The generalized reciprocal super Catalan matrix

Emrah KILIC\textsuperscript{1}, Talha ARIKAN\textsuperscript{2,}\textsuperscript{*}

\textsuperscript{1}Department of Mathematics, TOBB Economics and Technology University, Ankara, Turkey
\textsuperscript{2}Department of Mathematics, Hacettepe University, Ankara, Turkey

Received: 30.04.2015 \hspace{1em} Accepted/Published Online: 20.01.2016 \hspace{1em} Final Version: 21.10.2016

Abstract: The reciprocal super Catalan matrix studied by Prodinger is further generalized, introducing two additional parameters. Explicit formulae are derived for the $LU$-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use $q$-analysis and to leave the justification of the necessary identities to the $q$-version of Zeilberger’s celebrated algorithm.

Key words: Determinant, inverse matrix, $LU$ factorization, Gaussian $q$-binomial coefficient, Zeilberger’s algorithm

1. Introduction

As mentioned in [8], there are many combinatorial matrices defined by a given sequence $\{a_n\}$. One of them is known as the Hankel matrix and is defined as follows:

\[
\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

for more details see [6]. Considering some special number sequences instead of $\{a_n\}$, there are many special matrices with nice algebraic properties. Moreover, some authors, such as [10], studied the Hankel matrix considering the reciprocal sequence of $\{a_n\}$

\[
\begin{bmatrix}
\frac{1}{a_0} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots \\
\frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \\
\frac{1}{a_2} & \frac{1}{a_3} & \frac{1}{a_4} & \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

For the sequence $\{a_{i,j}\}$, a matrix can be defined by taking $(i,j)$th entries $a_{i,j}$. Well-known types of these sequences typically include binomial coefficients. As examples, we give the family of Pascal matrices whose entries are defined via the usual binomial coefficients [2, 3]. The Pascal matrices are mainly two kinds: the first is the left adjusted Pascal matrix $P_n = (p_{ij})$ and the second is the right adjusted Pascal matrix $Q_n = (m_{ij})$.

\textsuperscript{*}Correspondence: tarikan@hacettepe.edu.tr

2010 \textit{AMS Mathematics Subject Classification}: 15B36, 15A15, 15A23, 11B65.
where
\[
p_{ij} = \binom{i}{j} \quad \text{and} \quad m_{ij} = \binom{i}{n-1-j}, \quad 0 \leq i, j < n.
\]

The Gaussian $q$-binomial coefficients are defined by
\[
\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},
\]
where $(x; q)_n$ is the $q$-Pochhammer symbol defined by
\[
(x; q)_n = (1 - x) (1 - xq) \ldots (1 - xq^{n-1}).
\]

Note that
\[
\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k},
\]
where $\binom{n}{k}$ is the usual binomial coefficient.

We recall that one version of the **Cauchy binomial theorem** is given by
\[
\sum_{k=0}^{n} q^{k+1} \binom{n}{k}_q x^k = \prod_{k=1}^{n} (1 + xq^k),
\]
and **Rothe’s formula** [1] is
\[
\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q x^k = (x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k).
\]

Recently, Prodinger [8] defined a matrix whose entries consist of super Catalan numbers. He also defined its reciprocal analogue as well as its $q$-versions whose $(i, j)$th entries are defined for $0 \leq i, j < n$
\[
\binom{2i}{i}^{-1} \binom{2j}{j}^{-1} \binom{i+j}{i},
\]
\[
\binom{2i}{i} \binom{2j}{j} \binom{i+j}{i}^{-1},
\]
\[
\binom{2i}{i} q^{\binom{2j}{j}} \binom{i+j}{i}_q,
\]
and
\[
\binom{2i}{i}_q \binom{2j}{j}_q \binom{i+j}{i}_q^{-1},
\]
respectively. Then he gave some algebraic properties of these matrices.

Recently, Kılıç et al. [4] defined and studied a variant of the reciprocal super Catalan matrix with two additional parameters whose entries are defined as
\[
\binom{2i+r}{i}^{-1} \binom{2j+s}{j}^{-1} \binom{i+j}{i}^{-1}.
\]
Explicit formulae for its LU-decomposition, LU decomposition of its inverse, and the Cholesky decomposition are obtained. For all results, \( q \)-analogues are also presented.

In this paper, for nonnegative integers \( r \) and \( s \), we define two \( n \times n \) matrices \( M = [M_{kj}] \) and \( T = [T_{kj}] \) with entries

\[
M_{kj} = \binom{k+j}{k} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}
\]

and

\[
T_{kj} = \binom{2k+r}{k} \binom{2j+s}{j} \binom{k+j}{k}^{-1}
\]

for \( 0 \leq k, j < n \), respectively.

First, we give the matrices \( \mathcal{M} \) and \( \mathcal{T} \) which are the \( q \)-analogues of the matrices \( M \) and \( T \), respectively. For both matrices, we derive explicit expressions for the LU-decomposition, which leads to a formula for the determinant via \( \prod_{0 \leq i < n} U_{i,i} \). Further, we have expressions for \( L_1 \) and \( U_1 \), for LU-decomposition of the inverse matrix and their inverses, and for the Cholesky decomposition when the matrix is symmetric, that is, the case \( r = s \). Afterwards, when \( q \to 1 \), we get the results for the matrices \( M \) and \( T \). Our results generalize the results of [8] for the case \( r = s = 0 \).

Firstly, we list the result related to the matrix \( \mathcal{M} \) in the next section and secondly prove them in Section 3. Then we list results related to the matrix \( \mathcal{T} \) and then give related proofs in the next section. Finally, we give the results related to the matrices \( M \) and \( T \) as special cases of the results related to the matrices \( \mathcal{M} \) and \( \mathcal{T} \). To prove the claimed results, our main tool is to guess relevant quantities and then we will use the \( q \)-version of Zeilberger’s celebrated algorithm (for more details see [7, 9]) and Rothe’s formula to justify relevant equalities. All identities we will obtain hold for general \( q \) and generalized Fibonomial analogue of our results could be obtained by using the application of \( q \)-identities for Fibonacci numbers. We refer to [5] to give an idea.

2. The matrix \( \mathcal{M} \)

We denote matrices \( L \) and \( U \) by \( A \) and \( B \) in LU-decomposition of any inverse matrix, respectively, that is, \( \mathcal{M}^{-1} = AB \). For the Cholesky decomposition of a matrix \( G \), we will use the letter \( C \) such that \( G = CC^T \).

The matrix \( \mathcal{M} \) is defined with entries for \( 0 \leq k, j < n \),

\[
\mathcal{M}_{kj} = \binom{k+j}{k} \left[ \frac{2k+r}{k} \right]^{-1} \left[ \frac{2j+s}{j} \right]^{-1}_q.
\]

Firstly, we list here the formulae related to matrix \( \mathcal{M} \) that were found for \( 0 \leq k, j < n \):

\[
L_{kj} = \left[ \frac{2k+r}{k} \right]^{-1}_q \left[ \frac{2j+s}{j} \right]^{-1}_q \binom{k}{j}_q,
\]

\[
L_{kj}^{-1} = (-1)^{k+j} q^{k+j} \left[ \frac{2k+r}{k} \right]^{-1}_q \left[ \frac{2j+s}{j} \right]^{-1}_q \binom{k}{j}_q,
\]

\[
U_{kj} = q^{k+j} \left[ \frac{2k+r}{k} \right]^{-1}_q \left[ \frac{2j+s}{j} \right]^{-1}_q \binom{j}{k}_q.
\]
\[ U_{kj}^{-1} = (-1)^{k+j} q^{k(k+1)/2-j(j+1)/2-kj} \left[ \begin{array}{c} 2k+s \\ k' \end{array} \right]_q \left[ \begin{array}{c} 2j+r \\ j' \end{array} \right]_q \left[ \begin{array}{c} j \\ k \end{array} \right]_q , \]

\[ A_{kj} = (-1)^{k+j} q^{k(k+3)/2-j(j+3)/2-n(k-j)} \frac{1 - q^{2j+1}}{1 - q^{k+j+1}} \left[ \begin{array}{c} n-j-1 \\ k-j \end{array} \right]_q \left[ \begin{array}{c} 2k+s \\ k \end{array} \right]_q \]

\[ A_{kj}^{-1} = q^{(k-j)(k-n+1)} \left[ \begin{array}{c} k+j \\ k' \end{array} \right]_q \left[ \begin{array}{c} n-j-1 \\ k-j \end{array} \right]_q \left[ \begin{array}{c} 2j+s \\ j' \end{array} \right]_q \left[ \begin{array}{c} 2k+s \\ s \end{array} \right]_q \left[ \begin{array}{c} k+s \\ s \end{array} \right]_q , \]

\[ B_{kj} = (-1)^{k+j} q^{(j+1)(j+2)/2-n(k+j+1)+3k(k+1)/2} \left[ \begin{array}{c} 2j+r \\ j' \end{array} \right]_q \left[ \begin{array}{c} n+k \\ j+k+1 \end{array} \right]_q \left[ \begin{array}{c} j \\ k \end{array} \right]_q \]

\[ B_{kj}^{-1} = q^{(k+j+1)(n-j-1)} \frac{1 - q^{2j+1}}{1 - q^{n-k}} \left[ \begin{array}{c} 2k+r \\ k' \end{array} \right]_q \left[ \begin{array}{c} n+j \\ j+k \end{array} \right]_q \left[ \begin{array}{c} j \\ k \end{array} \right]_q \]

\[ \times \left[ \begin{array}{c} 2j+s \\ s \end{array} \right]_q \left[ \begin{array}{c} k+s \\ s \end{array} \right]_q , \]

for \( r = s \),

\[ C_{kj} = q^{j^2/2} \left[ \begin{array}{c} 2k+r \\ k' \end{array} \right]_q \left[ \begin{array}{c} j \\ k \end{array} \right]_q \]

and

\[ \det \mathcal{M} = q^{n(n-1)(2n-1)/6} \prod_{k=0}^{n-1} \left[ \begin{array}{c} 2k+r \\ k' \end{array} \right]_q \left[ \begin{array}{c} 2k+s \\ k \end{array} \right]_q^{-1} . \]

### 3. Proofs related to the matrix \( \mathcal{M} \)

For \( L \) and \( L^{-1} \),

\[ \sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \sum_{j \leq d \leq k} (-1)^{d+j} q^{(d-j)/2} \left[ \begin{array}{c} 2k+r \\ k' \end{array} \right]_q \left[ \begin{array}{c} 2d+r \\ d \end{array} \right]_q \left[ \begin{array}{c} k \\ d' \end{array} \right]_q \]

\[ \times \left[ \begin{array}{c} 2d+r \\ d \end{array} \right]_q \left[ \begin{array}{c} j \\ d \end{array} \right]_q \left[ \begin{array}{c} 2j+r \\ j \end{array} \right]_q \]

\[ = \left[ \begin{array}{c} 2k+r \\ k \end{array} \right]_q \left[ \begin{array}{c} 2j+r \\ j \end{array} \right]_q \sum_{0 \leq d \leq k-j} \left[ \begin{array}{c} k-j \\ d \end{array} \right]_q (-1)^d q^{(d)/2} . \]
By Rothe’s formula, if \( k \neq j \) then we have \((1;q)_{k-j} = 0\), and, if \( k = j \), then the last sum on the RHS of the above equation is equal to 1. Thus we conclude

\[
\sum_{j\leq d\leq k} L_{kd} L_{d-j}^{-1} = \delta_{k,j},
\]

where \( \delta_{k,j} \) is Kronecker delta, as claimed.

For \( U \) and \( U^{-1} \),

\[
\sum_{k\leq d\leq j} U_{kd} U_{d-j}^{-1} = q^{k^2 - \binom{j+1}{2}} \begin{bmatrix} 2k + r \choose k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j + r \choose j \end{bmatrix}_q \begin{bmatrix} j \choose k \end{bmatrix}_q \times q^{2(j+k)/2} (-1)^{k+j} \sum_{0\leq d\leq j-k} \begin{bmatrix} j - k \choose d \end{bmatrix}_q (-1)^d q^{\binom{d+1}{2} + d(k-j)}.
\]

By the Cauchy binomial theorem, if \( k \neq j \), then the last sum on the RHS of the above equation equals \( \prod_{d=1}^{j-k} (1 - q^{(k-j)+d}) = 0 \). Thus we have

\[
\sum_{k\leq d\leq j} U_{kd} U_{d-j}^{-1} = \delta_{k,j},
\]

as desired.

For \( LU \)-decomposition, we have to prove that

\[
\sum_{0\leq d\leq \min\{k,j\}} L_{kd} U_{d-j} = M_{kj}.
\]

Consider

\[
\sum_{0\leq d\leq \min\{k,j\}} L_{kd} U_{d-j} = \begin{bmatrix} 2k + r \choose k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j + r \choose j \end{bmatrix}_q (q; q)_k (q; q)_j \times \sum_{0\leq d\leq k} q^{d^2} \frac{1}{(q; q)_d^2 (q; q)_{k-d} (q; q)_{j-d}}.
\]

Denote the last sum in the equation just above by \( \text{SUM}_k \). The Mathematica version of the \( q \)-Zeilberger algorithm [7] produces the recursion

\[
\text{SUM}_k = \frac{1 - q^{j+k}}{(1 - q^k)^2} \text{SUM}_{k-1}.
\]

Since \( \text{SUM}_0 = (q; q)_k^{-1} (q; q)_{j}^{-1} \), we obtain

\[
\text{SUM}_k = (q; q)_k^{-1} (q; q)_{j}^{-1} \begin{bmatrix} k + j \choose k \end{bmatrix}_q.
\]

Therefore, we get

\[
\sum_{0\leq d\leq \min\{k,j\}} L_{kd} U_{d-j} = M_{kj},
\]
which completes the proof.

For $A$ and $A^{-1}$, consider

$$
\sum_{j \leq d \leq k} A_{kd} A_{dj}^{-1} = (-1)^k q^{k(k+3)/2-j+n(j-k)} \frac{(q;q)_{n-j-1}}{(q;q)_{n-k-1}}
\times \left[ \begin{array}{c}
2k + s \\
k 
\end{array} \right]_q \left[ \begin{array}{c}
2j + s \\
j 
\end{array} \right]_q^{-1} \left[ \begin{array}{c}
k \\
j 
\end{array} \right]_q
\times \sum_{j \leq d \leq k} \left[ \begin{array}{c}
k - j \\
d - j 
\end{array} \right]_q \left( \frac{1 - q^{2d+1}}{q; q}_{d-j} \right) \frac{1}{1 - q^{k+1}}.
$$

By the $q$-Zeilberger algorithm for the second sum in the last equation, we obtain that it is equal to 0 provided that $k \neq j$. If $k = j$, it is obvious that $A_{kk} A_{kk}^{-1} = 1$. Thus

$$
\sum_{j \leq d \leq k} A_{kd} A_{dj}^{-1} = \delta_{k,j},
$$

as claimed.

Similarly, we have

$$
\sum_{k \leq d \leq j} B_{kd} B_{dj}^{-1} = \delta_{k,j}.
$$

For the Cholesky decomposition, we examine the equation

$$
\sum_{0 \leq d \leq \min\{k, j\}} C_{kd} C_{jd} = M_{kj}.
$$

Here

$$
\sum_{0 \leq d \leq \min\{k, j\}} C_{kd} C_{jd} = \left[ \begin{array}{c}
2k + r \\
k 
\end{array} \right]_q^{-1} \left[ \begin{array}{c}
2j + s \\
j 
\end{array} \right]_q^{-1} \sum_{0 \leq d \leq \min\{k, j\}} q^{a} \left[ \begin{array}{c}
k \\
j 
\end{array} \right]_q \left[ \begin{array}{c}
d \\
j 
\end{array} \right]_q.
$$

Note that the sum on the RHS of the equation just above is the same as the sum in the $LU$-decomposition, which was proven before.

For the $LU$-decomposition of $M^{-1}$, we should show that $M^{-1} = AB$, which is same as $M = B^{-1} A^{-1}$. Hence, it is sufficient to show that

$$
\sum_{\max\{k, j\} \leq d \leq n-1} B_{kd}^{-1} A_{dj}^{-1} = M_{kj}.
$$

After some arrangements, we have

$$
\sum_{\max\{k, j\} \leq d \leq n-1} B_{kd}^{-1} A_{dj}^{-1} = \left[ \begin{array}{c}
2k + r \\
k 
\end{array} \right]_q^{-1} \left[ \begin{array}{c}
2j + s \\
j 
\end{array} \right]_q^{-1} \sum_{j \leq d \leq n-1} q^{(j+k+1)(n-1-d)}
\times \frac{1 - q^{2d+1}}{1 - q^{n-k}} \left[ \begin{array}{c}
k + d \\
k + d 
\end{array} \right]_q \left[ \begin{array}{c}
j \\
d 
\end{array} \right]_q \left[ \begin{array}{c}
q^{d-1} \\
d 
\end{array} \right]_q.
$$
which, by replacing \((n - 1)\) with \(n\) and apart from the constants factors, equals

\[
\sum_{j \leq d \leq n} q^{(d-k+1)(n-d)} \frac{1-q^{2d+1}}{1-q^{n+1-k}} \left[ \begin{array}{c} d \\ j \end{array} \right]_q \left[ \begin{array}{c} n+1+d \\ k+d \end{array} \right]^{-1}_q \left[ \begin{array}{c} d+j \\ d \end{array} \right]_q \left[ \begin{array}{c} n-j \\ d-j \end{array} \right]_q.
\]

Denote this sum by \(\text{SUM}_n\). The \(q\)-Zeilberger algorithm gives the following recursion provided that \(k \neq n\) and \(j \neq n\)

\[
\text{SUM}_n = \text{SUM}_{n-1}.
\]

Therefore, \(\text{SUM}_n = \text{SUM}_j = \left[ \begin{array}{c} k+j \\ k \end{array} \right]_q\) which completes the proof except for the case \((k, j) = (n - 1, n - 1)\), which could be easily checked. Thus the proof is complete.

4. The matrix \(T\)

The matrix \(T\) is defined with entries for \(0 \leq k, j < n\),

\[
T_{kj} = \left[ \begin{array}{c} 2k+r \\ k \end{array} \right]_q \left[ \begin{array}{c} 2j+s \\ j \end{array} \right]_q \left[ \begin{array}{c} k+j \\ k \end{array} \right]^{-1}_q.
\]

For \(0 \leq k, j < n\), we have

\[
L_{kj} = \frac{2k+r}{k+j} \left[ \begin{array}{c} k \\ j \end{array} \right]_q \left[ \begin{array}{c} 2j+r \\ r \end{array} \right]^{-1}_q \left[ \begin{array}{c} j+r \\ r \end{array} \right]_q,
\]

\[
L_{kj}^{-1} = (-1)^{k+j} q^{(k-j)/2} \frac{1-q^{2k}}{1-q^{k+j}} \left[ \begin{array}{c} k+j \\ k-j \end{array} \right]_q \left[ \begin{array}{c} 2k+r \\ r \end{array} \right]_q \left[ \begin{array}{c} k+r \\ r \end{array} \right]^{-1}_q \left[ \begin{array}{c} j+r \\ r \end{array} \right]^{-1}_q.
\]

\[
\times \left[ \begin{array}{c} j+r \\ r \end{array} \right]_q \text{ for } j \geq 1,
\]

\[
L_{k0}^{-1} = (-1)^k (1+q^k) q^{(k+1)/2} \left[ \begin{array}{c} 2k+r \\ r \end{array} \right]_q \left[ \begin{array}{c} k+r \\ r \end{array} \right]^{-1}_q \text{ and } L_{00}^{-1} = 1,
\]

\[
U_{kj} = (-1)^k q^{k(3k-1)/2} (1+q^k) \left[ \begin{array}{c} 2j+s \\ k+j \end{array} \right]_q \left[ \begin{array}{c} 2k+r \\ r \end{array} \right]_q \left[ \begin{array}{c} j+k+s \\ s \end{array} \right]_q
\]

\[
\times \left[ \begin{array}{c} k+r \\ r \end{array} \right]^{-1}_q \left[ \begin{array}{c} j+s \\ s \end{array} \right]^{-1}_q \text{ for } k \geq 1, \quad U_{0j} = \left[ \begin{array}{c} 2j+s \\ j \end{array} \right]_q,
\]

\[
U_{kj}^{-1} = (-1)^k q^{k(k+1)/2-j(k+j)} \frac{1-q^j}{1-q^{k+j}} \left[ \begin{array}{c} k+j \\ j-k \end{array} \right]_q \left[ \begin{array}{c} 2j+r \\ r \end{array} \right]^{-1}_q \left[ \begin{array}{c} j+r \\ r \end{array} \right]_q
\]

\[
\times \left[ \begin{array}{c} 2k+s \\ s \end{array} \right]^{-1}_q \left[ \begin{array}{c} k+s \\ s \end{array} \right]_q.
\]
\[ A_{kj} = (-1)^{k+j} q^{(k+1)(k+2)/2-(j+1)(j+2)/2+n(j-k)} \binom{k}{j} q^{n+k-1} \binom{n+k-1}{2k} q \times \binom{n+j-1}{2j} q^{k+s} \binom{2k+s}{2j} q^{j+s} \binom{2j+s}{s} q, \]

\[ A_{kj}^{-1} = q^{(k-j)(k-n+1)} \binom{k}{j} q^{n+k-1} \binom{n+j-1}{2j} q^{k+s} \binom{2k+s}{2j} q^{j+s} \binom{2j+s}{s} q, \]

\[ B_{kj} = q^{(j+1)(j+2)/2-(n-1)/2-jn+k^2-1} \binom{n+j-1}{2j} q^{k} \binom{2k+s}{k} q \]

\[ B_{kj}^{-1} = (-1)^{n+j+1} q^{k-kj-j(j+1)/2+kn+n(n-1)/2} \binom{j}{k} q^{n+k-1} \binom{2j+s}{s} q, \]

for \( r = s \) and \( j \geq 1, \)

\[ C_{kj} = i^{(1+q)j/2} q^{j(3j-1)/4} \binom{2k+r}{k+j} q^{k+r} \binom{k+r}{r} q^{k-j+r} q, \]

where \( i = \sqrt{-1} \) and for \( j = 0, \)

\[ C_{k0} = \binom{2k+r}{k} q \]

and

\[ \det \mathcal{T} = (-1)^{\binom{d}{2}} \prod_{k=1}^{n-1} q^{k(3k-1)/2} \binom{2k+s}{2k} q^{2k+r} \binom{k+r}{r} q^{k+s} q^{-1}. \]

5. Proofs related to the matrix \( \mathcal{T} \)

For \( L \) and \( L^{-1} \), it should be shown

\[ \sum_{j \leq d \leq k} L_{kd} L_{d,j}^{-1} = \delta_{k,j}. \]

By the definitions of the matrices \( L \) and \( L^{-1} \), for the case \( j = 0 \), we have

\[ \sum_{0 \leq d \leq k} L_{k,d} L_{d,0}^{-1} = L_{k0} L_{0,0}^{-1} + \sum_{1 \leq d \leq k} L_{k,d} L_{d,0}^{-1}. \]
If $k = 0$, we get 1 as $(0, 0)$th entry of matrix $LL^{-1}$. If $k > 0$, after some rearrangements we have

$$
\sum_{1 \leq d \leq k} L_{kd} L_{d0}^{-1} = \sum_{0 \leq d \leq k-1} L_{k,d+1} L_{d+1,0}^{-1} = \sum_{0 \leq d \leq n} L_{n+1,d+1} L_{d+1,0}^{-1}
$$

$$
= \sum_{0 \leq d \leq n} (-1)^{d+1} (1 + q^{d+1}) q^{(d^2+d)/2} \left[ \begin{array}{c}
2n + 2 + r \\
n + d + 2
\end{array} \right]_q
$$

$$
\times \left[ \begin{array}{c}
\frac{n+1}{d+1} q - 1 \\
\frac{n+1}{d+1}
\end{array} \right]_q^{-1}
$$

which, by using the $q$-Zeilberger algorithm, equals $-\left[ \begin{array}{c}
2n + 2 + r \\
n + 1
\end{array} \right]_q$. By changing $n + 1$ with $k$ again, we get $-\left[ \begin{array}{c}
2k + r \\
k
\end{array} \right]_q$. Finally if $k > 0$,

$$
\sum_{0 \leq d \leq k} L_{kd} L_{d0}^{-1} = \left[ \begin{array}{c}
2k + r \\
k
\end{array} \right]_q + \sum_{1 \leq d \leq k} L_{kd} L_{d0}^{-1}
$$

$$
= \left[ \begin{array}{c}
2k + r \\
k
\end{array} \right]_q - \left[ \begin{array}{c}
2k + r \\
k
\end{array} \right]_q = 0,
$$

as desired. For the case $j > 0$, we have

$$
\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \sum_{j \leq d \leq k} (-1)^{d+j} q^{(d-j)} \left[ \begin{array}{c}
1 - q^{2d} \\
1 - q^{d+j}
\end{array} \right]_q \left[ \begin{array}{c}
2k + r \\
k + d
\end{array} \right]_q^{-1} \left[ \begin{array}{c}
k \\
d
\end{array} \right]_q
$$

$$
\times \left[ \begin{array}{c}
k + r \\
d + j
\end{array} \right]_q^{-1} \left[ \begin{array}{c}
j + r \\
r
\end{array} \right]_q.
$$

By the $q$-Zeilberger algorithm, we obtain that it is equal to 0 provided that $k \neq j$. The case $k = j$ could be easily checked. Thus

$$
\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \delta_{k,j},
$$

which completes the proof.

Verification of the inverse of $U$ could be similarly done. Inverses of the matrices $A$ and $B$ could be shown as in Section 3.

For $LU$-decomposition, we have to prove that

$$
\sum_{0 \leq d \leq \min\{k, j\}} L_{kd} U_{dj} = T_{kj}.
$$
Without loss of generality, we may consider \( k \leq j \). Hence, consider the sum

\[
\sum_{-k \leq d \leq k} (-1)^d \left(1 + q^d\right) q^{(3d-1)d/2} \binom{2k}{k+d} q^{d-j} \binom{2j}{j+d}\frac{1}{q}.
\]

The \( q \)-Zeilberger algorithm gives the recurrence relation

\[
\sum_k = \frac{(1 + q^k) \left(1 - q^{2k-1}\right)}{(1 - q^{2k+1})} \sum_{k-1}.
\]

Since \( \sum_0 = 2 \binom{2k}{k} q^k \), we obtain

\[
\sum_k = 2 \binom{2k}{k} q^k \binom{2j}{j} q^{-j} \binom{k+j}{k}^{-1}.
\]

Since the summand of the \( \sum_k \) is symmetric with respect to \( k \) and \( -k \), we have

\[
\sum_{1 \leq d \leq k} (-1)^d \left(1 + q^d\right) q^{(3d-1)d/2} \binom{2k}{k+d} q^{d-j} \binom{2j}{j+d} = \frac{1}{2} \sum_k - \binom{2k}{k} q^j \frac{2j}{j} q.
\]

Finally consider

\[
\sum_{0 \leq d \leq k} L_{kd} U_{dj} = \binom{2k+r}{k} q^{j+s} \binom{2k+r}{j} q^{j+s} + \binom{2k+r}{k} q^{j+s} \binom{2k+r}{j} q^{j+s}
\]

\[
\times \left[ \binom{2k-r}{k} q^{j-s} \left( \frac{1}{2} \sum_k - \binom{2k}{k} q^j \frac{2j}{j} q \right) \right]^{-1} = \binom{2k+r}{k} q^{j+s} \binom{2k+r}{j} q^{j+s} = T_{kj},
\]

as desired.

For \( LU \)-decomposition of the inverse of the matrix \( T \), the argument in Section 3 could be similarly used. We omit it here.
6. The matrix $M$

Recall that the $n \times n$ matrix $M = [M_{kj}]$ is defined for $0 \leq k, j < n$ and nonnegative integers $r$ and $s,$

$$M_{kj} = \binom{k+j}{k}^{-1} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}.$$ 

In Section 2, by taking $q \rightarrow 1,$ we get the following results for $0 \leq k, j < n$:

$$L_{kj} = \binom{2k+r}{k}^{-1} \binom{2j+r}{j}^{-1} \binom{k}{k},$$

$$L_{kj}^{-1} = (-1)^{k+j} \binom{2k+r}{k}^{-1} \binom{2j+r}{j}^{-1} \binom{k}{k},$$

$$U_{kj} = \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1} \binom{j}{k},$$

$$U_{kj}^{-1} = (-1)^{k+j} \binom{2k+s}{k} \binom{2j+r}{j} \binom{j}{k},$$

$$A_{kj} = (-1)^{k+j} \binom{2j+s}{j}^{-1} \binom{k+j}{k}^{-1} \binom{n-j-1}{k-j} \binom{2k+s}{k} \binom{k+j}{k}^{-1} \times \binom{2j+s}{s}^{-1} \binom{j+s}{s},$$

$$A_{kj}^{-1} = \binom{k+j}{k} \binom{n-j-1}{k-j} \binom{2j+s}{j}^{-1} \binom{2k+s}{s} \binom{k+j}{k}^{-1} \binom{2k+s}{s} \binom{k+j}{k}^{-1},$$

$$B_{kj} = (-1)^{k+j} \binom{2j+r}{j} \binom{n+k}{k+j+1} \binom{j}{k} \binom{2k+s}{s} \binom{k+s}{s}^{-1},$$

$$B_{kj}^{-1} = \frac{2j+1}{n-k} \binom{2k+r}{k}^{-1} \binom{n+j}{k+j} \binom{j}{k}^{-1} \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

for $r = s,$

$$C_{kj} = \binom{2k+r}{k}^{-1} \binom{k}{j}$$

and

$$\det M = \prod_{k=0}^{n-1} \binom{2k+r}{k}^{-1} \binom{2k+s}{k}^{-1}.$$
7. The matrix $T$
Recall that the $n \times n$ matrix $T = [T_{kj}]$ is defined for $0 \leq k, j < n$, and nonnegative integers $r$ and $s$,

$$T_{kj} = \binom{2k+r}{k} \binom{2j+s}{j} \binom{k+j}{r}^{-1}.$$ 

In the Section 4, by taking $q \to 1$, we obtain the following results. For $0 \leq k, j < n$,

$$L_{kj} = \binom{2k+r}{k+j} \binom{k+r}{j} \binom{k+j}{r}^{-1} \binom{2j+r}{j} \binom{j+r}{r},$$

for $j \geq 1$,

$$L_{kj}^{-1} = (-1)^{k+j} \frac{2k}{k+j} \binom{k+j}{k-j} \binom{2k+r}{k-r} \binom{k+r}{r}^{-1} \binom{2j+r}{j} \binom{j+r}{r},$$

$L_{k0}^{-1} = 2(-1)^k \binom{2k+r}{k+r} \binom{k+r}{r}^{-1}$ and $L_{00}^{-1} = 1$,

for $k \geq 1$,

$$U_{kj} = (-1)^k 2 \binom{2j+s}{k+j} \binom{2k+r}{k-r} \binom{j-k+s}{s} \binom{k+r}{r}^{-1} \binom{j+s}{s}^{-1}$$

and $U_{0j} = \binom{2j+s}{j}$,

$$U_{kj}^{-1} = (-1)^k \frac{j}{k+j} \binom{k+j}{j-k} \binom{2j+r}{j-r} \binom{k+r}{r}^{-1} \binom{2k+s}{s} \binom{k+s}{s},$$

$$A_{kj} = (-1)^{k+j} \binom{k}{j} \binom{n+k-1}{2k} \binom{n+j-1}{2j} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \binom{j+s}{s}^{-1}$$

$$\times \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$A_{kj}^{-1} = \binom{k}{j} \binom{n+k-1}{2k} \binom{n+j-1}{2j} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \binom{j+s}{s}^{-1}$$

$$\times \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$B_{kj} = \binom{n+j-1}{2j} \binom{j}{k} \binom{2k+s}{s} \binom{k+r}{r} \binom{2j+r}{j}^{-1},$$

$$B_{kj}^{-1} = (-1)^{n+j+1} \binom{j}{k} \binom{n+k-1}{2k} \binom{2j+s}{s} \binom{2k+r}{r} \binom{k+r}{r}^{-1},$$

for $r = s$ and $j \geq 1$,

$$C_{kj} = (-2)^{j/2} \binom{2k+r}{k+j} \binom{k+r}{r}^{-1} \binom{k-j+r}{r},$$

971
for \( j = 0 \),

\[
C_{k0} = \binom{2k + r}{k}.
\]

Thus

\[
\det T = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} \binom{2k + s}{2k} \binom{2k + r}{r} \binom{k + r}{s}^{-1}.
\]

References


