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Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions

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Abstract: In this study, we introduce and investigate two new subclasses of the bi-univalent functions; both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric analytic functions. Among other results, upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are found in this investigation.

Key words: Univalent functions, bi-univalent functions, m-fold symmetric bi-univalent functions

1. Introduction

A function is said to be *univalent* (or schlicht) if it never takes the same value twice: $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$.

Let $\mathcal{A}$ denote the class of functions $f(z)$ that are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Moreover, let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$ (for details, see [5]).

The Koebe one quarter theorem (e.g., see [5]) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see [11] (see also [3, 8, 9, 14]).

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In fact, the aforecited work by Srivastava et al. [11] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Ali et al. [2], Srivastava et al. [12], and Jahangiri and Hamidi [7](see also [1, 4, 6, 10], and the references cited in each of them).

Let $m \in \mathbb{N}$. A domain E is said to be *m-fold symmetric* if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric $(m \in \mathbb{N})$ if

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).$$

In particular, every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{1.2}$$

In [13] Srivastava et al. defined m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$, in their study. Furthermore, for the normalized form of f given by (1.2), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \tag{1.3}$$

$$- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$.

The object of the present paper is to introduce new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient estimates for the function class $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$

We begin by introducing the function class $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$ by means of the following definition.

Definition 1 *A function $f(z)$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$ $(0 < \alpha \leq 1,\ \lambda \geq 0, m \in \mathbb{N})$ if the following conditions are satisfied:*

$$f \in \Sigma_m \ \text{ and } \ \left| arg(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{2.2}$$

where the function $g(w)$ is given by (1.3).

Theorem 1 *Let* $f \in \mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$ $(0 < \alpha \leq 1,\ \lambda \geq 0,\ m \in \mathbb{N})$ *be given by (1.2). Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m\lambda)^2 + \alpha m(1+2m\lambda - m\lambda^2)}} \tag{2.3}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2(m+1)}{(1+m\lambda)^2} + \frac{2\alpha}{1+2m\lambda}. \tag{2.4}$$

Proof From (2.1) and (2.2) we have

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = [p(z)]^{\alpha} \tag{2.5}$$

and for its inverse map, $g = f^{-1}$, we have

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(z) = [q(w)]^{\alpha} \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheodory class $\mathcal{P}$ (see for details [5]) and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$(1+m\lambda)a_{m+1} = \alpha p_m \tag{2.9}$$

$$(1+2m\lambda)a_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2 \tag{2.10}$$

$$-(1+m\lambda)a_{m+1} = \alpha q_m \tag{2.11}$$

$$(1+2m\lambda)\left[(m+1)a_{m+1}^2 - a_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2(1+m\lambda)^2 a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \tag{2.14}$$

Moreover, from (2.10), (2.12), and (2.14), we get

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{(1+m\lambda)^2 + \alpha m(1+2m\lambda - m\lambda^2)}. \tag{2.15}$$

Note that, according to the Caratheodory Lemma (see [5]), $|p_m| \leq 2$ and $|q_m| \leq 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Caratheodory Lemma for coefficients p_{2m} and q_{2m} we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m\lambda)^2 + \alpha m(1+2m\lambda - m\lambda^2)}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted (2.3).

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$2(1+2m\lambda)a_{2m+1} - (1+2m\lambda)(m+1)a_{m+1}^2 = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2 - q_m^2).$$

Upon substituting the value of a_{m+1}^2 from (2.14) and observing that $p_m^2 = q_m^2$, it follows that

$$a_{2m+1} = \frac{\alpha^2(m+1)p_m^2}{2(1+m\lambda)^2} + \frac{\alpha(p_{2m} - q_{2m})}{2(1+2m\lambda)}. \tag{2.16}$$

Taking the absolute value of (2.16) and applying the Caratheodory Lemma again for coefficients p_m, p_{2m}, and q_{2m} we obtain

$$|a_{2m+1}| \leq \frac{2\alpha^2(m+1)}{(1+m\lambda)^2} + \frac{2\alpha}{1+2m\lambda}.$$

This completes the proof of Theorem 1. □

3. Coefficient estimates for the function class $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$

Definition 2 *A function $f(z)$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$ $(0 < 1,\ \lambda \geq 0, 0 \leq \beta < 1,\ m \in \mathbb{N})$ if the following conditions are satisfied:*

$$f \in \Sigma_m \ \text{ and } \ Re\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right\} > \beta \qquad (z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right\} > \beta \qquad (w \in \mathbb{U}) \tag{3.2}$$

where the function $g(w)$ is given by (1.3).

Theorem 2 *Let $f \in \mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$ $(0 < 1,\ \lambda \geq 0,\ 0 \leq \beta < 1,\ m \in \mathbb{N})$ be given by (1.2). Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{1-\beta}{(1+2m\lambda)(m+1)}} \tag{3.3}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{(1+m\lambda)^2} + \frac{2(1-\beta)}{1+2m\lambda}. \tag{3.4}$$

Proof It follows from (3.1) and (3.2) that

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$(1+m\lambda)a_{m+1}=(1-\beta)p_m \tag{3.7}$$

$$(1+2m\lambda)a_{2m+1}=(1-\beta)p_{2m} \tag{3.8}$$

$$-(1+m\lambda)a_{m+1}=(1-\beta)q_m \tag{3.9}$$

$$(1+2m\lambda)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]=(1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_m=-q_m \tag{3.11}$$

and

$$2(1+m\lambda)^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Moreover, from (3.8) and (3.10), we obtain

$$(1+2m\lambda)(m+1)a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Thus we have

$$\left|a_{m+1}^2\right|\leq\frac{(1-\beta)}{(1+2m\lambda)(m+1)}\left(|p_{2m}|+|q_{2m}|\right).$$

$$=\frac{4(1-\beta)}{(1+2m\lambda)(m+1)},$$

which is the bound on $|a_{m+1}|$ as given in Theorem 2.

In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$2(1+2m\lambda)a_{2m+1}=(1-\beta)(p_{2m}-q_{2m})+(1+2m\lambda)(m+1)a_{m+1}^2$$

or equivalently

$$a_{2m+1}=\frac{(1-\beta)(p_{2m}-q_{2m})}{2(1+2m\lambda)}+\frac{(m+1)}{2}a_{m+1}^2.$$

Upon substituting the value of a_{m+1}^2 from (3.12), we get

$$a_{2m+1}=\frac{(1-\beta)(p_{2m}-q_{2m})}{2(1+2m\lambda)}+\frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{4(1+m\lambda)^2}.$$

Applying the Caratheodory Lemma for the coefficients p_m, q_m, p_{2m}, and q_{2m}, we find

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{(1+m\lambda)^2} + \frac{2(1-\beta)}{1+2m\lambda}.$$

which is the bound on $|a_{2m+1}|$ as asserted in Theorem 2. □

Remark For 1-fold symmetric bi-univalent functions, Theorem 1 and Theorem 2 reduce to results given by Frasin and Aouf [6]. In addition, for 1-fold symmetric bi-univalent functions, if we put $\lambda = 1$ in our Theorems, we obtain the results given by Srivastava et al.[11]. Furthermore, for m-fold symmetric bi-univalent functions, if we put $\lambda = 1$ in Theorem 1 and Theorem 2, we obtain to results which were given by Srivastava et al. [13].

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