Some upper bounds on the dimension of the Schur multiplier of a pair of nilpotent Lie algebras

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Abstract: Let \((L, N)\) be a pair of Lie algebras where \(N\) is an ideal of the finite dimensional nilpotent Lie algebra \(L\). Some upper bounds on the dimension of the Schur multiplier of \((L, N)\) are obtained without considering the existence of a complement for \(N\). These results are applied to derive a new bound on the dimension of the Schur multiplier of a nilpotent Lie algebra.

Key words: Pair of Lie algebras, Schur multiplier, nilpotent Lie algebra

1. Introduction
Throughout this paper, we denote by \((L, N)\) a pair of Lie algebras where \(N\) is an ideal of the Lie algebra \(L\). The Schur multiplier of the pair \((L, N)\) is defined to be the abelian Lie algebra \(\mathcal{M}(L, N)\), whose principal feature is the following natural exact sequence of Lie algebras:

\[
\begin{align*}
H_3(L) & \to H_3(L/N) \to \mathcal{M}(L, N) \to H_2(L) \to H_2(L/N) \\
& \to N/[N, L] \to H_1(L) \to H_1(L/N) \to 0,
\end{align*}
\]

where \(H_i(\cdot)\) is the \(i\)-th Chevalley–Eilenberg homology group of a Lie algebra. From the homotopical point of view, \(\mathcal{M}(L, N)\) is the second relative homology of \((L, N)\), see [3, 4] for more details and a brief introduction. Taking \(N = L\) we find that \(\mathcal{M}(L, N) = H_2(L)\), which is called the Schur multiplier of \(L\) and denoted by \(\mathcal{M}(L)\).

Determining bounds on the dimension of the Schur multiplier of a (nilpotent) Lie algebra was a hot topic in recent decades. Nilpotent Lie algebras have been widely discussed in the literature in order to be classified by their multipliers; however, there are many other interesting open problems on the dimension of the homology groups of nilpotent Lie algebras; see [1, 2, 5, 6, 8] for instance.

Most of the bounds that have been obtained on the dimension of the Schur multiplier of the pair \((L, N)\) are just generalizations of a previously known bound on the dimension of the Schur multiplier of \(L\). In the most discussed case, authors have considered that the ideal \(N\) is complemented in \(L\). Thus, the morphisms \(H_i(L) \to H_i(L/N)\) split for any \(i\), and \(\mathcal{M}(L, N)\) is a complement of \(H_2(L/N)\) in \(H_2(L)\). Therefore, if \(L \cong F/R\) and \(N \cong S/R\) are arbitrary free presentations of \(L\) and \(N\) respectively, then by Hopf’s formula we have

\[
\mathcal{M}(L) = (R \cap [F, F])/[R, F] \quad \text{and} \quad \mathcal{M}(L/N) = (S \cap [F, F])/[F, S].
\]

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This dedicates the free presentation \((R \cap [S, F])/[R, F]\) for \(\mathcal{M}(L, N)\) that applies to determine the bounds; see [9, 12] for instance.

By assuming that \(N\) admits a complement in \(L\), the following theorem was proved in [12]. We use different tools to eliminate this limitation and give a similar bound that can widely extend some results of [11, 12].

**Theorem A.** Let \(L\) be a finite dimensional nilpotent Lie algebra and \(N\) an ideal of \(L\). Then

\[
\dim(\mathcal{M}(L, N)) \leq \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + \dim([L, N]) (d(\frac{L}{Z(L, N)}) - 1),
\]

where \(d(X)\) is the minimal number of generators of a Lie algebra \(X\) and \(Z(L, N) = \{ n \in N \mid [l, n] = 0, \text{ for all } l \in L \} = Z(L) \cap N\).

It was shown in [13] that if \(L\) is a nilpotent Lie algebra then \(\dim(\mathcal{M}(L)) + \dim(L^2) \leq \dim(L) d(L)\). The following theorem can be a generalization of this bound on the dimension of \(\mathcal{M}(L, N)\).

**Theorem B.** Let \(L\) be a finite dimensional nilpotent Lie algebra with an ideal \(N\). Then

\[
\dim(\mathcal{M}(L, N)) + \dim([L, N]) \leq \dim(N)(d(N) + d(L/N)).
\]

We finally give the following theorem, which can be used to obtain a new bound for the Schur multiplier of a nilpotent Lie algebra.

**Theorem C.** Let \(L\) be a finite dimensional nilpotent Lie algebra and \(N\) be an ideal of \(L\) which is not central. Then

\[
\dim(\mathcal{M}(L, N)) \leq d(L)(\dim(N) - 1) - \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right).
\]

2. Proof of theorems

Let \(K, N\) be ideals of a Lie algebra \(L\). The nonabelian exterior product \(K \wedge N\) is the Lie algebra generated by the elements \(k \wedge n\) with \((k, n) \in K \times N\), subject to the relations

\[
\begin{align*}
c(k \wedge n) &= ck \wedge n = k \wedge cn, \\
(k + k' \wedge n) &= k \wedge n + k' \wedge n, \\
k \wedge (n + n') &= k \wedge n + k \wedge n', \\
x \wedge x &= 0,
\end{align*}
\]

for all \(x \in K \cap N\), \(k, k' \in K\), \(n, n' \in N\) and scalar \(c\). It follows from [4, Theorem 35] that the Schur multiplier of \((L, N)\) can be computed as

\[
\mathcal{M}(L, N) \equiv \ker(L \wedge N \xrightarrow{[\cdot, \cdot]} L),
\]

where \([\cdot, \cdot]\) is the commutator map defined on generators of \(L \wedge N\) by \([-,-](l \wedge n) = [l, n]\).

The following theorem plays a key role in our main results.

**Theorem 2.1** Let \(L\) be a Lie algebra and \(N, K\) be ideals of \(L\) such that \(K \subseteq N \cap Z(L)\). Then the following sequence is exact:

\[
K \wedge L \to \mathcal{M}(L, N) \to \mathcal{M}(L/K, N/K) \to K \cap [N, L] \to 0.
\]
Proof Using the functorial properties of the nonabelian exterior product, the short exact sequence of Lie algebras 0 → \( K \to L \xrightarrow{\pi} L/K \to 0 \) induces the exact sequence

\[
L \wedge K \to L \wedge N \xrightarrow{\pi \wedge \pi} L/K \wedge N/K \to 0.
\]

Now, we have the following diagram of Lie algebras

\[
\begin{array}{ccc}
L \wedge K & \longrightarrow & L \wedge N \\
\downarrow \{,\}_1 & & \downarrow \{,\}_2 \\
([L, N] \cap K) & \longrightarrow & [L, N]
\end{array}
\]

\[
\begin{array}{ccc}
& & \pi \\
\downarrow \{,\}_3 & & \\
L/K \wedge N/K & \longrightarrow & 0
\end{array}
\]

where the vertical arrows are the commutator maps; see [4]. In this diagram, the right-hand-side square is always commutative. Note that since \( K \) is a central ideal of \( L \) the commutator map \([ - , - ]_1\) is equal to the zero morphism and so the left-hand-side square is also commutative. Now the "Snake Lemma" yields that there is the following exact sequence:

\[
\ker([ , ]_1) \to \ker([ , ]_2) \to \ker([ , ]_3) \to \text{coker}([ , ]_1) \to 0.
\]

The last homomorphism is surjective because \([ - , - ]_2\) is onto. Finally, the result follows from (2). \( \square \)

**Remark 2.2** By taking \( N = L \) in Theorem 2.1, we can obtain the Ganea sequence in homology of Lie algebras; see [11, Proposition 4.1]. In the case that \( L \) splits over \( N \), a similar sequence was obtained in [9].

Using Theorem 2.1, we obtain the following corollary that generalizes [11, corollary 4.2] and [12, Proposition 2.2].

**Corollary 2.3** Let \( L \) be a finite dimensional Lie algebra and \( N, K \) be ideals of \( L \) such that \( K \subseteq N \cap Z(L) \). Then \( \dim(\mathcal{M}(L/K, N/K)) \leq \dim(\mathcal{M}(L, N)) + \dim([N, L] \cap K) \); in particular, if \( N \) is a central ideal of \( L \) then

\[
\dim(\mathcal{M}(L/K, N/K)) \leq \dim(\mathcal{M}(L, N)).
\]

Now, we are ready to prove the theorems.

**Proof** [Proof of Theorem A] The proof is stated on induction on \( \dim(L) \). If \( N \) is central then \([L, N] = 0 \) and there is nothing to prove. Therefore, suppose that \([L, N] \neq 0 \) and choose a one-dimensional ideal \( K \) of \( L \) such that \( K \subseteq Z(L) \cap [L, N] \). Thanks to Theorem 2.1 and applying the induction hypothesis, we have

\[
\dim(\mathcal{M}(L, N)) \leq \dim(\mathcal{M}(L/K, N/K)) + \dim(K \wedge L) - 1
\]

\[
\leq \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + (\dim([L, N]) - 1) \times
\]

\[
(d(\frac{L/K}{Z(L/K, N/K)}) - 1) + \dim(K \wedge L) - 1
\]

\[
\leq \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + (\dim([L, N]) - 1)(d(\frac{L}{Z(L, N)}) - 1) + d(L) - 1
\]

\[
= \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + \dim([L, N])(d(\frac{L}{Z(L, N)}) - 1),
\]
which completes the proof.

**Proof** [Proof of Theorem B] Similar to the previous proof, we proceed by induction on the dimension of $L$. Suppose that the result occurs for any Lie algebra of dimension less than $\dim(L)$. Choose a one-dimensional ideal $K$ such that $K \subseteq N \cap Z(L)$. Since $L$ is a finite dimensional nilpotent Lie algebra, $d(L)$ is equal to $\dim(L/L^2)$ and

$$d(L) \leq \dim(L) - \dim(L^2) + \dim(L^2 \cap N) - \dim(N^2) = d(L/N) + d(N).$$

Hence, the sequence (3) implies that

$$\dim(L \wedge N) \leq \dim(K \wedge L) + \dim(L/K \wedge N/K)$$

$$\leq d(L) + \dim(N/K)(d(N/K) + d(L/N))$$

$$\leq d(L/N) + d(N) + \dim(N) - 1)(d(N) + d(L/N))$$

$$= \dim(N)(d(N) + d(L/N)).$$

Since $\dim(M(L, N)) + \dim([L, N]) = \dim(L \wedge N)$ by (2) the proof completes. □

We can use a similar method of Theorem B to prove the following proposition.

**Proposition 2.4** Let $L$ be a finite dimensional nilpotent Lie algebra and $N$ be an ideal of $L$ that is not contained in $Z(L)$. Then

$$\dim(M(L, N)) \leq \dim(N)(d(N) + d(L/N) - 1).$$

**Proof** [Proof of Theorem C] Similarly, the proof is based on induction on $\dim(L)$. Suppose that $\dim(L) > 1$ and choose a one-dimensional ideal $K$ of $L$ such that $K \subseteq Z(L) \cap [L, N]$. Using Theorem 2.1 and applying the induction hypothesis, we have

$$\dim(M(L, N)) \leq \dim(M(L/K, N/K)) + \dim(K \wedge L)$$

$$\leq d(L/K)(\dim(N/K) - 1) - \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + \dim(K \wedge L)$$

$$\leq d(L)(\dim(N) - 2) - \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right) + d(L)$$

$$\leq d(L)(\dim(N) - 1) - \dim \left( \mathcal{M} \left( \frac{L}{[L, N]}, \frac{N}{[L, N]} \right) \right).$$

Note that since $K$ is a central ideal of $L$, the Lie actions of $K$ and $L$ on each other are trivial, and so $K \wedge L \cong K \wedge L/L^2$ and

$$\dim(K \wedge L) \leq \dim(L/L^2) = d(L).$$

□

Now we can derive a new bound for the dimension of the Schur multiplier of a nilpotent Lie algebra.

**Corollary 2.5** Let $L$ be a $d$-generator nilpotent Lie algebra of dimension $n$. Then

$$\dim(M(L)) \leq \frac{1}{2}d(2n - d - 1).$$
Proof If $L$ is an abelian Lie algebra then $d = n$, $\dim(M(L)) = \frac{1}{2}n(n-1)$ and the statement is obviously true. Hence, suppose that $L$ is not an abelian Lie algebra. Using the fact
\[ \dim(M(L/L^2, L/L^2)) = \dim(M(L/L^2)) = \frac{1}{2}d(d-1), \]
the desired result follows by taking $N = L$ in Theorem C.

Note that since $d(2n - d - 1) \leq n(n-1)$ for all integers $1 \leq d \leq n$, the upper bound obtained in Corollary 2.5 is sharper than the known bound $\dim(M(L)) \leq \frac{1}{2}n(n-1)$, which is due to Moneyhun [7].

Remark 2.6 Let $(L, N)$ be a pair of Lie algebras such that $N$ is of codimension less than two. Since $H_3(L/N) = 0$ in the sequence (1), one can deduce that $\dim(M(L, N)) = \dim(M(L))$. Hence any upper bound on the dimension of $M(L)$ can be considered as an upper bound for $\dim(M(L, N))$. In particular, if $N$ is an ideal of codimension one, then $M(L/N) = H_3(L/N) = 0$, which immediately implies $M(L) \cong M(L, N)$. Therefore, any upper and lower bound on $M(L)$ is a bound for $M(L, N)$. The result obtained in [12, Theorem D] is an example of the bound that was previously obtained by Jones (1974) on the dimension of $M(L)$.

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References