Structural stability analysis of solutions to the initial boundary value problem for a nonlinear strongly damped wave equation

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Abstract: In this paper the initial-boundary value problem for a nonlinear strongly damped wave equation is considered. We analyze the structural stability of solutions of the nonlinear strongly damped wave equation with coefficients from $H^1(\Omega)$.

Key words: Structural stability, continuous dependence, strongly damped, nonlinear wave equation

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ whose boundary $\partial \Omega$ is assumed to be class $C^2$. The model considered here is given as the following initial-boundary value problem:

$$u_{tt} - \Delta u + \beta |u_t|^2 u_t = \alpha \Delta u_t, \quad x \in \Omega, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega$$

$$u = 0, \quad x \in \partial \Omega, \quad t > 0,$$

where $\alpha$ and $\beta$ are positive constants.

Basically, in such a style of models, continuous dependence of solutions on the given coefficients reflects the effect of small changes on the solutions that eventually yields the structural stability result [4].

The term $\alpha \Delta u_t$ indicates that the stress is proportional not only to the strain, as with Hooke’s law, but also to the strain rate as in a linearized Kelvin material [9].

Many works on strongly damped nonlinear wave equations have been carried out at different levels. In 1980, Webb [14] considered the following problem:

$$w_{tt} - \alpha \Delta w_t - \Delta w = f(w), \quad t > 0,$$

$$w(x, 0) = \phi(x), \quad x \in \Omega$$

$$w_t (x, 0) = \psi(x), \quad x \in \Omega$$

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He firstly established the existence of a unique strong global solution to (1.4) and then he analyzed the behavior of this solution as \( t \to \infty \).

In [2], Massatt investigated both the existence and the limiting behavior for the equation \( u_{tt} + Au_t + Au = f(t, u, u_t) \), where \( A \) is a sectoral operator and \( f \) satisfies certain regularity and growth assumptions, being periodic in \( t \).

In [3], the authors considered the long-time behavior of a strongly damped nonlinear wave equation and showed that the initial boundary value problem has a global solution and that there exists a compact global attractor with finite dimension.

In [4], the authors investigated the existence and uniqueness of solutions of the following equation of hyperbolic type with a strong dissipation:

\[
\begin{align*}
\Delta u(t, x) &+ \beta \nu (t) \nabla u(t, x) \cdot \nabla u(t, x) = 0, & x \in \Omega, \ t \geq 0 \\
\partial_{\nu} u(t, x) &+ \alpha u(t, x) = 0, & x \in \partial \Omega.
\end{align*}
\]

Then, in [6], Çelebi and Ügür introduced and analyzed these solutions on coefficients \( \alpha \) and \( \beta \). The proof relies on energy-type a priori estimates.

Throughout this article we denote by \( \| \cdot \| \) the norm in \( L^2(\Omega) \).

2. Essential inequality

**Theorem 1** For every \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\), the solution \((u, u_t)\) to (1.1)-(1.3) satisfies the following inequality:

\[
\| \nabla u_t \|^2 \leq D_1.
\]  

(2.1)

Here \( D_1 \) is a positive constant, depending on the initial values of (1.1).

**Proof** We first multiply (1.1) by \(-\Delta u_t\) and integrate over \( \Omega \). Then we have

\[
\frac{d}{dt} \left[ \frac{1}{2} \| \nabla u_t \|^2 + \frac{1}{2} \| \Delta u \|^2 \right] \leq 0.
\]

(2.2)

It follows from (2.2) that

\[
E_1(t) = \frac{1}{2} \| \nabla u_t \|^2 + \frac{1}{2} \| \Delta u \|^2 \leq E_1(0).
\]

(2.3)

Hence, (2.1) follows from (2.3).
3. Continuous dependence on coefficient $\alpha$

We consider the following problems.

$$u_{tt} - \Delta u + \beta |u_t|^2 u_t = \alpha_1 \Delta u_t, \ x \in \Omega, t > 0 \quad (3.1)$$

$$u(x, 0) = 0, u_t(x, 0) = 0, \ x \in \Omega \quad (3.2)$$

$$w|_{\partial \Omega} = 0, \ x \in \partial \Omega, t > 0 \quad (3.3)$$

$$v_{tt} - \Delta v + \beta |v_t|^2 v_t = \alpha_2 \Delta v_t, \ x \in \Omega, t > 0 \quad (3.4)$$

$$v(x, 0) = 0, v_t(x, 0) = 0, \ x \in \Omega \quad (3.5)$$

$$v|_{\partial \Omega} = 0, \ x \in \partial \Omega, t > 0 \quad (3.6)$$

Assume that $u$ is a solution of (3.1)–(3.3) and $v$ is a solution of (3.4)–(3.6). We define the variables $w$ and $\alpha$ by $w = u - v$ and $\alpha = \alpha_1 - \alpha_2$. It is easy to see that $w$ satisfies the following initial boundary value problem:

$$w_{tt} - \Delta w + \beta \left(|u_t|^2 u_t - |v_t|^2 v_t \right) = \alpha_1 \Delta w_t + \alpha \Delta v_t, \ x \in \Omega, t > 0; \quad (3.7)$$

$$w(x, 0) = 0, w_t(x, 0) = 0, \ x \in \Omega; \quad (3.8)$$

$$w|_{\partial \Omega} = 0, \ x \in \partial \Omega, t > 0. \quad (3.9)$$

**Theorem 2** Let $w$ be the solution to (3.7)–(3.9). Then $w$ satisfies the estimate

$$\|w_t\|^2 + \|\nabla w\|^2 \leq M_1 (\alpha_1 - \alpha_2)^2 t, \forall t > 0,$$

where $M_1$ is a positive constant and depends on the initial data and the parameters of (1.1).

**Proof** If we multiply (3.7) by $w_t$ and integrate over $\Omega$ we get the relation

$$\frac{d}{dt} \left[ \frac{1}{2} \|w_t\|^2 + \frac{1}{2} \|\nabla w\|^2 \right] + \alpha_1 \|\nabla w_t\|^2 + \beta \int_{\Omega} \left(|u_t|^2 u_t - |v_t|^2 v_t \right) w_t dx +$$

$$\alpha \int_{\Omega} \nabla w_t \nabla v_t dx = 0. \quad (3.10)$$

It can be easily shown that

$$\left(|u_t|^2 u_t - |v_t|^2 v_t \right) w_t \geq 0. \quad (3.11)$$
Indeed,

\[
\left( |u_t|^2 u_t - |v_t|^2 v_t \right) w_t = \frac{1}{2} |u_t|^2 (u_t - v_t) w_t - \frac{1}{2} |v_t|^2 v_t w_t + \\
\frac{1}{2} |u_t|^2 u_t w_t + \frac{1}{2} |v_t|^2 (u_t - v_t - u_t) w_t
\]

\[
= \frac{1}{2} |u_t|^2 w_t w_t + \frac{1}{2} |v_t|^2 w_t w_t + \frac{1}{2} v_t w_t \left( |u_t|^2 - |v_t|^2 \right) + \frac{1}{2} u_t w_t \left( |u_t|^2 - |v_t|^2 \right)
\]

\[
= \frac{1}{2} w_t^2 \left( |u_t|^2 + |v_t|^2 \right) + \frac{1}{2} \left( |u_t| + |v_t| \right)^2 (|u_t| - |v_t|)^2.
\]

Now using Cauchy–Schwarz and \(\varepsilon\)-Young inequalities with (3.11) we obtain from (3.10) that

\[
\frac{d}{dt} E_2(t) + \left( \alpha_1 - \frac{\varepsilon}{2} \right) \| \nabla w_t \|^2 \leq \frac{\alpha_1^2}{2 \varepsilon_1} \| \nabla u_t \|^2,
\]

where

\[
E_2(t) = \frac{1}{2} \| w_t \|^2 + \frac{1}{2} \| \nabla w \|^2.
\]

Taking into account (2.1) we get

\[
\frac{d}{dt} E_2(t) \leq \frac{\alpha_1^2}{\alpha_1} D_1 t,
\]

which gives

\[
E_2(t) \leq \frac{\alpha_1^2}{\alpha_1} D_1 t.
\]

Hence, the statement of the theorem holds.

\[\square\]

4. Continuous dependence on coefficient \(\beta\)

We consider the following problems:

\[
u_{tt} - \Delta u + \beta_1 |u_t|^2 u_t = \alpha \Delta u_t, \quad x \in \Omega, t > 0; \tag{4.1}\]

\[
u(x, 0) = 0, u_t(x, 0) = 0, \quad x \in \Omega; \tag{4.2}\]

\[
u_{|\partial \Omega} = 0, \quad x \in \partial \Omega, t > 0; \tag{4.3}\]

\[
u_{tt} - \Delta v + \beta_2 |v_t|^2 v_t = \alpha \Delta v_t, \quad x \in \Omega, t > 0; \tag{4.4}\]

\[
u(x, 0) = 0, v_t(x, 0) = 0, \quad x \in \Omega; \tag{4.5}\]

\[
u_{|\partial \Omega} = 0, \quad x \in \partial \Omega, t > 0. \tag{4.6}\]

Let \(u\) be a solution of (4.1)–(4.3) and \(v\) be a solution of (4.4)–(4.6). Similar to the argument followed in the previous section, we define the variables \(w\) and \(\beta\) as \(w = u - v\) and \(\beta = \beta_1 - \beta_2\). Then \(w\) satisfies the following initial boundary value problem:

\[
w_{tt} - \Delta w + \beta_1 \left( |u_t|^2 u_t - |v_t|^2 v_t \right) + \beta |v_t|^2 v_t = \alpha \Delta w_t, \quad x \in \Omega, t > 0; \tag{4.7}\]
Now the following theorem establishes continuous dependence of the solution to (1.1)–(1.3) on the coefficient \( \beta \) in \( H^1(\Omega) \).

**Theorem 3** Let \( w \) be the solution to (4.7)–(4.9). Then \( w \) satisfies the estimate

\[
\| w_t \|^2 + \| \nabla w \|^2 \leq M_2(\epsilon^t - 1)(\beta_1 - \beta_2)^2, \forall t > 0
\]

where \( M_2 \) is a positive constant, depending on the parameters of (1.1).

**Proof** Multiplying (4.7) by \( w_t \) and integrating over \( \Omega \), we obtain

\[
\frac{d}{dt} E_2(t) + \alpha \| \nabla w_t \| - 2 + \beta_1 \int_\Omega \left( |u_t|^2 u_t - |v_t|^2 v_t \right) w_t dx + \beta \int_\Omega |v_t|^3 w_t dx = 0. \tag{4.10}
\]

Using (3.11) in (4.10) we obtain

\[
\frac{d}{dt} E_2(t) \leq |\beta| \int_\Omega |v_t|^3 |w_t| dx. \tag{4.11}
\]

Using the Cauchy–Schwarz and the Cauchy inequalities we can estimate the term \( |\beta| \int_\Omega |v_t|^3 |w_t| dx \) as follows:

\[
|\beta| \int_\Omega |v_t|^3 |w_t| dx \leq |\beta| \left( \int_\Omega |v_t|^6 dx \right)^{\frac{1}{2}} \left( \int_\Omega |w_t|^2 dx \right)^{\frac{1}{2}}
\leq \frac{|\beta|^2}{2} \int_\Omega |v_t|^6 dx + \frac{1}{2} \int_\Omega |w_t|^2 dx = \frac{|\beta|^2}{2} \| v_t \|_6^6 + \frac{1}{2} \| w_t \|^2. \tag{4.12}
\]

Taking into account (4.12) in (4.11) we get

\[
\frac{d}{dt} E_2(t) \leq E_2(t) + \frac{|\beta|^2}{2} \| v_t \|_6^6. \tag{4.13}
\]

If we use the Sobolev inequality for the second term of (4.13) and consider (2.1) we have from (4.13) that

\[
\| v_t \|_6^6 \leq c \| \nabla v_t \|_2^6 \leq c D_1^3 = c_1, \tag{4.14}
\]

since

\[
\frac{d}{dt} E_2(t) \leq E_2(t) + \beta^2 c_2 \tag{4.15}
\]

where \( c_2 = \frac{c}{4} \). Solving the first-order differential inequality (4.15), we obtain

\[
E_2(t) \leq c_2(\epsilon^t - 1)\beta^2,
\]

which gives that \( \| \nabla w \| \to 0 \) as \( \beta \to 0 \), \( t > 0 \) and hence the proof is completed. \( \square \)
5. Conclusion
In this article, by using the multiplier method, we conclude that the solution of the problem (1.1)–(1.3) describing a strongly damped nonlinear wave equation is continuously dependent on the coefficients $\alpha$ and $\beta$.

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References