{\mathbb{R}} \eta w(\xi)u^2(t,\xi)d\xi + \int_{\mathbb{R}} \frac{1}{\eta}\frac{w(\xi+1)}{w(\xi)} w(\xi)u^2(t,\xi)d\xi \\
=&2e^{-\lambda_*}\int_{\mathbb{R}} w(\xi)u^2(t,\xi)d\xi.
\end{aligned}
$$

Substituting the two inequalities above into (3.16), one has

$$
\begin{aligned}
&\frac{d}{dt}||\sqrt{w}u||_{L^2}^2 + \mathcal{A}||\sqrt{w}u||_{L^2}^2 \\
\le&\ 2\int_{\mathbb{R}} P(u_0(t-\tau,\xi-c_*\tau))w(\xi)u(t,\xi)d\xi,
\end{aligned}
\tag{3.18}
$$

where

$$\mathcal{A} = 2c_*\lambda_* - 2e^{\lambda_*} - 2e^{-\lambda_*} + 6.$$

From (2.2) we know that $\mathcal{A} = 2g'(0)e^{-\lambda_* c_* \tau} > 0$. Taking Young's inequality (3.17) with $\eta = 1$, we can see that

$$\begin{aligned} &2\left|\int_{\mathbb{R}} P(u_0(t-\tau, \xi - c_*\tau))w(\xi)u(t,\xi)d\xi\right| \\ \leq &2C\int_{\mathbb{R}} |u_0(t-\tau,\xi-c_*\tau)||u(t,\xi)|w(\xi)d\xi \\ \leq &\varepsilon||\sqrt{w}u||^2_{L^2} + \frac{C^2}{\varepsilon}||\sqrt{w}u_0(t-\tau,\xi-c_*\tau)||^2_{L^2}, \end{aligned} \tag{3.19}$$

for some small constant $\varepsilon > 0$. Substituting (3.19) into (3.18), one gets

$$\frac{d}{dt}||\sqrt{w}u||^2_{L^2} + (\mathcal{A}-\varepsilon)||\sqrt{w}u||^2_{L^2} \leq \frac{C^2}{\varepsilon}||\sqrt{w}u_0(t-\tau,\xi-c_*\tau)||^2_{L^2}. \tag{3.20}$$

Integrating (3.20) over $[0,t]$ for $t \in [0,\tau]$ and taking ε small enough to satisfy $\varepsilon < \mathcal{A}$, we obtain

$$\begin{aligned} &||\sqrt{w}u(t)||^2_{L^2} + C_2\int_0^t ||\sqrt{w}u(s)||^2_{L^2}ds \\ \leq &||\sqrt{w}u_0(0)||^2_{L^2} + C_3\int_0^t ||\sqrt{w}u_0(s-\tau,\xi-c_*\tau)||^2_{L^2}ds \\ \leq &C_3||\sqrt{w}u_0(0)||^2_{L^2} + C_3\int_{-\tau}^0 ||\sqrt{w}u_0(s)||^2_{L^2}ds \\ < &\infty \end{aligned} \tag{3.21}$$

for $t \in [0,\tau]$.

From (3.1) we know that

$$\frac{\partial u}{\partial t} + c_*\frac{\partial u}{\partial \xi} - \Delta_1 u(t,\xi) + u(t,\xi) = P(u_0(t-\tau,\xi-c_*\tau)) \leq C|u_0(t-\tau,\xi-c_*\tau)|, t \in [0,\tau].$$

Differentiating equation (3.1) with respect to ξ, and then multiplying it by $w(\xi)u_\xi(t,\xi)$, we have

$$wu_\xi(u_t)_\xi + c_*u_\xi u_{\xi\xi}w - \Delta_1 u_\xi u_\xi w + wu_\xi^2 \leq Cw(\xi)u_\xi(t,\xi)u_{0,\xi}(t-\tau,\xi-c_*\tau).$$

Then integrating the inequality above with respect to ξ over $\mathbb{R}$, and noting that

$$\left\{\frac{c_*}{2}wu_\xi^2\right\}\Bigg|_{\xi=-\infty}^{\xi=+\infty} = 0,$$

since $u \in H_w^2$, we can obtain

$$\begin{aligned}
&\frac{d}{dt}||\sqrt{w}u_\xi||_{L^2}^2 - c_* \int_{\mathbb{R}} \frac{w'}{w} w(\xi) u_\xi^2(t,\xi) d\xi - 2\int_{\mathbb{R}} w(\xi) u_\xi(t,\xi) \Delta_1 u_\xi(t,\xi) d\xi \\
&+ 2\int_{\mathbb{R}} w(\xi) u_\xi^2(t,\xi) d\xi \\
= \quad &\frac{d}{dt}||\sqrt{w}u_\xi||_{L^2}^2 - c_* \int_{\mathbb{R}} \frac{w'}{w} w(\xi) u_\xi^2(t,\xi) d\xi - 2\int_{\mathbb{R}} w(\xi) u_\xi(t,\xi) u_\xi(t,\xi+1) d\xi \\
&- 2\int_{\mathbb{R}} w(\xi) u_\xi(t,\xi) u_\xi(t,\xi-1) d\xi + 6\int_{\mathbb{R}} w(\xi) u_\xi^2(t,\xi) d\xi \\
\leq &2\int_{\mathbb{R}} Cw(\xi)|u_\xi(t,\xi)||u_{0,\xi}(t-\tau,\xi-c_*\tau)| d\xi. \qquad (3.22)
\end{aligned}$$

Using Young's inequality, and choosing $\eta = e^{\lambda_*}$, we can obtain

$$\begin{aligned}
&\left| 2\int_{\mathbb{R}} w(\xi) u_\xi(t,\xi) u_\xi(t,\xi+1) d\xi \right| \\
\leq &\int_{\mathbb{R}} \eta w(\xi) u_\xi^2(t,\xi) d\xi + \int_{\mathbb{R}} \frac{1}{\eta} w(\xi) u_\xi^2(t,\xi+1) d\xi \\
= &\int_{\mathbb{R}} \eta w(\xi) u_\xi^2(t,\xi) d\xi + \int_{\mathbb{R}} \frac{1}{\eta} \frac{w(\xi-1)}{w(\xi)} w(\xi) u_\xi^2(t,\xi) d\xi \\
= &2e^{\lambda_*} \int_{\mathbb{R}} w(\xi) u_\xi^2(t,\xi) d\xi.
\end{aligned}$$

Similarly, choosing $\eta = e^{-\lambda_*}$, we can obtain

$$\begin{aligned}
&\left| 2\int_{\mathbb{R}} w(\xi) u_\xi(t,\xi) u_\xi(t,\xi-1) d\xi \right| \\
\leq &\int_{\mathbb{R}} \eta w(\xi) u_\xi^2(t,\xi) d\xi + \int_{\mathbb{R}} \frac{1}{\eta} w(\xi) u_\xi^2(t,\xi-1) d\xi \\
= &\int_{\mathbb{R}} \eta w(\xi) u_\xi^2(t,\xi) d\xi + \int_{\mathbb{R}} \frac{1}{\eta} \frac{w(\xi+1)}{w(\xi)} w(\xi) u_\xi^2(t,\xi) d\xi \\
= &2e^{-\lambda_*} \int_{\mathbb{R}} w(\xi) u_\xi^2(t,\xi) d\xi.
\end{aligned}$$

Substituting both inequalities above into (3.22), we have

$$\begin{aligned}
&\frac{d}{dt}||\sqrt{w}u_\xi||_{L^2}^2 + \mathcal{A}||\sqrt{w}u_\xi||_{L^2}^2 \\
\leq &2\int_{\mathbb{R}} Cu_{0,\xi}(t-\tau,\xi-c_*\tau) w(\xi) u_\xi(t,\xi) d\xi \\
\leq &\varepsilon||\sqrt{w}u_\xi||_{L^2}^2 + \frac{C^2}{\varepsilon}||\sqrt{w}u_{0,\xi}(t-\tau,\xi-c_*\tau)||_{L^2}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}||\sqrt{w}u_\xi||^2_{L^2}+(\mathcal{A}-\varepsilon)\,||\sqrt{w}u_\xi||^2_{L^2}\\ &\le\frac{C^2}{\varepsilon}||\sqrt{w}u_{0,\xi}(t-\tau,\xi-c_*\tau)||^2_{L^2}.\end{aligned}\tag{3.23}$$

Integrating (3.23) over $[0,t]$, we have

$$\begin{aligned}&||\sqrt{w}u_\xi||^2_{L^2}+(\mathcal{A}-\varepsilon)\int_0^t||\sqrt{w}u_\xi(s)||^2_{L^2}ds\\ &\le||\sqrt{w}u_{0,\xi}(0)||^2_{L^2}+\frac{C^2}{\varepsilon}\int_{-\tau}^0||\sqrt{w}u_{0,\xi}(s)||^2_{L^2}ds\\ &<\infty.\end{aligned}\tag{3.24}$$

Now we prove that $\|u(t)\|_C<\infty$. By (3.1), one has

$$\frac{\partial u}{\partial t}+c_*\frac{\partial u}{\partial \xi}+3u(t,\xi)=P(u(t-\tau,\xi-c_*\tau))+u(t,\xi+1)+u(t,\xi-1).\tag{3.25}$$

The solution of equation (3.25) can be explicitly and uniquely solved by, for $t\in[0,\tau]$,

$$\begin{aligned}u(t,\xi)=&e^{-3t}u_0(0,\xi-c_*t)+e^{-3t}\int_0^t e^{3s}[P(u(s-\tau,\xi+c_*[(s-t)-\tau]))\\ &+u(s,\xi+1+c_*(s-t))+u(s,\xi-1+c_*(s-t))]ds.\end{aligned}$$

Then, since $|P(u)|\le C|u|$ from (3.2), we can obtain

$$\begin{aligned}||u(t)||_C\le\;&e^{-3t}||u_0(0)||_C+2e^{-3t}\int_0^t e^{3s}||u(s)||_Cds\\ &+e^{-3t}\int_0^t e^{3s}P(u_0(s-\tau,\xi+c_*[(s-t)-\tau]))ds\\ &\le||u_0(0)||_C+2\int_0^t e^{-3(t-s)}||u(s)||_Cds+C\int_0^t e^{-3(t-s)}||u_0(s-\tau)||_Cds\\ &\le||u_0(0)||_C+2\int_0^t ||u(s)||_Cds+C\int_0^t ||u_0(s-\tau)||_Cds,\quad t\in[0,\tau].\end{aligned}$$

It then follows that

$$||u(t)||_C\le\left(||u_0(0)||_C+C\tau\sup_{t\in[-\tau,0]}||u_0(t)||_C\right)+2\int_0^t||u(s)||_Cds.\tag{3.26}$$

Applying Gronwall's inequality to (3.26), we get

$$||u(t)||_C\le\left(||u_0(0)||_C+C\tau\sup_{t\in[-\tau,0]}||u_0(t)||_C\right)e^{2\tau}\quad t\in[0,\tau].\tag{3.27}$$

Note that $u_0 \in C_{unif}(-\tau,0)$, namely $\lim_{\xi\to\infty} u_0(t,\xi) =: u_{0,\infty}(t) \in C[-\tau,0]$ exists uniformly in t, and we are going to prove $u \in C_{unif}[0,\tau]$. We rewrite the solution of (3.1) as

$$\begin{aligned}u(t,\xi) =& e^{-t}u_0(0,\xi-c_*t) + e^{-t}\int_0^t e^s[u(s,\xi+1+c_*(s-t)) \\ & - 2u(s,\xi+c_*(s-t)) + u(s,\xi-1+c_*(s-t)) \\ & + P(u(s-\tau,\xi+c_*(s-t-\tau)))]\, ds.\end{aligned}$$

Then one has

$$\begin{aligned}\lim_{\xi\to+\infty} u(t,\xi) =& e^{-t}u_0(0,\infty) + e^{-t}\int_0^t e^s \lim_{\xi\to+\infty} P(u_0(s-\tau,\xi+c_*[(s-t)-\tau]))ds \\ =& e^{-t}u_{0,\infty}(0) + \int_0^t e^{-(t-s)}P(u_{0,\infty}(s-\tau))ds \\ =& \mathcal{Y}(t), \quad \text{uniformly with respect to } t\in[0,\tau].\end{aligned} \tag{3.28}$$

Therefore, (3.21), (3.24), (3.27), and (3.28) imply $u \in X_{loc}(0,\tau)$ and

$$\begin{aligned}& ||u(t)||_C^2 + ||\sqrt{w}u(t)||_{H^1}^2 + \int_0^t ||\sqrt{w}u(s)||_{H^1}^2 ds \\ \le & C\left(||u(0)||_C^2 + ||\sqrt{w}u_0(0)||_{H^1}^2 + \int_{-\tau}^0 ||\sqrt{w}u_0(s)||_{H^1}^2 ds\right), \quad t\in[0,\tau],\end{aligned}$$

for some $C > 0$.

When $t \in [\tau, 2\tau]$, the solution of $u(t,\xi)$ for $t \in [\tau,2\tau]$ is uniquely and explicitly given by

$$\begin{aligned}u(t,\xi) =& e^{-3t}u_0(\tau,\xi-c_*t) + e^{-3t}\int_\tau^t e^{3s}[P(u(s-\tau,\xi+c_*[(s-t)-\tau])) \\ & + u(s,\xi+1+c_*(s-t)) + u(s,\xi-1+c_*(s-t))]ds.\end{aligned}$$

Taking the same estimates as in (3.21)–(3.28), we can prove $u \in X_{loc}(\tau,2\tau)$ and

$$\begin{aligned}& ||u(t)||_C^2 + ||\sqrt{w}u(t)||_{H^1}^2 + \int_\tau^t ||\sqrt{w}u(s)||_{H^1}^2 ds \\ \le & C\left(||u(\tau)||_C^2 + ||\sqrt{w}u(\tau)||_{H^1}^2 + \int_0^\tau ||\sqrt{w}u(s)||_{H^1}^2 ds\right) \\ \le & C^2\left(||u(0)||_C^2 + ||\sqrt{w}u_0(0)||_{H^1}^2 + \int_{-\tau}^0 ||\sqrt{w}u_0(s)||_{H^1}^2 ds\right), \quad t\in[\tau,2\tau],\end{aligned}$$

and

$$\begin{aligned}\lim_{\xi\to+\infty} u(t,\xi) =& e^{-t}u_0(\tau,\infty) + e^{-t}\int_\tau^t e^s \lim_{\xi\to+\infty} P(u_0(s-\tau,\xi+c_*[s-(t+\tau)]))ds \\ =& e^{-t}\mathcal{Y}(\tau) + \int_\tau^t e^{-(t-s)}P(\mathcal{Y}(\tau))ds \\ =& \mathcal{Z}(t), \quad \text{uniformly with respect to } t\in[\tau,2\tau].\end{aligned}$$

Repeating the above produce, step by step, we can prove that $u \in X_{loc}((n-1)\tau, n\tau)$ uniquely exists and satisfies

$$\begin{aligned}&||u(t)||_C^2 + ||\sqrt{w}u(t)||_{H^1}^2 + \int_0^t ||\sqrt{w}u(s)||_{H^1}^2 ds \\ &\leq C^n \left(||u(0)||_C^2 + ||\sqrt{w}u_0(0)||_{H^1}^2 + \int_{-\tau}^0 ||\sqrt{w}u_0(s)||_{H^1}^2 ds \right),\end{aligned}$$

for $t \in [(n-1)\tau, n\tau]$. Finally, we prove that u is unique, and $u \in X_{loc}(0,\infty)$ with, for any $T > 0$, that

$$\begin{aligned}&||u(t)||_C^2 + ||\sqrt{w}u(t)||_{H^1}^2 + \int_0^t ||\sqrt{w}u(s)||_{H^1}^2 ds \\ &\leq C_T \left(||u(0)||_C^2 + ||\sqrt{w}u_0(0)||_{H^1}^2 + \int_{-\tau}^0 ||\sqrt{w}u_0(s)||_{H^1}^2 ds \right), \quad t \in [0,T].\end{aligned}$$

The proof is completed. □

3.2. Uniform boundedness

In this subsection, we are going to prove Theorem 3.2. For the global solution of equation (3.3), $u \in X_{loc}(0,\infty)$, when the initial perturbation $u_0 \in X_0(-\tau, 0)$ is small enough, we shall show $u \in X(0,\infty)$ by deriving the uniform boundedness (3.13).

Take the following transformation:

$$\overline{u}(t,\xi) = \sqrt{w(\xi)}u(t,\xi) = e^{-\lambda_* \xi}u(t,\xi).$$

Substituting $u(t,\xi) = w^{-1/2}(\xi)\overline{u}(t,\xi)$ to (3.3), then we derive the following equation for $\overline{u}(t,\xi)$:

$$\begin{cases} \frac{\partial \overline{u}}{\partial t} + c_* \frac{\partial \overline{u}}{\partial \xi} - [e^{\lambda_*}\overline{u}(t,\xi+1) + e^{-\lambda_*}\overline{u}(t,\xi-1)] + (c_*\lambda_* + 3)\overline{u}(t,\xi) \\ \qquad -g'(\phi(\xi - c_*\tau))\overline{u}(t-\tau, \xi - c_*\tau)e^{-\lambda_* c_* \tau} = \overline{Q}(\overline{u}(t-\tau, \xi - c_*\tau)), \\ \overline{u}(s,\xi) = \sqrt{w}u(s,\xi) = \overline{u}_0(s,\xi), \quad s \in [-\tau, 0], \xi \in \mathbb{R}, \end{cases} \tag{3.29}$$

where

$$\overline{Q}(\overline{u}) = e^{-\lambda_* \xi}Q(u). \tag{3.30}$$

By Taylor's expansion formula:

$$|\overline{Q}(\overline{u})| \leq Ce^{-\lambda_* \xi}|u|^2 = \frac{C}{\sqrt{w(\xi)}}|\overline{u}|^2. \tag{3.31}$$

Lemma 3.4 *It holds that*

$$\begin{aligned}&||\overline{u}(t)||_{L^2}^2 + \int_0^t \int_{\mathbb{R}} \mathcal{M}(\xi)|\overline{u}(s,\xi)|^2 d\xi ds \\ \leq& ||\overline{u}_0(0)||_{L^2}^2 + 2C \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}}|\overline{u}(s,\xi)||\overline{u}(s-\tau, \xi - c_*\tau)|^2 d\xi ds \\ &+ g'(0)e^{-\lambda_* c_* \tau} \int_{-\tau}^0 \int_{\mathbb{R}} |\overline{u}_0(s,\xi)|^2 d\xi ds,\end{aligned} \tag{3.32}$$

where

$$\mathcal{M}(\xi) := e^{-\lambda_* c_* \tau}\big[2g'(0) - |g'(\phi(\xi - c_*\tau))| - |g'(\phi(\xi))|\big]. \tag{3.33}$$

Proof Multiplying the equation (3.29) by $\overline{u}(t,\xi)$, then integrating it with the respect to ξ and t over $\mathbb{R}\times[0,t]$, we have

$$\begin{aligned}
&||\overline{u}(t)||_{L^2}^2 - 2(e^{\lambda_*} + e^{-\lambda_*})\int_0^t \int_{\mathbb{R}} \bar{u}(s,\xi)\bar{u}(s,\xi+1)d\xi ds \\
&+ 2(c_*\lambda_* + 3)\int_0^t \int_{\mathbb{R}} \overline{u}^2(s,\xi)d\xi ds \\
&- 2e^{-\lambda_* c_* \tau}\int_0^t \int_{\mathbb{R}} g'(\phi(\xi - c_*\tau))\overline{u}(s,\xi)\overline{u}(s-\tau,\xi - c_*\tau)d\xi ds \\
=&||\overline{u}_0(0)||_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}} \overline{Q}\left(\overline{u}(s-\tau,\xi - c_*\tau)\right)\overline{u}(s,\xi)d\xi ds.
\end{aligned} \tag{3.34}$$

By the Cauchy–Schwarz inequality $2xy \le x^2 + y^2$, we get

$$\begin{aligned}
&2\left|\int_0^t \int_{\mathbb{R}} \bar{u}(s,\xi)\bar{u}(s,\xi+1)d\xi ds\right| \\
\le&\int_0^t \int_{\mathbb{R}} |\bar{u}(s,\xi)|^2 d\xi ds + \int_0^t \int_{\mathbb{R}} |\bar{u}(s,\xi+1)|^2 d\xi ds \\
=&2\int_0^t \int_{\mathbb{R}} |\bar{u}(s,\xi)|^2 d\xi ds,
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
&2\left|\int_0^t \int_{\mathbb{R}} g'(\phi(\xi - c_*\tau))\overline{u}(s,\xi)\overline{u}(s-\tau,\xi - c_*\tau)d\xi ds\right| \\
\le&\int_0^t \int_{\mathbb{R}} |g'(\phi(\xi - c_*\tau))|\,|\overline{u}(s,\xi)|^2 d\xi ds + \int_0^t \int_{\mathbb{R}} |g'(\phi(\xi - c_*\tau))|\,\overline{u}(s-\tau,\xi - c_*\tau)|^2 d\xi ds \\
\le&\int_0^t \int_{\mathbb{R}} |g'(\phi(\xi - c_*\tau))|\,|\overline{u}(s,\xi)|^2 d\xi ds + \int_{-\tau}^{t-\tau} \int_{\mathbb{R}} |g'(\phi(\xi))|\,|\overline{u}(s,\xi)|^2 d\xi ds \\
\le&\int_0^t \int_{\mathbb{R}} \left(|g'(\phi(\xi - c_*\tau)| + |g'(\phi(\xi))|\right)|\overline{u}(s,\xi)|^2 d\xi ds \\
&+ g'(0)\int_{-\tau}^0 \int_{\mathbb{R}} |\overline{u}_0(s,\xi)|^2 d\xi ds.
\end{aligned} \tag{3.36}$$

The last inequality holds due to the fact that $|g'(\phi)| \le g'(0)$; see $(H3)$. On the other hand, by (3.31), the nonlinear term in (3.34) can be estimated as follows:

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} \overline{Q}\left(\overline{u}(s-\tau,\xi - c_*\tau)\right)\overline{u}(s,\xi)d\xi ds\right| \\
\le& C\int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}}|\overline{u}(s,\xi)||\overline{u}(s-\tau,\xi - c_*\tau)|^2 d\xi ds.
\end{aligned} \tag{3.37}$$

Substituting (3.35), (3.36), and (3.37) into (3.34), we have

$$
\begin{aligned}
&||\overline{u}(t)||_{L^2}^2 + 2(c_*\lambda_* + 3 - e^{\lambda_*} - e^{-\lambda_*}) \int_0^t \int_{\mathbb{R}} |\overline{u}(s,\xi)|^2 d\xi ds \\
&- e^{-\lambda_* c_* \tau} \int_0^t \int_{\mathbb{R}} (|g'(\phi(\xi - c_*\tau)| + |g'(\phi(\xi))|) \, |\overline{u}(s,\xi)|^2 d\xi ds \\
\leq & ||\overline{u}_0(0)||_{L^2}^2 + 2C \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\overline{u}(s,\xi)||\overline{u}(s-\tau, \xi - c_*\tau)|^2 d\xi ds \\
&+ g'(0) e^{-\lambda_* c_* \tau} \int_{-\tau}^0 \int_{\mathbb{R}} |\overline{u}_0(s,\xi)|^2 d\xi ds.
\end{aligned}
$$

Note that $c_*\lambda_* + 3 - e^{\lambda_*} - e^{-\lambda_*} = g'(0)e^{-\lambda_* c_* \tau}$ and let

$$
\mathcal{M}(\xi) := e^{-\lambda_* c_* \tau} \left[2g'(0) - |g'(\phi(\xi - c_*\tau))| - |g'(\phi(\xi))| \right].
$$

Then we obtain

$$
\begin{aligned}
&||\overline{u}||_{L^2}^2 + \int_0^t \int_{\mathbb{R}} \mathcal{M}(\xi) |\overline{u}(s,\xi)|^2 d\xi ds \\
\leq & ||\overline{u}_0(0)||_{L^2}^2 + 2C \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\overline{u}(s,\xi)||\overline{u}(s-\tau, \xi - c_*\tau)|^2 d\xi ds \\
&+ g'(0) e^{-\lambda_* c_* \tau} \int_{-\tau}^0 \int_{\mathbb{R}} |\overline{u}_0(s,\xi)|^2 d\xi ds.
\end{aligned}
$$

The proof is completed. □

By a similar argument as in [3], we obtain the estimate for $\mathcal{M}(\xi)$.

Lemma 3.5 *It holds that*

$$
\mathcal{M}(\xi) \geq C\phi(\xi) \geq 0
$$

for some positive constant C.

Based on Lemmas 3.4 and 3.5, we can get the following estimate.

Lemma 3.6 *There exists* $\delta_1 > 0$, *when* $\mathcal{N}_\infty \leq \delta_1$, *and then*

$$
\begin{aligned}
&||\overline{u}(t)||_{L^2}^2 + \int_0^t \int_{\mathbb{R}} \phi(\xi) w(\xi) |u(s,\xi)|^2 d\xi ds \\
&\leq C \left(||\overline{u}_0||_{L^2}^2 + \int_{-\tau}^0 ||\overline{u}_0(s)||_{L^2}^2 ds \right) \\
&\leq C\mathcal{N}_0^2, \quad t \in [0, \infty),
\end{aligned}
\tag{3.38}
$$

where C *is a positive constant.*

Proof The proof is similar to Lemma 4.3 in [3], so we omit it here. □

Similarly to lemma 3.6, we have the following result.

Lemma 3.7 *When* $\mathcal{N}_\infty \le \delta_1$, *then*

$$||\overline{u}_\xi||^2_{L^2} + \int_0^t ||\overline{u}_\xi(s,\xi)||^2_{L^2} ds \le C_4(\mathcal{N}_\infty + 1)\mathcal{N}_0^2, \tag{3.39}$$

and

$$\int_0^t \left| \frac{d}{dt} ||\overline{u}_\xi(s,\xi)||^2_{L^2} \right| ds \le C_5(\mathcal{N}_\infty + 1)\mathcal{N}_0^2, \tag{3.40}$$

where C_4 *and* C_5 *are positive constants.*

Proof Differentiating equation (3.29) with respect to ξ yields

$$\begin{aligned}
&(\overline{u}_\xi)_t + c_*\overline{u}_{\xi\xi} - \left[e^{\lambda_*}\overline{u}_\xi(t,\xi+1) + e^{-\lambda_*}\overline{u}_\xi(t,\xi-1)\right] \\
&+ (c_*\lambda_* + 3)\overline{u}_\xi - g''(\phi(\xi - c_*\tau))\phi'(\xi - c_*\tau)\overline{u}(t-\tau, \xi - c_*\tau)e^{-\lambda_* c_* \tau} \\
&- g'(\phi(\xi - c_*\tau))\overline{u}_\xi(t-\tau, \xi - c_*\tau)e^{-\lambda_* c_* \tau} \\
=&\partial_\xi \overline{Q}(\overline{u}(t-\tau, \xi - c_*\tau)).
\end{aligned} \tag{3.41}$$

Multiplying (3.41) by $\overline{u}_\xi(t,\xi)$ and integrating it with respect to ξ over $\mathbb{R}$, we further obtain

$$\begin{aligned}
&\frac{d}{dt}||\overline{u}_\xi||^2_{L^2} + 2\mathcal{B}\int_{\mathbb{R}} \overline{u}_\xi^2(t,\xi)d\xi \\
=&2e^{-\lambda_* c_* \tau}\int_{\mathbb{R}} g''(\phi(\xi - c_*\tau))\phi'(\xi - c_*\tau)\overline{u}(t-\tau, \xi - c_*\tau)\overline{u}_\xi(t,\xi)d\xi \\
&+ 2e^{-\lambda_* c_* \tau}\int_{\mathbb{R}} g'(\phi(\xi - c_*\tau))\overline{u}_\xi(t-\tau, \xi - c_*\tau)\overline{u}_\xi(t,\xi)d\xi \\
&+ 2\int_{\mathbb{R}} \overline{u}_\xi(t,\xi)\partial_\xi \overline{Q}(\overline{u}(t-\tau, \xi - c_*\tau)d\xi \\
=:&I_1(t) + I_2(t) + I_3(t),
\end{aligned} \tag{3.42}$$

where

$$\mathcal{B} = c_*\lambda_* + 3 - e^{\lambda_*} - e^{-\lambda_*} = g'(0)e^{-c_*\lambda_*\tau}.$$

Integrating (3.42) over $[0,t]$, we get

$$\begin{aligned}
&||\overline{u}_\xi(t)||^2_{L^2} + 2\mathcal{B}\int_0^t ||\overline{u}_\xi||^2_{L^2} ds \\
&= ||\overline{u}_{0,\xi}(0)||^2_{L^2} + \int_0^t \left[I_1(s) + I_2(s) + I_3(s)\right] ds.
\end{aligned} \tag{3.43}$$

By a similar argument as in the proof of Lemma 4.4 of [3], we obtain

$$\int_0^t I_1(s)ds \le C\mathcal{N}_0^2, \quad \int_0^t I_2(s)ds \le C\mathcal{N}_0^2, \tag{3.44}$$

and

$$\int_0^t I_3(s)ds \le C\mathcal{N}_\infty\mathcal{N}_0^2, \tag{3.45}$$

provided $\mathcal{N}_\infty \leq \delta_1$. Substituting (3.44) and (3.45) into (3.43), we have

$$\begin{aligned}
&||\overline{u}_\xi||_{L^2}^2 + 2\mathcal{B}\int_0^t ||\overline{u}_\xi||_{L^2}^2 ds \\
&\leq \left(||\overline{u}_0(0)||_{H^1}^2 + \int_{-\tau}^0 ||\overline{u}_0(s)||_{H^1}^2 ds \right) C_4\left(\mathcal{N}_\infty + 1\right) \\
&\leq C_4\left(\mathcal{N}_\infty + 1\right)\mathcal{N}_0^2,
\end{aligned} \tag{3.46}$$

for some constant $C_4 > 0$, provided $\mathcal{N}_\infty \leq \delta_1$. This proves (3.39).

Now we prove (3.40). From (3.42), we can see that

$$\left| \frac{d}{dt}||\overline{u}_\xi||_{L^2}^2 \right| \leq 2\mathcal{B}||\overline{u}_\xi||_{L^2}^2 + |I_1| + |I_2| + |I_3|.$$

Integrating it over $[0, t]$, we have

$$\int_0^t \left| \frac{d}{dt}||\overline{u}_\xi||_{L^2}^2 \right| ds \leq 2\mathcal{B}\int_0^t ||\overline{u}_\xi||_{L^2}^2 ds + \int_0^t (|I_1| + |I_2| + |I_3|) ds.$$

By (3.46), one has

$$\int_0^t \left| \frac{d}{dt}||\overline{u}_\xi(s,\xi)||_{L^2}^2 \right| ds \leq C_5(\mathcal{N}_\infty + 1)\mathcal{N}_0^2, \tag{3.47}$$

where $C_5 > 0$, $\mathcal{N}_\infty < \delta_1$. The proof is completed. □

Lemma 3.8 *Assume that* (H1) – (H3) *hold. Then*

$$|u(t,\infty)| = |\mathcal{U}(t)| \leq C\mathcal{N}_0 e^{-\mu t}, \quad t > 0,$$

for some $0 < \mu = \mu(\tau, g'(K)) < 1$, *provided with* $|\mathcal{U}_0| \ll 1$.

Proof Let $u(t,x) = v(t,x) - \phi(x + c_* t)$. From (1.1) and (1.2), one gets that $u(t,x)$ satisfies

$$\begin{cases} u_t(t,x) - \Delta_1 u(t,x) + u(t,x) - g'(\phi)u(t-\tau,x) = Q(u(t-\tau,x)), & t > 0, x \in \mathbb{R}, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in \mathbb{R}, \end{cases} \tag{3.48}$$

where $Q(u(t-\tau,x)) = g(\phi + u) - g(\phi) - g'(\phi)u(t-\tau,x)$. Since $u \in X(0,\infty)$, so $u \in C_{unif}[0,\infty)$, namely $\lim_{x\to+\infty} u(t,x) = u(t,\infty) =: \mathcal{U}(t)$ exists uniformly for $t \in (-\tau,\infty)$, and $\lim_{x\to+\infty} Q(u(t-\tau,x)) = Q(\mathcal{U}(t-\tau))$. Taking the limit to equation (3.48) as $x \to +\infty$, we have

$$\begin{cases} \mathcal{U}_t + \mathcal{U}(t) - g'(v_+)\mathcal{U}(t-\tau) = Q(\mathcal{U}(t-\tau)), & t > 0, \\ \mathcal{U}(s) = \mathcal{U}_0(s), \quad s \in [-\tau, 0]. \end{cases}$$

Applying the nonlinear Halanay inequality given in [11], we obtain that if (H1) – (H3) hold, then

$$|\mathcal{U}(t)| \leq \tilde{C}||\mathcal{U}_0||_{L^\infty(-\tau,0)} e^{-\mu t}, \tag{3.49}$$

where $0 < \mu < 1$ and $\tilde{C}$ is some positive constant. Then we get

$$|\mathcal{U}(t)| \le C\mathcal{N}_0 e^{-\mu t}, \quad t > 0,$$

for some positive constant C. Thus, we complete the proof. □

Now we can prove the boundedness of u in $C(\mathbb{R})$.

Lemma 3.9 *If* $\mathcal{N}_\infty \le \delta_1$, *then*

$$||u(t)||_C \le C\sqrt{\mathcal{N}_\infty + 1}\mathcal{N}_0, \quad t \in [0,\infty). \tag{3.50}$$

Proof From Lemma 3.8, we have

$$\lim_{\xi \longrightarrow +\infty} u(t,\xi) = u(t,\infty) =: \mathcal{U}(t)$$

uniformly with respect to $t \in [0,\infty)$. Then for any given $\varepsilon > 0$, there exists a very large number $x_0 = x_0(\xi) \gg 1$ such that when $\xi \ge x_0$,

$$|u(t,\xi) - \mathcal{U}(t)| < \varepsilon \quad \text{uniformly in} \quad t \in [0,\infty),$$

and

$$|u(t,\infty)| = |\mathcal{U}(t)| \le C\mathcal{N}_0 e^{-\mu t} \le C\mathcal{N}_0.$$

That is,

$$\sup_{x\in[x_0,\infty)} |u(t,\xi)| < C\mathcal{N}_0 \quad \text{uniformly in} \quad t \in [0,\infty), \tag{3.51}$$

and

$$\begin{aligned} \sup_{x\in(-\infty,x_0]} |u(t,\xi)| \le & \sup_{x\in(-\infty,x_0]} \left| \frac{\sqrt{w(\xi)}}{e^{-\lambda_* x_0}} u(t,\xi) \right| \\ =& e^{\lambda_* x_0} \sup_{x\in(-\infty,x_0]} \left| \sqrt{w(\xi)} u(t,\xi) \right| \\ \le& C||\sqrt{w}u(t)||_{H^1} \\ \le& C\sqrt{\mathcal{N}_\infty + 1}\mathcal{N}_0, \quad t \in [0,\infty). \end{aligned} \tag{3.52}$$

Combining (3.51) and (3.52), we obtain that (3.50) holds. The proof is completed. □

Proof [Proof of Theorem 3.2] Adding (3.38), (3.39), and (3.50) together, we have

$$\begin{aligned} &||u(t)||_C^2 + ||\overline{u}(t)||_{H^1}^2 + \int_0^t ||\sqrt{\phi w}u(s)||_{L^2}^2 ds + \int_0^t ||\overline{u}_\xi(s)||_{L^2}^2 ds \\ \le& C(\mathcal{N}_\infty + 1)\mathcal{N}_0^2, \quad t \in [0,\infty). \end{aligned} \tag{3.53}$$

In order to guarantee $\mathcal{N}_\infty \le \delta_1$, and $\mathcal{N}_\infty \le \sqrt{C_6(\mathcal{N}_\infty + 1)}\mathcal{N}_0$, we take $\delta_0 > 0$ in Theorem 3.2 as

$$\delta_0 = \frac{\delta_1}{\sqrt{C_6(\delta_1 + 1)}}.$$

Thus, when $\mathcal{N}_0 \le \delta_0$, we can guarantee

$$\mathcal{N}_\infty \le \sqrt{C_6(\mathcal{N}_\infty + 1)}\mathcal{N}_0 \le \sqrt{C_6(\delta_1 + 1)}\delta_0$$

and

$$\mathcal{N}_\infty^2 \le C_6(\mathcal{N}_\infty + 1)\mathcal{N}_0^2 \le C_6(\delta_1 + 1)\mathcal{N}_0^2 \le C_7\mathcal{N}_0^2,$$

provided $\mathcal{N}_\infty < \delta_1$. The proof is completed. □

3.3. Asymptotic stability

This subsection is devoted to proving the asymptotic stability of critical traveling waves.

Proof [Proof of Theorem 3.3] From (3.40) and (3.53), we have

$$0 \le ||\overline{u}_\xi||_{L^2}^2 \le C\mathcal{N}_0^2,$$

$$0 \le \int_0^\infty ||\overline{u}_\xi(t,\xi)||_{L^2}^2 dt \le C\mathcal{N}_0^2,$$

$$0 \le \int_0^\infty \left| \frac{d}{dt} ||\overline{u}_\xi(t,\xi)||_{L^2}^2 \right| dt \le C\mathcal{N}_0^2,$$

which implies

$$\lim_{t\to\infty} ||\overline{u}_\xi(t)||_{L^2}^2 = 0. \tag{3.54}$$

By the Sobolev inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, we further obtain

$$||\overline{u}(t)||_C \le \sqrt{2}||\overline{u}(t)||_{L^2}^{\frac{1}{2}} ||\overline{u}_\xi(t)||_{L^2}^{\frac{1}{2}}.$$

With the boundedness of $||\overline{u}(t)||_{L^2} = ||(\sqrt{w}u)(t)||_{L^2} \le C\mathcal{N}_0$ and the convergence of (3.54), we then prove

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |\sqrt{w(\xi)}u(\xi)| = \lim_{t\to\infty} ||\overline{u}(t)||_C = 0. \tag{3.55}$$

Next, we shall show the convergence

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |u(t,\xi)| = 0.$$

By Lemma 3.8, we have

$$|u(t,\infty)| = |\mathcal{U}(t)| \le C\mathcal{N}_0 e^{-\mu t}, \quad t > 0.$$

It is easy to see that (3.48) is equivalent to

$$e^t[u_t(t,x) - \Delta_1 u(t,x) + u(t,x) - g'(\phi)u(t-\tau,x)] = e^t Q(u(t-\tau,x)),$$

that is,

$$\{e^t u(t,x)\}_t + e^t[-\Delta_1 u(t,x) - g'(\phi)u(t-\tau,x)] = e^t Q(u(t-\tau,x)).$$

Integrating the above equation with respect to t over $[0,t]$, we get

$$u(t,x) = e^{-t}u_0(0,x) + e^{-t}\int_0^t e^s[\Delta_1 u(s,x) + g'(\phi)u(s-\tau,x) + Q(u(s-\tau,x))]ds.$$

Thus, one has, for $0 < \mu < 1$,

$$\begin{aligned} e^{\mu t}u(t,x) =& e^{(\mu-1)t}u_0(0,x) \\ &+ e^{(\mu-1)t}\int_0^t e^s[\Delta_1 u(s,x) + g'(\phi)u(s-\tau,x) + Q(u(s-\tau,x))]ds. \end{aligned} \tag{3.56}$$

Taking the limit to (3.56) as $x \to \infty$, and noting that all these limits are uniformly in t, then applying the fact $|Q(\mathcal{U}(t))| \le C|\mathcal{U}|^2$ and the decay estimate (3.49) for $\mathcal{U}(t)$, we have

$$\begin{aligned} &\lim_{x\to\infty} e^{\mu t}u(t,x) \\ =& \lim_{x\to\infty} e^{(\mu-1)t}u_0(0,x) \\ &+ \lim_{x\to\infty} e^{(\mu-1)t}\int_0^t e^s\big[\Delta_1 u(s,x) + g'(\phi)u(s-\tau,x) + Q(u(s-\tau,x))\big]ds \\ =& e^{(\mu-1)t}\mathcal{U}_0(0) + e^{(\mu-1)t}\int_0^t e^s[g'(v_+)\mathcal{U}(s-\tau) + Q(\mathcal{U}(s-\tau))]ds \\ \le& e^{(\mu-1)t}\mathcal{U}_0(0) + e^{(\mu-1)t}\int_0^t e^s[|\mathcal{U}(s-\tau)| + C|\mathcal{U}(s-\tau)|^2]ds \\ \le& e^{(\mu-1)t}\mathcal{U}_0(0) + e^{(\mu-1)t}\int_0^t e^s[e^{-\mu(s-\tau)} + Ce^{-2\mu(s-\tau)}]ds \\ \le& C, \quad \text{uniformly in } t. \end{aligned}$$

This implies that there exists a number $x_1 \gg 1$ (independent of t), such that when $\xi \ge x_1$, then

$$\sup_{\xi\in[x_1,\infty)} |u(t,\xi)| \le Ce^{-\mu t}, \quad t \in (0,\infty). \tag{3.57}$$

Notice that $\sqrt{w(\xi)} = e^{-\lambda_*\xi} \ge e^{-\lambda_* x_1}$ for $\xi \in (-\infty, x_1]$. Then (3.55) implies

$$\begin{aligned} \lim_{t\longrightarrow\infty} \sup_{\xi\in(-\infty,x_1]} |u(t,\xi)| \le& \lim_{t\longrightarrow\infty} \sup_{\xi\in(-\infty,x_1]} |\frac{\sqrt{w(\xi)}}{e^{-\lambda_* x_1}}u(t,\xi)| \\ \le& e^{\lambda_* x_1} \lim_{t\longrightarrow\infty} \sup_{\xi\in\mathbb{R}} |\sqrt{w(\xi)}u(t,\xi)| \\ =& 0. \end{aligned} \tag{3.58}$$

From (3.57) and (3.58), we have

$$\lim_{t\longrightarrow\infty} \sup_{\xi\in\mathbb{R}} |u(t,\xi)| = 0.$$

The proof is completed. □

Acknowledgments

The second author was supported by SRFDP (20126203120006), NSF of China (11401478), Gansu Provincial Natural Science Foundation (145RJZA220).

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