Isometric $N$-Jordan weighted shift operators

Saeed YARMAHMOODI$^1$, Karim HEDAYATIAN$^{2,*}$

$^1$Department of Mathematics, Marvdasht University, Islamic Azad University, Marvdasht, Iran
$^2$Department of Mathematics College of Sciences, Shiraz University, Shiraz, Iran

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Abstract: A bounded linear operator $T$ on a Hilbert space is an isometric $N$-Jordan operator if it can be written as $A+Q$, where $A$ is an isometry and $Q$ is a nilpotent of order $N$ such that $AQ = QA$. In this paper, we will show that the only isometric $N$-Jordan weighted shift operators are isometries. This answers a question recently raised.

Key words: Isometric $N$-Jordan operator, nilpotent, weighted shift operator

1. Introduction and preliminaries

Let $H$ be a Hilbert space and $B(H)$ stand for the space of all bounded linear operators on $H$. An operator $T$ in $B(H)$ is called an isometric $N$-Jordan operator if $T = A + Q$, where $A$ is an isometry and $Q$ is a nilpotent operator of order $N$, that is, $Q^N = 0$ but $Q^{N-1} \neq 0$, and $AQ = QA$. Note that the notions of isometric 1-Jordan and isometry coincide. It follows from Proposition 1.1 of [11] that the operator $T$ is injective. The dynamic and spectral properties of $T$ have been studied in [11]. We note that $T^*T$ is invertible. Indeed, by Corollary 1.2 of [11] the operator $T$ is bounded below, and so for every $h \in H$,

$$\|T^*Th\| \geq |(T^*Th, h)| = \|Th\|^2 \geq c\|h\|^2$$

for some $c > 0$, which implies that $T^*T$ is also bounded below and so is injective and has closed range. However, $H = (\ker(T^*T))^\perp = \overline{\text{ran}(T^*T)} = \text{ran}(T^*T)$ implies that $T^*T$ is invertible. It is easy to see that if $A$ is a unitary operator then

$$(T^*T)^{-1} = 3I - 3TT^* + T^2T^*.$$ 

For a positive integer $m$ an operator $S \in B(H)$ is an $m$-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k} \binom{m}{k} S^k S^k = 0.$$ 

The operator $S$ is called a strict $m$-isometry if it is not an $(m-1)$-isometry. These operators have been introduced by Agler in [1] and have been studied extensively by Agler and Stankus in three papers [2–4].

*Correspondence: hedayati@shirazu.ac.ir

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Recently, such operators have been considered by several authors. It is proved in [7] that every isometric \(N\)-Jordan operator is a strict \((2N-1)\)-isometry. The authors asked about the validity of the converse. In this paper, we prove that the answer is negative.

2. Main results
Suppose that \(H\) is a separable Hilbert space with orthonormal basis \(\{e_n\}_{n \geq 0}(\{e_n\}_{n \in \mathbb{Z}})\). An operator \(S \in B(H)\) is called a unilateral (bilateral) weighted forward shift, provided that for every \(n \in \mathbb{Z}\), \(Se_n = w_ne_{n+1}\), where \((w_n)_n\) is a sequence of bounded complex numbers. Note that \(S\) is injective if and only if \(w_n \neq 0\), for every \(n\). It is known that \(S\) is an isometry if and only if \(|w_n| = 1\) for all \(n\), and is hyponormal if and only if its weight sequence is increasing [9]. Furthermore, \(m\)-isometric weighted shifts are discussed in [6, 8, 10]. Recall that \(S^*\), the adjoint of \(S\), is called a unilateral (bilateral) weighted backward shift. In this section, we will show that the only isometric \(N\)-Jordan weighted shift operators are isometries.

**Theorem 1** There is no isometric \(N\)-Jordan weighted shift operator when \(N > 1\).

**Proof** In contrast, assume that \(T = A + Q\) is an isometric \(N\)-Jordan weighted shift operator. In the proof of Theorem 2.2 of [7], it is shown that
\[
\sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \|T^k h\|^2 = \frac{(2N-2)!}{(N+1)!} \|Q^{N-1} h\|^2.
\]
Let \(J\) be the set \(\mathbb{N} \cup \{0\}\) or \(\mathbb{Z}\) and suppose that the operator \(T\) is a forward shift operator with weight sequence \((w_n)_n\). Put \(Q^{N-1} e_0 = \sum_{n \in J} c_n e_n\). Thus the above equality shows that
\[
0 = \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \|T^k (Q^{N-1} e_0)\|^2
\]
\[
= \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \left\| \sum_{n \in J} c_n T^k e_n \right\|^2
\]
\[
= \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \left\| \sum_{n \in J} c_n \prod_{i=0}^{k-1} w_{n+i} e_{n+k} \right\|^2
\]
\[
= \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \sum_{n \in J} |c_n|^2 \left\| \prod_{i=0}^{k-1} w_{n+i} \right\|^2
\]
\[
= \sum_{n \in J} |c_n|^2 \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} \|T^k e_n\|^2
\]
\[
= \sum_{n \in J} |c_n|^2 \frac{(2N-2)!}{(N+1)!} \|Q^{N-1} e_n\|^2.
\]
On the other hand, for every \(n \in J\)
\[
Q^{N-1} A e_n = w_n Q^{N-1} e_{n+1},
\]
and so
\[ \|Q^{n-1}e_n\| = |w_n| \|Q^{n-1}e_{n+1}\|. \] (1)

Therefore, if \(Q^{n-1}e_0 = 0\) then \(Q^{n-1}e_n = 0\) for every \(n \in J\); hence \(Q^{n-1} \equiv 0\), which is a contradiction. Moreover, if \(Q^{n-1}e_0\) is nonzero then there is \(n_0 \in J\) such that \(e_{n_0} \neq 0\) and the previous argument shows that \(Q^{n-1}e_{n_0} = 0\). Thus, (1) shows that \(Q^{n-1}e_n = 0\) for every \(n \in J\); hence \(Q^{n-1} \equiv 0\), which is again a contradiction. Now suppose that \(T e_n = w_n e_{n-1}\) (\(n \in \mathbb{Z}\)) is a bilateral backward shift operator. Define the unitary operator \(U\) on \(H\) by \(U (\sum_{n \in \mathbb{Z}} \beta_n e_n) = \sum_{n \in \mathbb{Z}} \beta_n e_{-n}\). It is easily seen that \(SU = UT\), where \(S\) is the bilateral forward shift defined by \(S e_n = w_{n} e_{n+1}\). Put \(B = UAU^{-1}\) and \(P = UQU^{-1}\); therefore, \(S = B + P\) is an isometric \(N\)-Jordan operator which is impossible. Lastly, since every unilateral backward shift is not injective, we conclude that \(T\) cannot be a unilateral weighted backward shift. \(\square\)

For a positive integer \(m\) let \(T\) be the unilateral weighted shift with weight sequence \(w_n = \sqrt{\frac{n+m}{n+1}}\), \(n \geq 0\). It is known that \(T\) is a strict \(m\)-isometric operator (see [5, Proposition 8]). Moreover, it is proved in [8] that for every odd number \(m\), there is an invertible bilateral weighted shift that is a strict \(m\)-isometry. Thus, we have the following corollary that answers the question posed in [7].

**Corollary 1** For a fixed \(m > 1\), there is a strict \(m\)-isometric operator \(T\) so that it is not an isometric \(N\)-Jordan operator for every \(N \geq 1\).

Recall that an operator is a co-isometry if its adjoint is an isometry.

**Corollary 2** If the operator \(S = B + P\) is a weighted shift where \(B\) is a co-isometry, \(P\) is a nilpotent operator and \(BP = PB\); then \(P = 0\).

**Proof** Apply the preceding theorem for \(S^* = B^* + P^*\). \(\square\)

Note that the commutativity of \(A\) and \(Q\) is essential in the preceding theorem as the following example shows.

**Example 1** Let \(\{e_n\}_n\) be an orthonormal basis for the Hilbert space \(H\). Define the isometric operator \(A\) by \(A e_n = e_{n+1}\) for all \(n\) and the weighted shift operator \(Q\) by \(Q e_n = v_n e_{n+1}\), where \(v_{2n} = \frac{1}{2n+1}\) and \(v_{2n-1} = 0\). Note that \(Q^2 = 0\) and \(AQ \neq QA\). Moreover, \(T = A + Q\) is a forward weighted shift with weight sequence \(w_{2n} = 1 + \frac{1}{2n+1}\) and \(w_{2n+1} = 1\).

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**References**


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