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A new aspect to Picard operators with simulation functions

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Abstract: In the present paper, considering the simulation function, we give a new class of Picard operators on complete metric spaces. We also provide a nontrivial example that shows the aforementioned class properly contains some earlier such classes.

Key words: Fixed point, Picard operators, simulation functions

1. Introduction

Let (X,d) be a metric space and $T: X \to X$ be a mapping; then T is called a Picard operator on X, if T has a unique fixed point and the sequence of successive approximation for any initial point converges to the fixed point. The concept of Picard operators is closely related to that of contractive-type mappings on metric spaces. It is well known that almost all contractive-type mappings are Picard operators on complete metric spaces. (See for more details [2–6]).

In the present paper, considering the simulation function, we give a new class of Picard operators on complete metric spaces. The concept of simulation functions is given by [8] in fixed point theory.

Let $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a mapping; then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$

(ζ_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by Z. For example, $\zeta(t,s) = \lambda s - t$ with $0 \le \lambda < 1$ belonging to Z. Many different examples of simulations functions can be found in Example 2.2 of [8].

Before we give our main result we recall the following definition and theorem presented in [8].

Definition 1 ([8]) *Let (X,d) be a metric space, $T: X \to X$ be a mapping, and $\zeta \in$ Z. Then T is called a Z-contraction with respect to ζ if the following condition is satisfied:*

$$\zeta(d(Tx,Ty), d(x,y)) \ge 0 \text{ for all } x, y \in X.$$

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Taking into account Definition 1 we can say that every Banach contraction is a Z-contraction with respect to $\zeta(t,s) = \lambda s - t$ with $0 \leq \lambda < 1$. Moreover, it is clear from the definition of the simulation function that $\zeta(t,s) < 0$ for all $t \geq s > 0$. Therefore, if T is a Z-contraction with respect to $\zeta \in Z$ then

$$d(Tx,Ty) < d(x,y) \text{ for all distinct } x, y \in X.$$

This shows that every Z-contraction mapping is contractive; therefore it is continuous.

Theorem 1 *Every* Z*-contraction on a complete metric space has a unique fixed point and moreover every Picard sequence converges to the fixed point.*

If we consider the concept of Picard operator, every Z-contraction on a complete metric is a Picard operator.

2. Main Result

First we introduce the concept of generalized Z-contraction on metric spaces.

Definition 2 *Let (X,d) be a metric space, $T : X \to X$ be a mapping, and $\zeta \in$ Z. Then T is called generalized* Z*-contraction with respect to ζ if the following condition is satisfied*

$$\zeta(d(Tx,Ty), M(x,y)) \geq 0 \text{ for all } x, y \in X, \tag{2.1}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}.$$

Remark 1 *Every generalized Z-contraction on a metric space has at most one fixed point. Indeed, let z and w be two fixed points of T, which is a generalized Z-contraction self map of a metric space (X,d). Then*

$$0 \leq \zeta(d(Tz,Tw), M(z,w)) = \zeta(d(z,w), d(z,w)),$$

which is a contradiction.

Now we give our main theorem.

Theorem 2 *Every generalized Z-contraction on a complete metric space is a Picard operator.*

Proof Let (X,d) be a complete metric space and $T : X \to X$ be a generalized Z-contraction with respect to $\zeta \in Z$. First, we show that T has a fixed point.

Let $x_0 \in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then x_{n_0} is a fixed point of T. Now suppose $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$ and define $d_n = d(x_n, x_{n+1})$. Then, since

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{ \begin{array}{c} d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \\ \frac{1}{2}[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \end{array} \right\} \\ &= \max\{d_{n-1}, d_n\} \end{aligned}$$

from (2.1) we get

$$
\begin{aligned}
0 &\leq \zeta(d(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})) \\
&= \zeta(d_n, \max\{d_{n-1}, d_n\}).
\end{aligned} \tag{2.2}
$$

Suppose that $d_n \geq d_{n-1}$ for some $n \in \mathbb{N}$; then from (2.2)

$$
0 \leq \zeta(d_n, \max\{d_{n-1}, d_n\}) = \zeta(d_n, d_n),
$$

which is a contradiction. Thus $d_n < d_{n-1}$ for all $n \in \mathbb{N}$ and

$$
0 \leq \zeta(d_n, d_{n-1}). \tag{2.3}
$$

Therefore, the sequence $\{d_n\}$ is a decreasing sequence of nonnegative reals and so it must be convergent. Let $\lim_{n\to\infty} d_n = r \geq 0$. If $r > 0$ then from (2.3) and (ζ_3) we have

$$
0 \leq \lim_{n\to\infty} \sup \zeta(d_n, d_{n-1}) < 0,
$$

which is a contradiction. Therefore, we have $r = 0$, that is, $\lim_{n\to\infty} d_n = 0$.

Now we show that the Picard sequence $\{x_n\}$ is bounded. Assume that $\{x_n\}$ is not bounded. Without loss of generality we can assume that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Since $\{x_n\}$ is not bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and, for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$
d(x_{n_{k+1}}, x_{n_k}) > 1
$$

and

$$
d(x_m, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1.
$$

Therefore, by the triangular inequality we have

$$
\begin{aligned}
1 &< d(x_{n_{k+1}}, x_{n_k}) \\
&\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\
&\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1.
\end{aligned}
$$

Letting $k \to \infty$ we get

$$
\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1.
$$

Now, since

$$\begin{aligned} 1 &< d(x_{n_{k+1}}, x_{n_k}) \leq M(x_{n_{k+1}-1}, x_{n_k-1}) \\ &= \max\left\{ \begin{array}{c} d(x_{n_{k+1}-1}, x_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})] \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}), \\ d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})] \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} 1 + d(x_{n_k}, x_{n_k-1}), \\ d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})] \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} 1 + d(x_{n_k}, x_{n_k-1}), \\ d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[1 + d(x_{n_k-1}, x_{n_{k+1}})] \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} 1 + d(x_{n_k}, x_{n_k-1}), \\ d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[1 + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})] \end{array} \right\}, \end{aligned}$$

taking $k \to \infty$ we get

$$1 \leq \lim_{k\to\infty} M(x_{n_{k+1}-1}, x_{n_k-1}) \leq 1,$$

that is, $\lim_{k\to\infty} M(x_{n_{k+1}-1}, x_{n_k-1}) = 1$. By (2.1) we have

$$0 \leq \lim_{k\to\infty} \sup \zeta(d(x_{n_{k+1}}, x_{n_k}), M(x_{n_{k+1}-1}, x_{n_k-1})) < 0,$$

which is a contradiction. This result proves that $\{x_n\}$ is bounded. Now we shall show that the sequence $\{x_n\}$ is a Cauchy sequence. For this, consider the real sequence

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}.$$

Note that the sequence $\{C_n\}$ is a decreasing sequence of nonnegative reals. Thus there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. We shall show that $C = 0$. If $C > 0$ then by the definition of C_n, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Hence

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C. \tag{4}$$

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C.$$

$$
\begin{aligned}
d(x_{m_k-1}, x_{n_k-1}) &\leq M(x_{m_k-1}, x_{n_k-1}) \\
&= \max\left\{\begin{array}{c} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[d(x_{m_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k-1})] \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}[d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) \\ +d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1})] \end{array}\right\}
\end{aligned}
$$

Letting $k \to \infty$ we get

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = C.$$

Using (2.1), we have

$$0 \leq \lim_{k\to\infty} \sup \zeta(d(x_{m_k}, x_{n_k}), M(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Therefore, $C = 0$. That is $\{x_n\}$ is a Cauchy sequence; since X is complete there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of T. Suppose that $Tu \neq u$; then $d(u, Tu) > 0$. Using (2.1), $(\zeta 2)$, $(\zeta 3)$, we have

$$0 \leq \lim_{n\to\infty} \sup \zeta(d(Tx_n, Tu), M(x_n, u)) < 0,$$

since $\lim_{n\to\infty} M(x_n, u) = d(u, Tu)$. This contradiction shows that $d(u, Tu) = 0$, that is, $Tu = u$. If we consider the proof, we can see that every Picard sequence converges to the fixed point of T. Therefore, T is a Picard operator. □

The following example shows that our main theorem is a generalization of Theorem 2.8 of [8].

Example 1 *Let $X = [0, 1]$ and d is a usual metric on X. Define a mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{2}{5} &, \quad x \in [0, \frac{2}{3}) \\ \\ \frac{1}{5} &, \quad x \in [\frac{2}{3}, 1] \end{cases}.$$

Since T is not continuous, then it is not a Z-contraction. Thus considering Theorem 1, we cannot guarantee that T is a Picard operator. Now we claim that T is a generalized Z-contraction with respect to a simulation function defined by $\zeta(t, s) = \frac{6}{7}s - t$. By Example 1.3.1 of [9], we get

$$
\begin{aligned}
d(Tx, Ty) &\leq \frac{3}{7}[d(x, Tx) + d(y, Ty)] \\
&\leq \frac{6}{7}\max\{d(x, Tx), d(y, Ty)\} \\
&\leq \frac{6}{7}M(x, y)
\end{aligned}
$$

for all $x, y \in X$. That is, we have

$$\zeta(d(Tx, Ty), M(x, y)) = \frac{6}{7}M(x, y) - d(Tx, Ty) \geq 0$$

for all $x, y \in X$. Thus, taking into account Theorem 2, we can say that T is a Picard operator.

In the next example, T is a Z-contraction and also a generalized Z-contraction with respect to the same $\zeta \in Z$. However, T is not a Ćirić-type generalized contraction.

Example 2 *Let $X = [0, 1]$ and d is a usual metric on X. Define a mapping $T : X \to X$ as $Tx = \frac{x}{1+x}$. By Example 2.9 of [8] we get T is a Z-contraction with respect to $\zeta \in Z$ where*

$$\zeta(t, s) = \frac{s}{1+s} - t \text{ for all } t, s \in [0, \infty).$$

Therefore, for all $x, y \in X$, we get

$$\begin{aligned} 0 &\leq \zeta(d(Tx, Ty), d(x, y)) \\ &= \frac{d(x, y)}{1 + d(x, y)} - d(Tx, Ty) \\ &\leq \frac{M(x, y)}{1 + M(x, y)} - d(Tx, Ty) \\ &= \zeta(d(Tx, Ty), M(x, y)). \end{aligned}$$

This shows that T is a generalized Z-contraction with respect to the same $\zeta \in Z$. On the other hand, since

$$\sup_{n \in \mathbb{N}} \frac{d(T\frac{1}{n}, T0)}{M(\frac{1}{n}, 0)} = 1,$$

we cannot find $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda M(x, y)$$

for all $x, y \in X$. That is, T is not a Ćirić-type generalized contraction (see for details [1, 7]).

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