A remark on singularity of homeomorphisms and Hausdorff dimension

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Abstract: We prove that there is a homeomorphism of the unit interval onto itself that is so singular that it maps some set $E$ of $\dim_H E = 0$ onto a set $F$ of $\dim_H [0, 1] \setminus F = 0$.

Key words: Singularity, homeomorphism, Hausdorff dimension

1. Introduction

Let $E \subset \mathbb{R}^d$, $s > 0$, and $\delta > 0$. A family of sets $\{U_i\}_{i=1}^\infty$ is called a $\delta$-covering of the set $E$ if $\bigcup_{i=1}^\infty U_i \supset E$ and $0 < |U_i| \leq \delta$ for all $i$, where $|U_i|$ denotes the diameter of $U_i$. The $s$-dimension Hausdorff measure of the set $E$ is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(E),$$

(1)

where

$$\mathcal{H}_\delta^s(E) = \inf\{\sum |U_i|^s : \{U_i\}_{i=1}^\infty \text{ is a } \delta\text{-covering of } E\}.$$

There is a unique value of $s$ such that $\mathcal{H}^s(E)$ jumps from $\infty$ to 0. This value, denoted by $\dim_H E$, is called the Hausdorff dimension of $E$. Thus,

$$\dim_H E = \sup\{s : \mathcal{H}^s(E) > 0\} = \inf\{s : \mathcal{H}^s(E) < \infty\}.$$  

(2)

For the properties of the Hausdorff dimension we refer to [1].

A homeomorphism $f : [0, 1] \to [0, 1]$ is singular if it maps some set of Lebesgue measure zero onto a set of Lebesgue measure 1. Singular homeomorphisms can be used to construct measurable sets that are not Borel. They also act as examples of increasing functions satisfying

$$\int_0^1 f'(x)dx < f(1) - f(0).$$

We may give a finer description for the singularity of a homeomorphism by means of Hausdorff dimension. We say that a homeomorphism is $(\alpha, \beta)$-singular if it maps some set $E$ of $\dim_H E \leq \alpha$ onto a set $F$ of...
dim_H[0,1] \setminus F \leq \beta$, where $\alpha, \beta \in [0,1]$. It is known that for any $\alpha, \beta \in (0,1)$ there are $(\alpha, \beta)$-singular quasisymmetric homeomorphisms (see [4]). However, quasisymmetric homeomorphisms are not $(0,0)$-singular because they preserve sets of dim$_H = 0$ by their Hölder-continuity (see [3]). In this note, we shall prove that a general homeomorphism can be $(0,0)$-singular.

**Theorem 1** There is a homeomorphism $f : [0,1] \to [0,1]$ and a set $E \subset [0,1]$ such that $\dim_H E = \dim_H [0,1] \setminus f(E) = 0$.

2. Proof of Theorem 1

We start by recalling middle interval Cantor sets. Let $E_0 = [a; b]$ be a closed interval. Let $\{\lambda_i\}_{i=1}^\infty$ be a sequence of numbers in $(0,1)$. Removing an open interval of length $(b-a)$ from the middle of $[a; b]$, we get a set $E_1$ consisting of 2 intervals each of length $(b-a)/2$. Removing an open interval of length $\lambda_2|I|$ from the middle of every component interval $I$ of $E_1$, we get a set $E_2$ consisting of $2^2$ intervals each of length $(1-\lambda_2)(1-\lambda_2)(b-a)$. Proceeding infinitely, we get a nested sequence of compact sets $\{E_i\}_{i=0}^\infty$. The set

$$E := \bigcap_{i=0}^\infty E_i$$

is called a middle interval Cantor set in $[a; b]$. In this case, we also say that $E$ is a $\{\lambda_i\}_{i=1}^\infty$-Cantor set. Obviously, the set $E$ is totally disconnected and has no isolated points. From the definition, the set $E_n$ consists of $2^n$ disjoint closed intervals each of length

$$\delta_n = \frac{b-a}{2^n} \prod_{i=1}^n (1 - \lambda_i).$$

The Hausdorff dimension of the $\{\lambda_i\}_{i=1}^\infty$-Cantor set $E$ is

$$\dim_H E = \lim_{n \to \infty} \frac{n \log 2}{-\log \delta_n}.$$  \hspace{1cm} (5)

We refer to [2] for a proof of (5). For the $\{\lambda_i\}_{i=1}^\infty$-Cantor set $E$ in $[a, b]$ it follows from (4) and (5) that

$$\dim_H E = \lim_{n \to \infty} \frac{n \log 2}{-\log \frac{(b-a)}{2^n} (n+1)!} = 0.$$  \hspace{1cm} (6)

We shall use this fact in the following construction.

For a $\{\lambda_i\}_{i=1}^\infty$-Cantor set $C$ in $[a, b]$ we denote by $\mathcal{I}_n(C)$ the set of components of $E_n$ and let $\mathcal{I}(C) = \bigcup_{n=1}^\infty \mathcal{I}_n(C)$. Denote by $\mathcal{G}(C)$ the set of components of $[a, b] \setminus C$. An element in $\mathcal{I}(C)$ will be called a basic interval of $C$ and an element in $\mathcal{G}(C)$ will be called a gap of $C$.

Now we introduce composite Cantor sets. Let $C_1$ be a $\{\lambda_i^{(1)}\}_{i=1}^\infty$-Cantor set in $[0,1]$. For every gap $J \in \mathcal{G}(C_1)$ we take a $\{\lambda_i^{(2)}\}_{i=1}^\infty$-Cantor set $C_J$ in the closure $\overline{J}$ and write

$$C_2 = C_1 \cup \bigcup_{J \in \mathcal{G}(C_1)} C_J.$$
\[ \mathcal{G}(C_2) = \bigcup_{J \in \mathcal{G}(C_1)} \mathcal{G}(C_J), \]

\[ \mathcal{I}(C_2) = \mathcal{I}(C_1) \cup \bigcup_{J \in \mathcal{G}(C_1)} \mathcal{I}(C_J). \]

Proceeding infinitely, we get three sequences \( \{\mathcal{I}(C_k)\}_{k=1}^{\infty} \), \( \{\mathcal{G}(C_k)\}_{k=1}^{\infty} \), and \( \{C_k\}_{k=1}^{\infty} \). We call the set

\[ C := \bigcup_{k=1}^{\infty} C_k \quad (7) \]

a composite Cantor set in \([0, 1]\). Clearly, a composite Cantor set consists of a countable number of middle interval Cantor sets. It is not a compact set.

Note that a composite Cantor set \( C \) may not be dense in \([0, 1]\). The following lemma gives some necessary and sufficient conditions under which \( C \) is dense in \([0, 1]\).

**Lemma 1** Let \( C \) be a composite Cantor set in \([0, 1]\). The following statements are equivalent:

(i) \( C \) is dense in \([0, 1]\).

(ii) \( \frac{1}{2} \in \overline{C} \), where \( \overline{C} \) is the closure of \( C \).

(iii) \( \prod_{k=1}^{\infty} \lambda_1^{(k)} = 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). This is obvious.

(ii) \( \Rightarrow \) (iii). For every \( i \geq 1 \) let \( J_i \) denote the gap of \( C_i \) in the midst of \([0, 1]\). Then

\[ |J_i| = \prod_{j=1}^{i} \lambda_1^{(j)}. \]

Let

\[ x_i = \frac{1}{2} \left( 1 - \prod_{j=1}^{i} \lambda_1^{(j)} \right), \quad i \geq 1. \]

We see that \( x_i \) is the left endpoint of the gap \( J_i \). By (ii) and the construction of \( C \), we have \( \lim_{i \to \infty} x_i = \frac{1}{2} \), which implies

\[ \prod_{k=1}^{\infty} \lambda_1^{(k)} = 0. \]

(iii) \( \Rightarrow \) (i). Let \( k \geq 1 \) and let \( J \in \mathcal{G}(C_k) \) be given. Denote by \( M(J) \) the middle point of \( J \). We have \( M(J) = \inf J + \frac{1}{2}|J| \). Let

\[ x_i = \inf J + \frac{1}{2} \left( 1 - \prod_{j=1}^{i} \lambda_1^{(k+j)} |J| \right), \quad i \geq 1. \quad (8) \]

We see that \( x_i \) is the left endpoint of the gap of \( C_{k+i} \) in the midst of \( J \). By (iii), we have \( \prod_{j=1}^{\infty} \lambda_1^{(k+j)} = 0 \), so \( \lim_{i \to \infty} x_i = M(J) \). This implies \( M(J) \in \overline{C} \). Next we show that \( J \subset \overline{C} \). In fact, given \( u \in J \setminus C \), we have
from the construction of the composite Cantor set $C$ that
\[ \text{dist}(u, C) \leq |x_i - M(J)| \]
for all $x_i$ in (8), which gives $u \in \overline{C}$, and thus $J \subset \overline{C}$. Finally, since $J \subset \overline{C}$ for all $J \in \cup_{k=1}^\infty \mathcal{G}(C_k)$, it is easy to see that $C$ is dense in $[0, 1]$.

\[ \square \]

**Proof of Theorem 1.** Note that if $X, Y$ are two dense composite Cantor sets in $[0, 1]$ then we have a unique increasing homeomorphism $f : [0, 1] \rightarrow [0, 1]$ such that $f(X) = Y$. Therefore, to prove Theorem 1, it suffices to construct two dense composite Cantor sets $X$ and $Y$ in $[0, 1]$ such that $\dim_H X = \dim_H [0, 1] \setminus Y = 0$.

The dense composite Cantor set $X$ is constructed as follows: let all middle interval Cantor sets be chosen to be $\{\frac{i}{2^i}\}_{i=1}^\infty$-Cantor sets, and then it follows from (6) that $\dim_H X = 0$ because $X$ consists of a countable number of sets of Hausdorff dimension zero. It is also clear that $X$ is dense in $[0, 1]$ by Lemma 1.

Finally we construct a dense composite Cantor set $Y$ such that $\dim_H [0, 1] \setminus Y = 0$. We will use the following simple fact repeatedly: for any $s, \alpha > 0$ and for any closed interval $I$ there is a $f_i$-Cantor set $C$ in $I$ with $\lambda_i \in (0, \frac{1}{2})$ such that
\[ \sum_{J \in \mathcal{G}(C)} |J|^s \leq \alpha. \]  
\[ \text{(9)} \]

Let $\{s_i\}_{i=1}^\infty$ and $\{\alpha_i\}_{i=1}^\infty$ be two fixed sequences of positive numbers such that $s_i$ is decreasing to zero and $\sum \alpha_i = 1$. The desired composite of Cantor sets $Y$ can be inductively constructed as follows:

Choose a $\{\lambda_i^{(1)}\}_{i=1}^\infty$-Cantor set $C_1$ in $[0, 1]$ with $\lambda_i^{(1)} \in (0, \frac{1}{2})$ such that
\[ \sum_{J \in \mathcal{G}(C_1)} |J|^{s_1} \leq 1. \]

Let $\{J_i\}_{i=1}^\infty$ be an enumeration of members of $\mathcal{G}(C_1)$. For every $J_i$ choose a $\{\lambda_i^{(2)}\}_{i=1}^\infty$-Cantor set $C_{J_i}$ in the closure $\overline{J_i}$ with $\lambda_i^{(2)} \in (0, \frac{1}{2})$ such that
\[ \sum_{J \in \mathcal{G}(C_{J_i})} |J|^{s_2} \leq \alpha_i. \]

Take $C_2 = C_1 \cup \cup_{i=1}^\infty C_{J_i}$. Since $\sum \alpha_i = 1$ is assumed, it follows that
\[ \sum_{J \in \mathcal{G}(C_2)} |J|^{s_2} = \sum_{i=1}^\infty \sum_{J \in \mathcal{G}(C_{J_i})} |J|^{s_2} \leq 1. \]

Proceeding infinitely, we get an increasing sequence $\{C_k\}_{k=1}^\infty$ of sets such that
\[ \sum_{J \in \mathcal{G}(C_k)} |J|^{s_k} \leq 1 \text{ for all } k \geq 1. \]  
\[ \text{(10)} \]

Let $Y = \cup_{k=1}^\infty C_k$. Then $Y$ is a composite of Cantor sets and $Y$ is dense in $[0, 1]$ by Lemma 1. To complete this proof we are going to show $\dim_H [0, 1] \setminus Y = 0$. By the construction, $\mathcal{G}(C_k)$ is a covering of $[0, 1] \setminus Y$ and
every member of $G(C_k)$ has diameter of at most $2^{-k}$. Therefore,

$$H_{2^{-k}}^s([0, 1] \setminus Y) \leq \sum_{J \in \mathcal{G}(C_k)} |J|^s_i$$

for all $i, k \geq 1$. Given $i \geq 1$, since $s_k$ has been assumed to be decreasing, it follows from (10) that

$$H_{2^{-k}}^s([0, 1] \setminus Y) \leq \sum_{J \in \mathcal{G}(C_k)} |J|^s_k \leq 1$$

for any $k > i$, and so $H^s([0, 1] \setminus Y) \leq 1$. Since $s_i$ is also assumed to tend to zero, we get $\dim_H[0, 1] \setminus Y = 0$. This completes the proof of Theorem 1.

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