Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 368 – 380
© TÜBİTAK
doi:10.3906/mat-1601-34

On certain GBS-Durrmeyer operators based on q-integers

Dan BĂRBOSU[1], **Ana-Maria ACU**[2], **Carmen Violeta MURARU**[3,*]
[1]Department of Mathematics and Informatics, North University Center at Baia Mare, Technical University of Cluj-Napoca, Baia Mare, Romania
[2]Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Sibiu, Romania
[3]Department of Mathematics, Informatics and Education Sciences, "Vasile Alecsandri" University of Bacău, Bacău, Romania

Received: 10.01.2016 • **Accepted/Published Online:** 12.05.2016 • **Final Version:** 03.04.2017

Abstract: In the present paper we introduce the GBS (Generalized Boolean Sum) operators of Durrmeyer type based on q-integers and the approximation of B-continuous functions using the above operators is studied. In addition, a uniform convergence theorem is established and the degree of approximation in terms of mixed modulus of continuity is evaluated. The study contains in the last section numerical considerations regarding the constructed operators based on MATLAB algorithms.

Key words: Positive linear operator, Durrmeyer-type operators, GBS operator, B-continuous function, mixed modulus of continuity, q-integers

1. Introduction

In [12, 13], Karl Bögel introduced the concepts of B-continuous and B-differentiable functions. The GBS (Generalized Boolean Sum) operators are used in the uniform approximation of B-continuous functions. The term GBS operators was introduced by Badea et al. in [6]. In recent years, several researchers have made significant contributions in this area of approximation theory [1, 3, 11, 18, 21-24, 27, 29]. In this paper we study the uniform approximation of B-continuous functions using GBS operators of Durrmeyer type based on q-integers. The notions like GBS operators, B-continuous functions, and mixed modulus of continuity are considered in this section. A Korovkin-type theorem for the approximation of B-continuous functions using GBS operators and a Shisha–Mond-type theorem to estimate the degree of approximation in terms of mixed modulus of continuity are also presented.

Let I and J be compact real intervals and $A = I \times J$. A function $f : A \to \mathbb{R}$ is a B-continuous function at $(x_0, y_0) \in A$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta_{x,y} f[x_0, y_0; x, y] = 0,$$

where

$$\Delta_{x,y} f[x_0, y_0; x, y] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

The set of B-continuous functions in A is denoted by $C_b(A)$.

*Correspondence: carmen_7419@yahoo.com

2010 *AMS Mathematics Subject Classification:* 41A10, 41A36.

The function $f : A \to \mathbb{R}$ is B-bounded on A if there exists a constant $M > 0$ such that

$$|\Delta_{x,y} f[s,t;x,y]| < M$$

holds for all $(x,y),(s,t) \in A$. Throughout this paper, $B_b(A)$ denotes all B-bounded functions on A, equipped with the norm $\|f\|_B = \sup_{(x,y),(s,t)\in A} |\Delta_{x,y} f[s,t;x,y]|$. Let $B(A)$, $C(A)$ be the space of all bounded functions and the space of all continuous functions endowed with the sup-norm $\|\cdot\|_\infty$.

A function $f : A \to \mathbb{R}$ is B-differentiable function at $(x_0,y_0) \in A$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta_{x,y} f[x_0,y_0;x,y]}{(x-x_0)(y-y_0)}$$

exists and is finite. The limit is the B-derivative of f at the point (x_0,y_0) and is denoted by $D_B f(x_0,y_0)$. The space of all B-differentiable functions is denoted by $D_b(A)$.

The function $f : A \to \mathbb{R}$ is uniform B-continuous on A if and only if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $(x,y),(s,t) \in A$ satisfying $\max\{|x-s|,|y-t|\} < \delta$ hence

$$|\Delta_{x,y} f[s,t;x,y]| < \epsilon. \tag{1}$$

Clearly, if f is uniform B-continuous on A then $f \in C_b(A)$.

Let $\mathbb{R}^A = \{f \mid f : A \to \mathbb{R}\}$ and $L : \mathbb{R}^A \to \mathbb{R}^A$ be a bivariate linear positive operator. If $f \in \mathbb{R}^A$, $(x,y) \in A$, the bivariate operator $U : \mathbb{R}^A \to \mathbb{R}^A$ defined as

$$U(f;x,y) = L\left[f(\cdot,y) + f(x,\cdot) - f(\cdot,\cdot)\right](x,y) \tag{2}$$

is called the GBS operator associated with L.

Badea et al. in [5] established the following Korovkin-type result to approximate B-continuous functions using GBS operators.

Theorem 1.1 *Let $(L_{m,n})_{m,n\in\mathbb{N}}$, $L_{m,n} : C_b(A) \to B(A)$ be a sequence of bivariate linear positive operators, $U_{m,n}$ be the GBS operators associated with $L_{m,n}$, and $e_{ij} : A \to \mathbb{R}$, $e_{ij}(x,y) = x^i y^j$ (i,j-nonnegative integers such that $0 \le i+j \le 2$) be the test functions. Suppose that the following relations*

(i) $L_{m,n}(e_{00};x,y) = 1$,

(ii) $L_{m,n}(e_{10};x,y) = x + u_{m,n}(x,y)$,

(iii) $L_{m,n}(e_{01};x,y) = y + v_{m,n}(x,y)$,

(iv) $L_{m,n}(e_{20}+e_{02};x,y) = x^2 + y^2 + w_{m,n}(x,y)$,

hold for all $(x,y) \in A$ and $m,n \in \mathbb{N}\setminus\{0\}$. If the sequences $(u_{m,n})$, $(v_{m,n})$, $(w_{m,n})$ converge to zero uniformly on A, then the sequence $(U_{m,n}f)$ converges to f uniformly on A for all $f \in C_b(A)$.

In order to evaluate the approximation degree of B-continuous functions using GBS operators, an important tool is the mixed modulus of continuity, for which we will recall the definition.

Let $\mathbb{R}_+ := [0, \infty)$ and f be a B-bounded function on A. The mixed modulus of continuity of f is the function $\omega_{mixed} : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) = \sup\{|\Delta_{x,y}[f; s, t; x, y]| \,:\, |s - x| \leq \delta_1, |t - y| < \delta_2\} \tag{3}$$

for all $\delta_1, \delta_2 \in \mathbb{R}_+$, $(x, y), (s, t) \in A$. The properties of mixed modulus of continuity were investigated in [4, 7].

Shisha and Mond [28] established an important result meant to evaluate the approximation degree of continuous and differentiable univariate real valued functions by linear positive operators using the modulus of continuity. Following the ideas from [28], Badea et al. in [7] proved the following Shisha–Mond-type theorem to evaluate the approximation degree for B-continuous functions using GBS operators.

Theorem 1.2 *Let $L : C_b(A) \to C_b(A)$ be a bivariate linear positive operator and $U : C_b(A) \to C_b(A)$ be the associated GBS operator. The following inequality*

$$\begin{aligned} |f(x,y) - U(f; x, y)| \leq |f(x, y)|\,|1 - L(1; x, y)| + \big\{L(1; x, y) \\ + \delta_1^{-1}\sqrt{L((s - x)^2; x, y)} + \delta_2^{-1}\sqrt{L((t - y)^2; x, y)} \\ + \delta_1^{-1}\delta_2^{-1}\sqrt{L((s - x)^2; x, y)L((t - y)^2; x, y)}\big\}\omega_{mixed}(f; \delta_1, \delta_2) \end{aligned} \tag{4}$$

holds for all $f \in C_b(A)$, $(x, y) \in A$, and $\delta_1, \delta_2 \in \mathbb{R}_+$.

Recall that $\omega_{mixed}(f; 0, 0) = 0$ and ω_{mixed} is a B-continuous function. These properties of mixed modulus of continuity and the inequality (4) make it possible to obtain the uniform convergence for the sequence defined by the GBS operators to a B-continuous function.

Example 1.1 In 1967, Durrmeyer [17] introduced the operators $D_m : L_1[0, 1] \to C[0, 1]$ defined for each positive integer m, $f \in L_1[0, 1]$ and $x \in [0, 1]$ by the formula

$$D_m(f; x) = (m + 1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t)f(t)dt, \tag{5}$$

where $p_{m,k}(x) = \binom{m}{k}x^k(1 - x)^{m-k}$ are the Bernstein fundamental polynomials and $L_1[0, 1]$ is the space of Lebesque integrable functions. The operators (5) are known as the Durrmeyer operators and were intensively studied by Derriennic in [14].

Let $\alpha, \beta \in \mathbb{R}$ be real parameters independently on the positive integer m such that $0 \leq \alpha \leq \beta$. In 1969, Stancu [30] introduced the operators $P_m^{(\alpha,\beta)} : C[0, 1] \to C[0, 1]$ defined for any positive integer m, $f \in C[0, 1]$, $x \in [0, 1]$ as follows:

$$P_m^{(\alpha,\beta)}(f; x) = \sum_{k=0}^{m} p_{m,k}(x)f\left(\frac{k + \alpha}{m + \beta}\right). \tag{6}$$

The operators (6) are known as the Stancu operators and they contain as a particular case the classical Bernstein operators (when $\alpha = \beta = 0$).

Pop and Barbosu [26] introduced the Durrmeyer–Stancu operators $D_m^{(\alpha,\beta)} : L_1[0, 1] \to C[0, 1]$ defined as

$$D_m^{(\alpha,\beta)}(f; x) = (m + 1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t)f\left(\frac{mt + \alpha}{m + \beta}\right)dt \tag{7}$$

for all positive integer m, $f \in L_1[0,1]$ and $x \in [0,1]$.
When $\alpha = \beta = 0$, the operators (7) reduce to the classical Durrmeyer operators (5).

In 2008, using the method of parametric extensions, Pop and Barbosu [27] introduced the bivariate Durrmeyer–Stancu operators $D_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ defined as

$$D_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;x,y) = (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) \cdot \int_0^1\int_0^1 p_{m,k}(s)p_{n,j}(t) f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right) ds\, dt \tag{8}$$

for all positive integers m,n, $f \in L_1([0,1]\times[0,1])$, $(x,y) \in [0,1]\times[0,1]$, where $\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbb{R}$ such that $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$.

The GBS associated with the operators $U_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : L_1([0,1]\times[0,1] \to C([0,1]\times[0,1])$ was introduced in [27] as follows:

$$U_{m,n}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;x,y) = (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)\int_0^1\int_0^1 p_{m,k}(s)p_{n,j}(t) \cdot \left\{ f\left(\frac{ms+\alpha_1}{m+\beta_1}, y\right) + f\left(x, \frac{nt+\alpha_2}{n+\beta_2}\right) - f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right)\right\} dsdt. \tag{9}$$

It was proved that the sequence having the general term (9) converges uniformly to f on $[0,1]\times[0,1]$, for each $f \in C_b([0,1]\times[0,1])$ and the rate of convergence in terms of mixed modulus of continuity was evaluated. Note that for $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 0$ the operator (9) reduces to the classical GBS Durrmeyer operators [9].

Dirik and Demirci [15] obtained important results regarding the approximation in a statistical sense of n-variate B-continuous functions, while Karakus and Demirci [20] report results regarding the statistical τ-approximation to B-continuous functions. Barbosu and Muraru investigated in [11] the approximation of B-continuous functions using q-GBS operators of Bernstein–Schurer–Stancu type. The GBS operators associated with the bivariate Chlodowsky–Szasz–Charlier-type operators and the bivariate Bernstein–Schurer–Kantorovich were investigated by Agrawal and Ispir [3] and Sidharth et al. [29].

In this study we introduce the GBS operators of bivariate Durrmeyer type and some estimates for these operators are obtained in the space of Bögel continuous functions. The paper has the following structure. Section 2 contains some general results regarding q-calculus and also the construction of GBS-Durrmeyer operators based on q-integers. The main results of the paper, which consist of some approximation properties of operators, are contained in Section 3. A uniform convergence theorem and an estimation of the degree of approximation in terms of mixed modulus of continuity are proved. Numerical considerations on the present topic are also presented.

2. Results regarding q-calculus and GBS Durrmeyer operators based on q-integers

In recent decades the applications of q-calculus have represented an important area of research in approximation theory using linear and positive operators. In the following we will recall some achievements in the topic on the bivariate case.

We mention that [10] was probably the first paper focused on Bernstein bivariate operators. Dogru and Gupta [16] constructed and studied a generalization of Meyer-König and Zeller bivariate operators based on q-integers. Agratini [2] presented two-dimensional extensions of some approximation processes, expressed by series. Orkcü [25] studied q-Szasz–Mirakjan–Kantorovich operators.

For any real number $q > 0$, the q-integer k, $k \in \mathbb{N}$, $k \geq 1$ is defined by

$$[k]_q = \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

The q-factorial and the q-binomial coefficients are introduced as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \dots [1]_q, & k = 1, 2, \dots, \\ 1, & k = 0 \end{cases}$$

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[k]_q!\,[m-k]_q!} \qquad (m \geq k \geq 0).$$

The q-analogue of $(x-a)^m$ is the polynomial

$$(x-a)_q^m = \begin{cases} (x-a)(x-qa)\dots(x-q^{m-1}a), & \text{if } m > 0, \\ 1, & \text{if } m = 0. \end{cases}$$

The q-analogue of integration in the interval $[0, a]$ is defined by

$$\int_0^a f(s) d_q s = (1-q) \sum_{m=1}^{\infty} f(aq^m) q^m, \quad 0 < q < 1.$$

Suppose $q_1 > 0$, $q_2 > 0$. The q-analogue of the double integral on the bivariate interval $[0, a] \times [0, b]$ is defined as

$$\int_0^a \int_0^b f(s,t) dq_1 s \, dq_2 t = (1-q_1)(1-q_2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(aq_1^m, bq_2^n) q_1^m q_2^n, \ \ 0 < q_1 < 1, \ 0 < q_2 < 1.$$

Denote

$$p_{m,k}(q,x) = \begin{bmatrix} m \\ k \end{bmatrix}_q x^k (1-x)_q^{m-k}.$$

Gupta [19] introduced the q-Durrmeyer operators as

$$D_{m,q}(f;x) = [m+1]_q \sum_{k=0}^{m} q^{-m} p_{m,k}(q;x) \int_0^1 f(s) p_{m,k}(q;qs) d_q s \tag{10}$$

for all positive integer m and $f \in C[0,1]$.

Let $I = [0,1]$, $q_1, q_2 \in (0,1)$ and $f \in C(I^2)$. The parametric extensions of (10) are the operators D^x_{m,q_1}, $D^y_{n,q_2} : C(I^2) \to C(I^2)$, defined for positive integers m, n, $f \in C(I^2)$, $(x,y) \in I^2$ as follows (see [8]):

$$D^x_{m,q_1}(f;x,y) = [m+1]_{q_1} \sum_{k=0}^{m} p_{m,k}(q_1;x) \int_0^1 f(s,y) p_{m,k}(q_1;q_1 s) d_{q_1} s, \tag{11}$$

$$D^{y}_{n,q_2}(f;x,y)=[n+1]_{q_2}\sum_{j=0}^{n}p_{n,j}(q_2;y)\int_0^1 f(x,t)p_{n,j}(q_2;q_2t)d_{q_2}t. \tag{12}$$

Lemma 2.1 *The operators (11), (12) are linear and positive. They commute on $C(I^2)$ and their product is the bivariate linear-positive operator $D_{m,n,q_1,q_2}:C(I^2)\to C(I^2)$ defined for positive integers m,n, $f\in C(I^2)$, and $(x,y)\in I^2$ as*

$$D_{m,n,q_1,q_2}(f;x,y)=[m+1]_{q_1}[n+1]_{q_2}\sum_{k=0}^{m}\sum_{j=0}^{n}q_1^{-k}q_2^{-j}p_{m,k}(q_1;x)p_{n,j}(q_2;y)$$

$$\cdot\int_0^1\int_0^1 f(s,t)p_{m,k}(q_1;q_1s)p_{n,j}(q_2;q_2t)d_{q_1}s\,d_{q_2}t. \tag{13}$$

Proof The assertions follow by direct computation taking into account the definitions (11) and (12). □

Remark 2.1 When $q_1=q_2=1$, the operators (13) reduce to the classical Durrmeyer bivariate operators from [9].

We define the GBS operator associated with the operator D_{m,n,q_1,q_2} as follows:

$$U_{m,n,q_1,q_2}(f;x,y)=D_{m,n,q_1,q_2}\left(f(\cdot,y)+f(x,\cdot)-f(\cdot,\cdot);x,y\right), \tag{14}$$

where $U_{m,n,q_1.q_2}$ is defined from the space $C_b(I^2)$ on itself and $f\in C_b(I^2)$.

Lemma 2.2 *For positive integers m,n, $f\in\mathbb{R}^I$, and $(x,y)\in I$ the following statement*

$$U_{m,n,q_1,q_2}(f;x,y)=[m+1]_{q_1}[n+1]_{q_2}\sum_{k=0}^{m}\sum_{j=0}^{n}q_1^{-k}q_2^{-j}p_{m,k}(q_1;x)p_{n,j}(q_2;y)$$

$$\cdot\int_0^1\int_0^1\{f(s,y)+f(x,t)-f(s,t)\}\,p_{m,k}(q_1;q_1s)p_{n,j}(q_2;q_2t)d_{q_1}sd_{q_2}t \tag{15}$$

holds.

Proof Taking into account the linearity of D_{m,n,q_1,q_2} and the definitions of parametric extensions D^x_{m,q_1}, D^y_{n,q_2}, from (14) one obtains

$$U_{m,n,q_1,q_2}(f;x,y)=D^x_{m,q_1}(f;x,y)+D^y_{n,q_2}(f;x,y)-D_{m,n,q_1,q_2}(f;x,y).$$

Next one makes use of (11) and (12). □

Remark 2.2 It can be easily observed that the GBS q-Durrmeyer operator is the Boolean sum of parametric extensions D^x_{m,q_1}, D^y_{n,q_2}, i.e.

$$U_{m,n,q_1,q_2}=D^x_{m,q_1}\oplus D^y_{n,q_2}=D^x_{m,q_1}+D^y_{n,q_2}-D_{m,n,q_1,q_2}.$$

3. A convergence theorem for the sequence U_{m,n,q_1,q_2}

In order to prove some approximation properties of the GBS q-Durrmeyer operators, we need the following result obtained by Gupta in [19].

Lemma 3.1 *The univariate q-Durrmeyer operators (10) verify*

i) $D_{m,q}(e_0;x) = 1$,

ii) $D_{m,q}(e_1;x) = \dfrac{1+qx[m]_q}{[m+2]_q}$,

iii) $D_{m,q}(e_2;x) = \dfrac{q^3x^2[m]_q([m]_q-1)+(1+q)^2qx[m]_q+1+q}{[m+3]_q[m+2]_q}$.

Farther on, we evaluate the images of the test functions by the bivariate q-Durrmeyer operator D_{m,n,q_1,q_2}.

Lemma 3.2 *For positive integers m,n, and $(x,y)\in I^2$ the operators D_{m,n,q_1,q_2} verify*

(i) $D_{m,n,q_1,q_2}(1;x,y) = 1$,

(ii) $D_{m,n,q_1,q_2}(s;x,y) = \dfrac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}$,

(iii) $D_{m,n,q_1,q_2}(t;x,y) = \dfrac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}$,

(iv) $D_{m,n,q_1,q_2}(st;x,y) = \dfrac{1+q_1x[m]_{q_1}}{[m+2]_{q_1}}\cdot\dfrac{1+q_2y[n]_{q_2}}{[n+2]_{q_2}}$,

(v) $D_{m,n,q_1,q_2}(s^2;x,y) = \dfrac{q_1^3x^2[m]_{q_1}([m]_{q_1}-1)+(1+q_1)^2q_1x[m]_{q_1}+1+q_1}{[m+3]_{q_1}[m+2]_{q_1}}$,

(vi) $D_{m,n,q_1,q_2}(t^2;x,y) = \dfrac{q_2^3y^2[n]_{q_2}([n]_{q_2}-1)+(1+q_2)^2q_2x[n]_{q_2}+1+q_2}{[n+3]_{q_2}[n+2]_{q_2}}$.

Corollary 3.1 *The following identities hold true:*

$$D_{m,n,q_1,q_2}((s-x)^2;x,y) = \frac{x^2\{[m]_{q_1}([m]_{q_1}-1)q_1^3-2[m]_{q_1}[m+3]_{q_1}q_1+[m+2]_{q_1}[m+3]_{q_1}\}}{[m+2]_{q_1}[m+3]_{q_1}} + \frac{x\{[m]_{q_1}q_1(1+q_1)^2-2[m+3]_{q_1}\}+1+q_1}{[m+2]_{q_1}[m+3]_{q_1}},$$

$$D_{m,n,q_1,q_2}((t-y)^2;x,y) = \frac{y^2\{[n]_{q_2}([n]_{q_2}-1)q_2^3-2[n]_{q_2}[n+3]_{q_2}q_2+[n+2]_{q_2}[n+3]_{q_2}\}}{[n+2]_{q_2}[n+3]_{q_2}} + \frac{y\{[n]_{q_2}q_2(1+q_2)^2-2[n+3]_{q_2}\}+1+q_2}{[n+2]_{q_2}[n+3]_{q_2}}.$$

The next result regarding the central moment operators can be obtained.

Lemma 3.3 *Let $m, n \in \mathbb{N}$, and $q_1, q_2 \in (0,1)$. Then*

i) $D_{m,n,q_1,q_2}\left((s-x)^2; x, y\right) \leq \dfrac{2}{[m+2]_{q_1}} \delta^2_{m,q_1}(x),$

ii) $D_{m,n,q_1,q_2}\left((t-y)^2; x, y\right) \leq \dfrac{2}{[n+2]_{q_2}} \delta^2_{n,q_2}(y),$

where $\delta^2_{m,q_1}(x) = \varphi^2(x) + \dfrac{3}{2[m+2]_{q_1}}$, $\delta^2_{n,q_2}(y) = \varphi^2(y) + \dfrac{3}{2[n+2]_{q_2}}$, $\varphi^2(x) = x(1-x)$, $x \in [0,1]$.

Theorem 3.1 *If $q_{1m}, q_{2n} \in (0,1)$ such that $\lim\limits_{m\to\infty} q_{1m} = 1$, $\lim\limits_{n\to\infty} q_{2n} = 1$, then the sequence $(U_{m,n,q_1,q_2} f)$ converges uniformly to f on I^2, for each $f \in C_b(I^2)$.*

Proof Applying Lemma 3.2 we get that $D_{m,n,q_{1m},q_{2n}}(1; x, y) = 1$, $\lim\limits_{m,n\to\infty} D_{m,n,q_{1m},q_{2n}}(s; x, y) = x$, $\lim\limits_{m,n\to\infty} D_{m,n,q_{1m},q_{2n}}(t; x, y) = y$, $\lim\limits_{m,n\to\infty} D_{m,n,q_{1m},q_{2n}}(s^2+t^2; x, y) = x^2 + y^2$, uniformly on I^2. Next one applies Theorem 1.1 and the result is obtained. □

In the next part of this section we will consider some numerical results, which show the rate of convergence of the GBS operator associated with the classical Durrmeyer bivariate operator D_{m,n,q_1,q_2} to a certain function, using MATLAB algorithms.

Example 3.1 *We consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2x^2y^2 + x^2y - 3y^2$. The convergence of the bivariate q-Durrmeyer operator to the function f is illustrated in Figures 1 and 2, respectively, for $m = n = 40, q_1 = q_2 = 0.6$ and $m = n = 600, q_1 = q_2 = 0.9$, respectively. We remark that as the values of m and n increase, the error in the approximation of the function by the operator becomes smaller.*

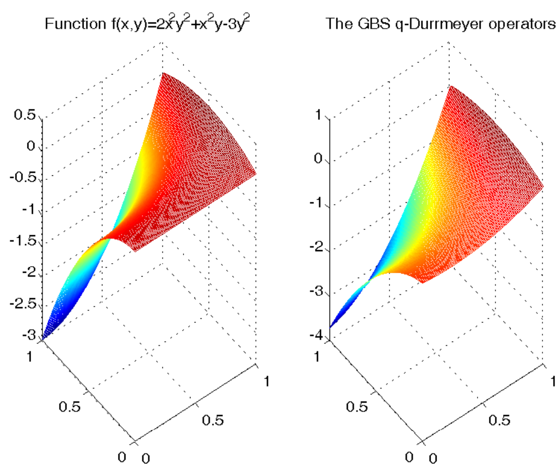

Figure 1. Approximation process by U_{m,n,q_1,q_2} for $m = n = 40$ and $q_1 = q_2 = 0.6$.

Example 3.1 *We consider $f(x,y) = 2x^2y^2 + x^2y - 3y^2$, $m = n = 50$, and $q_1 = q_2 = 0.9$. In the Table we computed the error of approximation for U_{m,n,q_1,q_2} and D_{m,n,q_1,q_2} at certain points.*

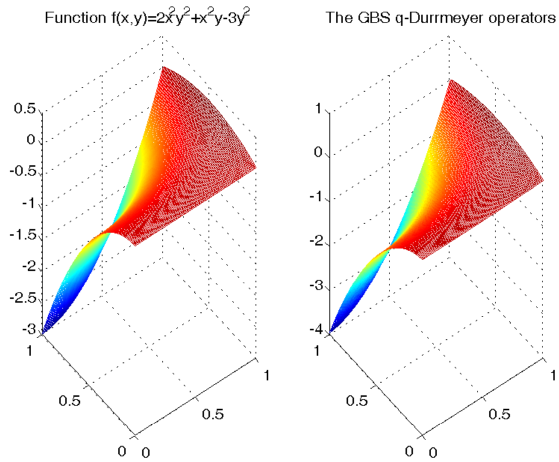

Figure 2. Approximation process by U_{m,n,q_1,q_2} for $m=n=600$ and $q_1=q_2=0.9$.

Table. Error of approximation for U_{m,n,q_1,q_2} and D_{m,n,q_1,q_2}.

(x,y)	$\lvert U_{m,n,q_1,q_2}(f;x,y)-f(x,y)\rvert$	$\lvert D_{m,n,q_1,q_2}(f;x,y)-f(x,y)\rvert$
$(0,0)$	0.021441576378817	0.036013234514013
$(0.8,0)$	0.013884346886559	0.045211880347349
$(1,0)$	0.000105774629589	0.081161827530114
$(1,0.2)$	0.000097946005677	0.009417677736995
$(1,0.4)$	0.000005127145007	0.034705500790987
$(1,0.6)$	0.000203444822466	0.051207708053833
$(0.4,0.8)$	0.533813793623424	0.576111343867509
$(0.60.8)$	0.396373758962003	0.457661346082196
$(0.8,0.8)$	0.218601496763094	0.252261250109179
$(1,0.8)$	0.000497007026698	0.040088944051543
$(0,1)$	0.979937914813845	1.040046583201666
$(0.2,1)$	0.888576699988353	1.102577828625508
$(0.4,1)$	0.744990882469993	1.029703872138851
$(0.6,1)$	0.549180462258765	0.821424713741695
$(0.8,1)$	0.301145439354669	0.477740353434038
$(1,1)$	0.000885813757705	0.001349208784118

From the above results it follows that U_{m,n,q_1,q_2} *converge faster than* D_{m,n,q_1,q_2} *to the function* $f(x,y)=2x^2y^2+x^2y-3y^2$ *at certain points.*

Regarding the rate of convergence, we have the next result.

Theorem 3.2 *Suppose* $q_{1m},q_{2n}\in(0,1)$ *such that* $\lim_{m\to\infty} q_{1m}=1$, $\lim_{n\to\infty} q_{2n}=1$. *The following inequality*

$$|f(x,y)-U_{m,n,q_{1m},q_{2n}}(f;x,y)|\leq 4\omega_{mixed}(f;\delta_{1m}(x)^{1/2},\delta_{2n}(y)^{1/2})$$

holds for each $f \in C_b(I^2)$, $(x,y) \in I^2$, *where*

$$\delta_{1m}(x) = \frac{2}{[m+2]_{q_{1m}}}\left(x(1-x) + \frac{3}{2[m+2]_{q_{1m}}}\right),$$

$$\delta_{2n}(y) = \frac{2}{[n+2]_{q_{2n}}}\left(y(1-y) + \frac{3}{2[n+2]_{q_{2n}}}\right).$$

Proof Applying Theorem 1.2 one obtains

$$|f(x,y) - U_{m,n,q_{1m},q_{2n}}(f;x,y)|$$

$$\leq \Big\{1 + \delta_1^{-1}\sqrt{D_{m,n,q_{1m},q_{2n}}((s-x)^2;x,y)} + \delta_2^{-1}\sqrt{D_{m,n,q_{1m},q_{2n}}((t-y)^2;x,y)}$$

$$+\delta_1^{-1}\delta_2^{-1}\sqrt{D_{m,n,q_{1m},q_{2n}}((s-x)^2;x,y)D_{m,n,q_{1m},q_{2n}}((t-y)^2;x,y)}\Big\}\omega_{mixed}(f;\delta_1,\delta_2)$$

for all $f \in C_b(I^2)$, $(x,y) \in I^2$, and $\delta_1, \delta_2 \in [0,+\infty)$.

Considering Lemma 3.3 we obtain the degree of local approximation for B-continuous functions. □

Let

$$Lip_M(\alpha,\beta) = \left\{f \in C_b(I^2) : |\Delta_{x,y}f[s,t;x,y]| \leq M|t-x|^\alpha|s-y|^\beta, \text{ for } (s,t),(x,y) \in I^2\right\}$$

be the Lipschitz class of B-continuous functions. The next result gives the rate of convergence of the operators $U_{m,n,q_{1m},q_{2n}}$ in terms of the Lipschitz class.

Theorem 3.3 *Let* $q_{1m}, q_{2n} \in (0,1)$ *such that* $\lim\limits_{m\to\infty} q_{1m} = 1$, $\lim\limits_{n\to\infty} q_{2n} = 1$. *If* $f \in Lip_M(\alpha,\beta)$, *then*

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| \leq M(\delta_{1m}(x))^{\alpha/2}(\delta_{2n}(y))^{\beta/2},$$

where $\delta_{1m}(x)$ *and* $\delta_{2n}(y)$ *are defined in Theorem 3.2.*

Proof Using the definition of the operators $U_{m,n,q_{1m},q_{2n}}$ we can write

$$U_{m,n,q_{1m},q_{2n}}(f;x,y) = D_{m,n,q_{1m},q_{2n}}(f(s,y) + f(x,t) - f(s,t);x,y)$$

$$= f(x,y)D_{m,n,q_{1m},q_{2n}}(e_{00};x,y) - D_{m,n,q_{1m},q_{2n}}\left(\Delta_{x,y}f[s,t;x,y];x,y\right).$$

Therefore,

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| \leq D_{m,n,q_{1m},q_{2n}}\left(|\Delta_{x,y}f[s,t;x,y]|;x,y\right)$$

$$\leq MD_{m,n,q_{1m},q_{2n}}(|t-x|^\alpha|s-y|^\beta;x,y)$$

$$\leq MD_{m,q_{1m}}\left(|t-x|^\alpha;x\right)D_{n,q_{2n}}(|s-y|^\beta;y).$$

Applying Hölder's inequality with $(p,q) = \left(\frac{2}{\alpha}, \frac{2}{2-\alpha}\right)$, respectively $(p,q) = \left(\frac{2}{\beta}, \frac{2}{2-\beta}\right)$, we have

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| \leq M\left[D_{m,q_{1m}}\left((t-x)^2;x\right)\right]^{\alpha/2}\left[D_{n,q_{2n}}((s-y)^2;y)\right]^{\beta/2}$$

$$\leq M(\delta_{1m}(x))^{\alpha/2}(\delta_{2n}(y))^{\beta/2}.$$

□

Theorem 3.4 *Suppose* $q_{1m}, q_{2n} \in (0,1)$, $\lim\limits_{m\to\infty} q_{1m} = \lim\limits_{n\to\infty} q_{2n} = 1$, $\lim\limits_{m\to\infty} q_{1m}^m = a_1 \in [0,1)$, $\lim\limits_{n\to\infty} q_{2n}^n = a_2 \in [0,1)$. *Let* $f \in D_b(I^2)$ *with* $D_B f \in B(I^2)$. *There exists a constant* $M > 0$ *such that for each* $(x,y) \in I^2$ *we have*

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| \le \frac{M}{[m]_{q_{1m}}^{1/2}[n]_{q_{2n}}^{1/2}}\left\{\|D_B f\|_\infty + \omega_{mixed}\left(D_B f; [m]_{q_{1m}}^{1/2}, [n]_{q_{2n}}^{1/2}\right)\right\}.$$

Proof Using the linearity of $D_{m,n,q_{1m},q_{2n}}$ we obtain

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| = |D_{m,n,q_{1m},q_{2n}}\left(\Delta_{x,y} f[s,t;x,y];x,y\right)|. \tag{16}$$

Since $D_B f(s,t) = \lim\limits_{(x,y)\to(s,t)} \dfrac{\Delta_{x,y} f[s,t;x,y]}{(x-s)(y-t)}$ for $f \in D_b(I^2)$ with $D_B f \in B(I^2)$, it follows

$$\Delta_{x,y} f[s,t;x,y] = (x-s)(y-t) D_B f(\xi,\eta),$$

where ξ is between x and s and η is between y and t.

Using the following relation

$$D_B f(\xi,\eta) = \Delta_{x,y} D_B f\left[\xi,\eta;x,y\right] + D_B f(\xi,y) + D_B f(x,\eta) - D_B f(x,y),$$

we can write

$$\begin{aligned}
|D_{m,n,q_{1m},q_{2n}}\left(\Delta_{x,y}[f;s,t;x,y];x,y\right)| &= |D_{m,n,q_{1m},q_{2n}}\left((s-x)(t-y)D_B f(\xi,\eta);x,y\right)| \\
&\le D_{m,n,q_{1m},q_{2n}}\left(|s-x||t-y||\Delta_{x,y} D_B f[\xi,\eta;x,y]|;x,y\right) \\
&+ D_{m,n,q_{1m},q_{2n}}\left(|s-x||t-y|\left(|D_B f(\xi,y)| + |D_B f(x,\eta)| + |D_B f(x,y)|\right);x,y\right) \\
&\le D_{m,n,q_{1m},q_{2n}}\left(|s-x||t-y|\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|);x,y\right) \\
&+ 3\|D_B f\|_\infty D_{m,n,q_{1m},q_{2n}}\left(|s-x||t-y|;x,y\right).
\end{aligned}$$

The mixed modulus of smoothness ω_{mixed} verified

$$\begin{aligned}
\omega_{mixed}\left(D_B f;|\xi-x|,|\eta-y|\right) &\le \omega_{mixed}\left(D_B f;|s-x|,|t-y|\right) \\
&\le \left(1+\delta_m^{-1}|s-x|\right)\left(1+\delta_n^{-1}|t-y|\right)\omega_{mixed}(D_B f;\delta_m,\delta_n).
\end{aligned}$$

Therefore

$$\begin{aligned}
|D_{m,n,q_{1m},q_{2n}}\left(\Delta_{x,y} f[s,t;x,y];x,y\right)| &\le D_{m,n,q_{1m},q_{2n}}\Big((|s-x||t-y| + \delta_n^{-1}|s-x||t-y|^2 \\
&+ \delta_m^{-1}|s-x|^2|t-y| + \delta_m^{-1}\delta_n^{-1}|s-x|^2|t-y|^2)\omega_{mixed}(D_B f;\delta_m,\delta_n);x,y\Big) \\
&+ 3\|D_B f\|_\infty D_{m,n,q_{1m},q_{2n}}\left(|s-x||t-y|;x,y\right)
\end{aligned} \tag{17}$$

From the relations (16) and (17) and applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| \le \Big\{\left[D_{m,n,q_{1m},q_{2n}}((s-x)^2(t-y)^2;x,y)\right]^{1/2} \\
&+ \delta_n^{-1}\left[D_{m,n,q_{1m},q_{2n}}\left((s-x)^2(t-y)^4;x,y\right)\right]^{1/2} + \delta_m^{-1}\left[D_{m,n,q_{1m},q_{2n}}\left((s-x)^4(t-y)^2;x,y\right)\right]^{1/2} \\
&+ \delta_m^{-1}\delta_n^{-1}\left[D_{m,n,q_{1m},q_{2n}}\left((s-x)^4(t-y)^4;x,y\right)\right]^{1/2}\Big\}\omega_{mixed}(D_B f;\delta_m,\delta_n) \\
&+ 3\|D_B f\|_\infty\left[D_{m,n,q_{1m},q_{2n}}\left((s-x)^2(t-y)^2;x,y\right)\right]^{1/2}.
\end{aligned}$$

If $0 < q_m < 1$, $q_m \to 1$ and $q_m^m \to a$, $a \in [0,1)$ as $m \to \infty$, then

$$\lim_{n\to\infty} [m]_{q_m} D_{m,q_m}((s-x)^2;x) = 2x(1-x),$$
$$\lim_{n\to\infty} [m]_{q_m}^2 D_{m,q_m}((s-x)^4;x) = 12x^2(1-x)^2.$$

Choosing $\delta_m = \frac{1}{[m]_{q_m}^{1/2}}$, $\delta_n = \frac{1}{[n]_{q_n}^{1/2}}$ we obtain

$$|U_{m,n,q_{1m},q_{2n}}(f;x,y) - f(x,y)| = \mathcal{O}\left(\frac{1}{[m]_{q_m}^{1/2}}\right)\mathcal{O}\left(\frac{1}{[n]_{q_n}^{1/2}}\right)\omega_{mixed}\left(D_B f; [m]_{q_m}^{1/2}, [n]_{q_n}^{1/2}\right)$$
$$+ 3\|D_B f\|_\infty \mathcal{O}\left(\frac{1}{[m]_{q_m}^{1/2}}\right)\mathcal{O}\left(\frac{1}{[n]_{q_n}^{1/2}}\right),$$

and the Theorem is proved. □

Acknowledgment

The second author was financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2015-01, No.2032/7. The work of the third author was supported by the project "Bacău and Lugano - Teaching Informatics for a Sustainable Society", co-financed by Switzerland through the Swiss–Romanian Cooperation Programme to reduce economic and social disparities within the enlarged European Union.

References

[1] Acu AM, Muraru C, Radu V, Sofonea F. Some approximation properties of a Durrmeyer variant of q-Bernstein-Schurer operators. Math Methods Appl Sci; DOI: 10.1002/mma.3949.

[2] Agratini, O. Bivariate positive operators in polynomial weighted spaces. Abstr Appl Anal 2013; 8 pp.; Article ID 850760.

[3] Agrawal PN, Ispir N. Degree of approximation for bivariate Chlodowsky-Szasz-Charlier type operators. Results Math; Doi:10.1007/s00025-015-0495-6.

[4] Badea I. Modulul de continuitate în sens Bögel şi unele aplicaţii în aproximarea printr-un operator Bernstein. Stud Univ Babeş-Bolyai Math 1973; 4: 69-78 (article in Romanian).

[5] Badea C, Badea I, Gonska H. A test function theorem and approximation by pseudopolynomials. Bull Aust Math Soc 1986; 34: 53-64.

[6] Badea C, Cottin C. Korovkin type theorems for generalized Boolean Sum Operators. Approximation Theory-Conference Proceedings, Kecskemet, Hungary 1990. Colloquia Mathematica Soc Janos Bolyai 1991; 58: 51-67.

[7] Badea C, Badea I, Cottin C, Gonska H. Notes on the degree of approximation of B-continuous and B-differentiable functions. J Approx Theory Appl 1988; 4: 95-108.

[8] Barbosu D. Aproximarea funcţiilor de mai multe variabile prin sume booleene de operatori liniari de tip interpolator. Cluj-Napoca: Ed Risoprint, 2002 (book in Romanian).

[9] Barbosu D. Polynomial approximation by means of Schurer-Stancu type operators. Baia Mare: Ed Univ de Nord, 2006 (book in Romanian).

[10] Barbosu D. Some generalized bivariate Bernstein operators. Miskolc Math Notes 2000; 1: 3-10.

[11] Barbosu D, Muraru CV. Approximating B-continuous functions using GBS operators of Bernstein-Schurer-Stancu type based on q-integers. Appl Math Comput 2015; 259: 80-87.

[12] Bögel K. Mehrdimensionale Differentiation von Funktionen mehrer Verenderlicher. J Reine Angew Math 1934; 170: 197-217.

[13] Bögel K. Über die mehrdimensionale differentiation, integration und beschränkte variation. J Reine Angew Math 1935; 173: 5-29.

[14] Derriennic MM. Sur l'approximation de fonctions integrable sur $[0, 1]$ par des polynômes de Bernstein modifies. J Approx Theory 1981; 31: 325-343 (in French).

[15] Dirik F, Demirci K. Approximation in statistical sense to n-variate B-continuous functions by positive linear operators. Math Slovaca 2010; 60: 877-886.

[16] Dogru O, Gupta V. Korovkin-type approximation properties of bivariate q-Meyer-König and Zeller operators. Calcolo 2006; 43: 51-63.

[17] Durrmeyer KL. Une formule d'inversion de la transformée de Laplace: Applications á la theorie des moments, Thése de 3 e cycle, Faculté des Sciences de l'Université de Paris, 1967 (in French).

[18] Gairola AR, Deepmala, Mishra LN. Rate of Approximation by Finite Iterates of q-Durrmeyer Operators, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 2016, doi: 10.1007/s40010-016-0267-z.

[19] Gupta V. Some approximation properties of q-Durrmeyer operators. Appl Math Comput 2008; 197: 172-178.

[20] Karakus S, Demirci K. Statistical τ-approximation to Bögel type continuous functions. Studia Sci Math Hungar 2011; 48: 475-488.

[21] Mishra VN, Khatri K, Mishra LN, Deepmala. Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators. J Inequal Appl 2013; 2013:586; doi:10.1186/1029-242X-2013-586.

[22] Mishra VN, Khatri K, Mishra LN. Statistical approximation by Kantorovich type Discrete $q-$Beta operators. Adv Difference Equ 2013; 2013:345; DOI: 10.1186/10.1186/1687-1847-2013-345.

[23] Mishra VN, Khan HH, Khatri K, Mishra LN. Hypergeometric representation for Baskakov-Durrmeyer-Stancu type operators. Bull Math Anal Appl 2013; 5: 18-26.

[24] Muraru C, Acu AM. Some approximation properties of q-Durrmeyer-Schurer operators. Sci Stud Res Ser Math Inform 2013; 23: 77-84.

[25] Orkcű M. q-Szasz-Mirakjan-Kantorovich operators of functions of two variables in weighted spaces. Abstr Appl Anal 2013; Article ID 823803, 8 pp.

[26] Pop OT, Barbosu D. Durrmeyer-Stancu type operators. Rev Anal Numer Theor Approx 2010; 39: 150-155.

[27] Pop OT, Barbosu D. GBS operators of Durrmeyer-Stancu type. Miskolc Math Notes 2008; 9: 53-60.

[28] Shisha O, Mond B. The degree of convergence of linear positive operators. Proc Natl Acad Sci USA 1968; 60: 1196-1200.

[29] Sidharth M, Ispir N, Agrawal PN. GBS operators of Bernstein-Schurer-Kantorowich type based on q-integers. Appl Math Comput; Doi: 10.1016/j.amc.2015.07.052.

[30] Stancu DD. Asupra unei generalizări a polinoamelor lui Bernstein. Stud Univ Babeş-Bolyai Math 1969; 14: 31-45 (article in Romanian).