On the comaximal ideal graph of a commutative ring

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Abstract: Let $R$ be a commutative ring with identity. We use $\Gamma(R)$ to denote the comaximal ideal graph. The vertices of $\Gamma(R)$ are proper ideals of $R$ that are not contained in the Jacobson radical of $R$, and two vertices $I$ and $J$ are adjacent if and only if $I + J = R$. In this paper we show some properties of this graph together with the planarity and perfection of $\Gamma(R)$.

Key words: Chromatic number, clique number, planar graph, perfect graph

1. Introduction

For the sake of completeness, we explain some definitions and points used throughout this paper. A graph with vertex set $V$ is said to be a graph on $V$. The vertex set of a graph $G$ is referred to as $V(G)$ and its edge set as $E(G)$. Let $v$ be a vertex of $G$. The neighbourhood of $v$ is the set $N_G(v) = \{u \in G|uv \in G\}$. For a graph $G$, the degree of a vertex $v$ in $G$, $\deg(v)$, is the number of edges of $G$ incident with $v$. A graph $G$ is said to be connected if there is at least one path between every pair of vertices in $G$ and the distance between two vertices $v$ and $w$, $d(v,w)$, is the length of the shortest path connecting them. The diameter of a connected graph is the maximum of the distances between vertices. A loop of $G$ is an edge that joins a vertex to itself. Multiple edges are two or more edges connecting the same two vertices within a multigraph. A simple graph is an unweighted, undirected graph containing no loops or multiple edges. A connected acyclic graph is called a tree. Acyclic graphs are usually called forests. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We denote by $K_n$ a complete graph with $n$ vertices. A complete bipartite graph is a bipartite graph (i.e. a set of graph vertices decomposed into two disjoint sets $X$ and $Y$ such that no two graph vertices within the same set are adjacent) such that all pairs of graph vertices in the two sets are adjacent. We denote by $K_{n,m}$ a complete bipartite graph with $|X| = n$ and $|Y| = m$. We define a coloring of $G$ to be an assignment of colors to the vertices of $G$, one color to each vertex, so that adjacent vertices are assigned distinct colors. If $n$ colors are used, then the coloring is referred to as $n -$ coloring. If there exists $n -$ coloring of $G$, then $G$ is called $n -$ colorable. The minimum $n$ for which $G$ is $n -$ colorable is called the chromatic number of $G$, and is denoted by $\chi(G)$. A subset $S$ of the set of vertices of $G$ is said to be a clique in $G$ if every pair of distinct elements $x$ and $y$ of $S$ is adjacent in $G$. The clique number of $G$ is the maximum of the cardinality of all cliques in $G$ and is denoted by clique$(G)$.

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Let $\mathcal{G}$ be the graph with the same vertex-set as $G$, where two distinct vertices are adjacent whenever they are nonadjacent in $G$. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Kuratowski’s theorem says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$[1, Theorem 4.4.6]. A subgraph of $G$ is a graph $H$ such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. The subgraph of $G$ induced by a subset $S$ of vertices of $G$ is the subgraph whose vertex set is $S$ and whose edges are all the edges of $G$ with both ends in $S$[5]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertices set $V_i$ and edges set $E_i$. The union of $G_1$ and $G_2$ is denoted by $G = G_1 \cup G_2$ with vertices set $V_1 \cup V_2$ and edges set $E_1 \cup E_2$. The join of $G_1$ and $G_2$ is denoted by $G = G_1 \vee G_2$ with vertices set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{xy|x \in V_1 \text{ and } y \in V_2\}$.

From now on let $R$ be a commutative ring with identity. In [4], Sharma and Bhatwadekar defined a graph on $R$, with vertices as elements of $R$, where two distinct vertices $a$ and $b$ are adjacent if and only if $Ra + Rb = R$.

Later, Maimani et al. [2] studied the graph structure defined by Sharma and Bhatwadekar and named such graph structure "Comaximal Graphs". They considered the subgraph of Sharma’s graph, $\Gamma_2(R)$, which consists of nonunit elements.

In [6], Ye and Wu defined comaximal ideal graph, $\Gamma(R)$, with vertices as proper ideals of $R$ that are not contained in the Jacobson radical of $R$, and two vertices $I$ and $J$ are adjacent if and only if $I + J = R$.

Some results of this paper for the graph $\Gamma(R)$ are similar to the results in [3] for the graph $\Gamma_2(R) \setminus J(R)$.

In this paper, we consider some properties of $\Gamma(R)$ and we investigate the planarity and perfection of this graph.

2. Properties of $\Gamma(R)$

Let $J(R)$ be Jacobson radical of $R$. $R$ is said to be local if it has a unique maximal ideal. Let Max($R$) be the set of maximal ideals of $R$ and $|\text{Max}(R)|$ denote the number of maximal ideals of $R$. For any maximal ideal $M$ of $R$, $\mathcal{M}$ denotes the set of nonzero ideals contained in $M$ and $|\mathcal{M}|$ denotes the number of ideals contained in $M$.

In [6], Ye and Wu showed that $\Gamma(R)$ has distance of at most 3. In what follows, first we characterize the cases in which two vertices have distance 1, 2, or 3. For any ideal $I$ of $R$, let

$$M(I) = \{M \in \text{Max}(R) : I \subseteq M\}.$$ 

**Lemma 2.1** The elements $I$ and $J$ are adjacent in $\Gamma(R)$ if and only if there does not exist a maximal ideal $M$ that contains both of them, that is,

$$\{I, J\} \in E(\Gamma(R)) \iff M(I) \cap M(J) = \emptyset.$$

**Proof** Assume $I, J \subseteq M$, where $M \in \text{Max}(R)$; then $I + J \subseteq M$ and so $I$ and $J$ cannot be adjacent. Conversely, if $I$ and $J$ are not adjacent, then $I + J$ is a proper ideal of $R$; hence there exists a maximal ideal $M$ such that $I + J \subseteq M$, and therefore $M(I) \cap M(J) \neq \emptyset$. $\square$
Theorem 2.2 ([6], Theorem 2.4) For a ring $R$, $\Gamma(R)$ is a simple, connected graph with diameter less than or equal to three.

Proposition 2.3 Let $G = \Gamma(R)$ and $I, J, K \in G$ be distinct elements. Then the following are equivalent:
(a) $K \in N_G(I) \cap N_G(J)$;
(b) $K \in N_G(IJ)$;
(c) $K \in N_G(I \cap J)$.

Proof (a) $\Rightarrow$ (b): Suppose $K \in N_G(I) \cap N_G(J)$. Then $K + I = R = K + J$. Thus $k_1 + i = 1$ and $k_2 + j = 1$ for some $k_1, k_2 \in K, i \in I$, and $j \in J$. Therefore, $1 = ij + ik_2 + jk_1 + k_1k_2$, which implies that $IJ + K = R$. Hence $K \in N_G(IJ)$. 

(b) $\Rightarrow$ (c): Assume $K + IJ = R$. As $IJ \subseteq I \cap J$, and so $K + (I \cap J) = R$, and $K \in N_G(I \cap J)$.

(c) $\Rightarrow$ (a): If $K + (I \cap J) = R$, then $K + I = R$ and $K + J = R$, which means that $K \in N_G(I)$ and $K \in N_G(J)$. Thus $K \in N_G(I) \cap N_G(J)$.

Theorem 2.4 Let $G = \Gamma(R)$ and $I, J \in G$ be distinct elements. Then the following hold.
(a) $d(I, J) = 1$ if and only if $M(I) \cap M(J) = \emptyset$.
(b) $d(I, J) = 2$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \nsubseteq J(R)$.
(c) $d(I, J) = 3$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \subseteq J(R)$.

Proof (a): By Lemma 2.1. (b): Assume that $d(I, J) = 2$. Then $M(I) \cap M(J) \neq \emptyset$, by Lemma 2.1 and there is a $K$ in $\Gamma(R)$ such that $K \in N_G(I)$ and $K \in N_G(J)$. Thus $K \in N_G(IJ)$, by Proposition 2.3, which implies that $\deg(IJ) > 0$. Therefore $IJ \nsubseteq J(R)$, by [6, Proposition 2.1(2)]. Conversely, if $IJ \nsubseteq J(R)$, then $\deg(IJ) > 0$ and there is a $K$ in $\Gamma(R)$ such that $K + IJ = R$. Again according to Proposition 2.3, $K \in N_G(I) \cap N_G(J)$. Since $M(I) \cap M(J) \neq \emptyset$, $d(I, J) = 1$. Thus $d(I, J) = 2$.

(c): According to Theorem 2.2 and (b), $d(I, J) = 3$ if and only if $M(I) \cap M(J) \neq \emptyset$ and $IJ \subseteq J(R)$.

In what follows, we investigate the condition that $\Gamma(R)$ is a planar graph.

Lemma 2.5 If $\Gamma(R)$ is planar, then $|\text{Max}(R)| \leq 4$.

Proof Assume the contrary that $|\text{Max}(R)| \geq 5$. Let $M_1, \ldots, M_5$ be distinct maximal ideals of $R$. As every two maximal ideals are comaximal, $K_5$ is a subgraph of $\Gamma(R)$. Therefore $\Gamma(R)$ is not planar, by Kuratowski’s theorem, which is a contradiction. Hence $|\text{Max}(R)| \leq 4$.

If $|\text{Max}(R)| = 1$, then $\Gamma(R)$ is an empty graph, by [6, Proposition 2.1(1)] and it is planar. Suppose that $|\text{Max}(R)| = 2$. Then $\Gamma(R)$ is a complete bipartite graph, by [6, Lemma 4.1]. Thus $\Gamma(R)$ is planar if and only if $|M_1 \setminus M_2| \leq 2$ or $|M_2 \setminus M_1| \leq 2$. Otherwise it has $K_{3,3}$ as a subgraph and so it is not planar.

Assume that $|\text{Max}(R)| = 3$ and $M_1, M_2,$ and $M_3$ are distinct maximal ideals of $R$. Set $V_i := M_i \setminus \bigcup_{j \neq i} M_j$, $V_{i_1 i_2} := (M_{i_1} \cap M_{i_2}) \setminus M_j$ for $j \neq i_1, i_2$ and $1 \leq i_1 < i_2 \leq 3$. It is obvious that $|V_i| \geq 1$, since $M_i \in V_i$.

By the above notations, we have the following theorem.

Theorem 2.6 Assume that $|\text{Max}(R)| = 3$. Then $\Gamma(R)$ is planar if and only if one of the following conditions hold.
(a) For only one $V_i$, $|V_i| \geq 3$ and for $j \neq i$, $|V_j| = 1$. Moreover, $V_{jk} = \emptyset$ for distinct $j, k$, where $1 \leq j, k \leq 3$.
(b) $|V_i| = 2$ for all $1 \leq i \leq 3$, and $V_{ij} = \emptyset$ for all $1 \leq i < j \leq 3$. 


(c) $|V_i| = |V_j| = 2$, $|V_k| = 1$ with $\{i, j, k\} = \{1, 2, 3\}$ and $V_{ki} = \emptyset$ or $V_{kj} = \emptyset$.

(d) There exists only one $V_i$ with $|V_i| = 2$ and $|V_j| = 1$ for all $j \neq i$, where $1 \leq i, j \leq 3$.

(e) $|V_i| = 1$ for all $1 \leq i \leq 3$.

Proof (\(\Rightarrow\)): Recall that each ideal in $V_i$ is adjacent to all ideals of $V_j$, $j \neq i$, and all ideals in $V_{jk}$, $j \neq i$ and $k \neq i$, by the definition of $\Gamma(R)$. There is no edge between ideals of $V_i$ and $V_{ik}$. There is also no edge between an ideal of $V_{ik}$ and an ideal of $V_{jk}$. According to the given conditions, in all cases, graphs can be drawn in the plane. See Figures 1–5.

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)

![Figure 5](image5.png)

Figure 1. (a). $|V_1| \geq 3, |V_2| = |V_3| = 1$, and $V_{23} = \emptyset$. 
Therefore, in this case \( R \) is not planar. 

Figure 9. Hence, according to Theorem 2.6, in the conditions (a) and (b) of the theorem, we can draw the diagram of \( R \) as shown in Figure 8 and Figure 9. Hence \( \Gamma(R) \) is planar.

\[
\begin{align*}
\text{Case 1.} & \text{ If for distinct } i \text{ and } j \text{ with } 1 \leq i, j \leq 3, |V_i|, |V_j| \geq 3, \text{ then we have } K_{3,3} \text{ in } \Gamma(R) \text{ and so it is not planar.} \\
\text{Case 2.} & \text{ Let there exist only one } V_i \text{ such that } |V_i| \geq 3. \text{ Without loss of generality, we assume that } |V_1| \geq 3. \text{ If } |V_2 \cup V_3| \geq 3 \text{ or } |V_2 \cup V_3 \cup V_{23}| \geq 3, \text{ then } K_{3,3} \text{ is the subgraph of } \Gamma(R) \text{ and so it is not planar.} \\
\text{Therefore, in this case } \Gamma(R) \text{ is planar if } |V_2| = |V_3| = 1 \text{ and } V_{23} = \emptyset. \\
\text{Case 3.} & \text{ Assume that } |V_i| \leq 2 \text{ for all } 1 \leq i \leq 3. \text{ First suppose that } |V_i| = 2 \text{ for all } 1 \leq i \leq 3. \text{ Let there exist } V_{ij}, \text{ say } V_{12}, \text{ such that } V_{12} \neq \emptyset. \text{ Let } V_1 = \{I_1, J_1\}, V_2 = \{I_2, J_2\}, V_3 = \{I_3, J_3\}, \text{ and } K \in V_1. \text{ As each ideal in } V_1 \text{ is adjacent to all ideals of } V_j \text{ for } i \neq j, \text{ and all ideals in } V_{jk} \text{ for } j, k \neq i, \text{ } \Gamma(R) \text{ has a subdivision of } K_{3,3} \text{ (Figure 6) and it is not planar. Therefore in this case } |V_i| = 2 \text{ and } V_{ij} = \emptyset \text{ for all distinct } i \text{ and } j. \\
\end{align*}
\]

![Figure 6](image1.png) ![Figure 7](image2.png)

Now let without loss of generality, \(|V_1| = |V_3| = 2\) and \(|V_2| = 1\). Let \(V_{12}, V_{23} \neq \emptyset, V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3, J_3\}, I \in V_{12}, \text{ and } J \in V_{23}. \text{ Then the subgraph generated by } \{I_1, J_1, I_2, I_3, J_3, I, J\} \text{ is a subdivision of } K_5 \text{ as shown in Figure 7. Therefore } \Gamma(R) \text{ is not planar.}

At the end, let for a unique \( V_i, |V_i| = 2 \) and \(|V_j| = 1 \) for \( 1 \leq j \neq i \leq 3 \) or \(|V_i| = 1 \) for all \( 1 \leq i \leq 3 \). It is obvious that in these cases \( \Gamma(R) \) is planar. \(\square\)

Now suppose that \(|\text{Max}(R)| = 4\). Set \( V_i := M_{i1} \setminus \bigcup_{j \neq i} M_j, V_{i12} := (M_{i1} \cap M_{i2}) \setminus \bigcup_{j \neq i_1, i_2} M_j, V_{i12i3} := (M_{i1} \cap M_{i2} \cap M_{i3}) \setminus M_j \) for \( j \neq i_1, i_2, i_3, 1 \leq i, j \leq 4 \), where \( 1 \leq i_1 < i_2 < i_3 \leq 4 \).

**Theorem 2.7** Assume that \(|\text{Max}(R)| = 4\). Then \( \Gamma(R) \) is planar if and only if one of the following conditions hold.

(a) There exists only one \( V_i \) with \(|V_i| = 2\). Also \( V_{jk} = \emptyset \) and \( V_{jkl} = \emptyset \) for distinct \( 1 \leq i, j, k, l \leq 4 \).

(b) \(|V_i| = 1 \) for all \( 1 \leq i \leq 4 \).

**Proof** \((\Rightarrow): \text{ Note that each ideal in } V_i \text{ is adjacent to all ideals of } V_j \text{ for } 1 \leq i \neq j \leq 4, \text{ all ideals in } V_{jk} \text{ for } 1 \leq j, k \neq i \leq 4, \text{ and all ideals in } V_{jkl} \text{ for } 1 \leq j, k, l \neq i \leq 4, \text{ according to the definition of } \Gamma(R). \text{ Similar to Theorem 2.6, in the conditions (a) and (b) of the theorem, we can draw the figure of } \Gamma(R) \text{ as Figure 8 and Figure 9. Hence } \Gamma(R) \text{ is planar.} \)
Figure 8. (a). \( V_1 = \{I_1, J_1\}, \ V_2 = \{I_2\}, \ V_3 = \{I_3\}, \ V_4 = \{I_4\}, \ V_{12}, V_{13}, V_{14}, V_{123}, V_{124}, V_{134} \neq \emptyset, \) and \( V_{23}, V_{24}, V_{34}, V_{234} = \emptyset. \)

Figure 9. (b). \( V_1 = \{I_1\}, \ V_2 = \{I_2\}, \ V_3 = \{I_3\}, \ V_4 = \{I_4\}, \) and \( V_{12}, V_{13}, V_{14}, V_{23}, V_{24}, V_{34}, V_{123}, V_{124}, V_{134}, V_{234} \neq \emptyset. \)

\((\Rightarrow)\): Assume that for some \(i\) with \(1 \leq i \leq 4, \ |V_i| \geq 2.\) Let \( |V_1| \geq 2.\) We have the following cases:

Case 1. Let for some \(j\) with \(2 \leq j \leq 4, \ |V_j| \geq 2.\) Without loss of generality, let \( |V_2| \geq 2.\) Then we have the subdivision of \(K_{3,3}\) in \(\Gamma(R),\) where \(V_1 = \{I_1, J_1\}, \ V_2 = \{I_2, J_2\}, \ K \in V_3, \) and \(L \in V_4.\) Hence \(\Gamma(R)\) is not planar.

Case 2. Assume that for only one \(V_i, \ |V_i| = 2, \ |V_j| = 1,\) for all \(1 \leq j \neq i \leq 4\) and for some \(1 < j < k, \ V_{jk} \neq \emptyset.\) Let \(i = 1\) and \(V_{24} \neq \emptyset\) or \(V_{234} \neq \emptyset.\) If \(V_{24} \neq \emptyset,\) then \(\Gamma(R)\) has the subdivision of \(K_{3,3},\) where \(V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3\}, V_4 = \{I_4\}, \) and \(I \in V_{24}\) (Figure 10). Hence \(\Gamma(R)\) is not planar. Now let \(V_{234} \neq \emptyset. V_1 = \{I_1, J_1\}, V_2 = \{I_2\}, V_3 = \{I_3\}, V_4 = \{I_4\}, \) and \(I \in V_{234}.\) Then \(\Gamma(R)\) has the subdivision of \(K_5\) (Figure 11). Thus \(\Gamma(R)\) is not planar.

\[\square\]

3. Perfect comaximal ideal graph of a commutative ring

In this section, we investigate the perfection of \(\Gamma(R).\) Firstly we recall some definitions and notations on perfect graphs.

**Definition 3.1** ([1]) A graph \(G\) is perfect if for every induced subgraph \(H\) of \(G, \ \chi(H) = \text{clique}(H).\)
Definition 3.2 ([1]) A graph is chordal (or triangulated) if each of its cycles of length at least 4 has a chord, i.e. if it contains no induced cycles other than triangles.

([1]) Let $G$ be a graph with induced subgraphs $G_1$ and $G_2$ such that $G = G_1 \cup G_2$. Let $S = G_1 \cap G_2$; we say that $G$ arises from $G_1$ and $G_2$ by pasting these graphs together along $S$ (Figure 12).

![Figure 12](image_url)

Figure 12.

Proposition 3.3 ([1], Proposition 5.5.1) A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.

Proposition 3.4 ([1], Proposition 5.5.2) Every chordal graph is perfect.

([1]) Complete graphs, empty graphs, and complete $k$-partite graphs are perfect.

([1]) If $G$ is obtained from two chordal graphs $G_1$ and $G_2$ by pasting them together along a complete subgraph $S$, then $G$ is chordal.

Theorem 3.5 ([1], Berge 1966) A graph $G$ is perfect if and only if neither $G$ or $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.

Theorem 3.6 If $|\text{Max}(R)| \leq 4$, then $\Gamma(R)$ is a perfect graph.

Proof Case 1. Let $R$ be a local ring. Then $\Gamma(R)$ is an empty graph, by [6, Proposition 2.1(1)] and so it is perfect.

Case 2. Assume that $R$ has only two maximal ideals. Then $\Gamma(R)$ is a complete bipartite graph and so it is perfect.

Case 3. Let $|\text{Max}(R)| = 3$. We show that $\overline{\Gamma(R)}$ is chordal. Let $V_i$ and $V_{ij}$ be defined as in Section 2. The connection between these sets is as Figure 13.

![Figure 13](image_url)

Figure 13. $V_i$’s are independent of each other and $V_{jk}$ too, $j \neq i$ and $k \neq i$, each $V_i$ is a complete graph, by the definition of $\overline{\Gamma(R)}$. 

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Consider $G_1 = \Gamma(R)[V_1 \cup V_{12} \cup V_{13}]$ and $G_2 = \Gamma(R)[V_3 \cup V_{13} \cup V_{23}]$. Both $G_1$ and $G_2$ are complete graphs. $G'$ arises from $G_1$ and $G_2$ by pasting these graphs together along $S = \Gamma(R)[V_{13}]$. Thus $G'$ is a chordal graph, since $\Gamma(R)[V_{13}]$ is complete. Now define $G'' = \Gamma(R)[V_2 \cup V_{12} \cup V_{23}]$. Then $G''$ is a complete graph. $\Gamma(R)$ arises from $G'$ and $G''$ by pasting these graphs together along $S = \Gamma(R)[V_{12} \cup V_{23}]$. Since $\Gamma(R)[V_{12} \cup V_{23}]$ is complete, $\Gamma(R)$ is a chordal graph and so it is perfect.

Case 4. Suppose that $|\text{Max}(R)| = 4$. We show that $\Gamma(R)$ is chordal. Consider $V_i$, $V_{ij}$, and $V_{ijk}$ as in Section 2.

Let $G$ denote the induced subgraph of $\Gamma(R)$ generated by $V_1 \cup V_2 \cup V_3 \cup V_{12} \cup V_{13} \cup V_{14} \cup V_{23} \cup V_{24} \cup V_{34}$. $G$ is denoted in Figure 14.

We show that this graph has no odd cycle of length at least 5 as an induced subgraph. Assume to the contrary that there is an induced 5-cycle, $C$, in graph $G$. Let there exist two vertices $I$ and $J$, one in $V_i$ and the other in $V_j$ in $C$. Without loss of generality, let $I \in V_1$, $J \in V_2$ and the neighbour of $J$ be in $V_{13}$ or in $V_{14}$. If $K \in V_{13}$ and $JK \in E(C)$, every neighbour of $K$ is in $V_4$ or $V_{24}$, which are joined to $I$. Thus $C$ has a chord, which is a contradiction. A similar case will occur if $K \in V_{14}$.

Now let $C$ have only one vertex from $\bigcup_{i=1}^5 V_i$. Without loss of generality, assume that $I \in V_1$, $I \in C$, $J \in V_{23}$ is adjacent vertex of $I$ in $C$ and $C : I - J - K - L - M - \cdots$. If $K \in V_{14}$, then $L \in V_2$ or $V_3$. Thus $IL \in E(C)$ is a chord in $C$, which is a contradiction. If $K \in V_4$, then $IK \in E(C)$ is a chord, a contradiction.

Therefore $C$ has no vertex in $\bigcup_{i=1}^5 V_i$. However, the induced subgraph of $\bigcup_{1 \leq i < j \leq 4} V_{ij}$, $G$, is a forest and has no cycle and so $G$ is perfect. Now $\Gamma(R)$ is the graph shown in Figure 15.

![Figure 14.](image1.png)

![Figure 15.](image2.png)

It is obvious that $\Gamma(R)$ has no odd cycle of length at least 5 as an induced subgraph. Thus $\Gamma(R)$ is perfect. □

For $|\text{Max}(R)| \geq 5$, maybe $\Gamma(R)$ is not perfect.

**Example 3.7** Consider the ring $R = \mathbb{Z}_{2310}$ with $\text{Max}(R) = \{M_1 = < 2 >, M_2 = < 3 >, M_3 = < 5 >, M_4 = < 7 >, M_5 = < 11 >\}$ and $< 6 > \in V_{12}$, $< 15 > \in V_{23}$, $< 35 > \in V_{34}$, $< 77 > \in V_{45}$, $< 22 > \in V_{15}$.

Clearly, the above ideals are distinct and they are not contained in $J(R)$. It is easy to check that the subgraph $G$ of $\Gamma(R)$ induced on $\{< 6 >, < 15 >, < 35 >, < 77 >, < 22 >\}$ is a $C_5$. Therefore, $\Gamma(R)$ is imperfect.
Now we give the main result on $\Gamma(R)$.

**Corollary 3.8** If $\Gamma(R)$ is planar, then $\Gamma(R)$ is also perfect.

**Proof** By Lemma 2.5 and Theorem 3.6.

The converse of the above corollary does not hold in general.

**Example 3.9** Consider the ring $R = \mathbb{Z}_{36}$. Clearly $\text{Max}(R) = \{<2>, <3>\}$ with $|<2> \setminus <3>| \geq 3$ and $|<3> \setminus <2>| \geq 3$. Then $\Gamma(R)$ is perfect but it is not planar, since it has a subdivision of $K_{3,3}$.

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