The inclusion theorems for variable exponent Lorentz spaces

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Abstract: Let \((X, \Sigma, \mu)\) and \((X, \Sigma, \nu)\) be measure spaces. Assume that \(L^{p_1(q_1)}(X, \mu)\) and \(L^{p_2(q_2)}(X, \nu)\) are two variable exponent Lorentz spaces where \(p, q \in P_0([0, l])\). In this paper we investigated the existence of the inclusion \(L^{p_1(q_1)}(X, \mu) \subset L^{p_2(q_2)}(X, \nu)\) under what conditions for two measures \(\mu\) and \(\nu\) on \((X, \Sigma)\).

Key words: Inclusion, variable exponent Lorentz space

1. Introduction

Let \((X, \Sigma, \mu)\) be a measure space. The distribution function of \(f\) is defined by

\[
\lambda_f(y) = \mu(\{x \in X : |f(x)| > y\}) = \int_{\{x \in X : |f(x)| > y\}} d\mu(x) \quad [4, 6].
\]

The rearrangement function of \(f\) is defined by

\[
f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\} = \sup \{y > 0 : \lambda_f(y) > t\}, \quad t \geq 0 \quad [4, 6].
\]

Moreover, the average function of \(f^*\) is given by

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.
\]

Let \(0 < l \leq \infty\). We put

\[
p_- = \inf_{x \in [0,l]} p(x), \quad p^+ = \sup_{x \in [0,l]} p(x).
\]

In this paper, we shall also use the notation

\[
P_a = \{p : a < p_- \leq p^+ < \infty\}, \quad a \in \mathbb{R}.
\]

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The set $IP([0, \infty])$ is the family of $p \in L^\infty([0, \infty])$ such that there exist the limits $p(0) = \lim_{x \to 0} p(x)$, $p(\infty) = \lim_{x \to \infty} p(x)$ and we have

$$|p(x) - p(0)| \leq \frac{C}{\ln \frac{1}{|x|}}, \quad |x| \leq \frac{1}{2} \quad (C > 0)$$

and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln (e + |x|)}, \quad |x| > 2 \quad (C > 0). \quad (1.1)$$

We also denote $IP_a([0, l]) = IP([0, l]) \cap P_a([0, l])$. If $l = \infty$, it is enough that the inequality (1.1) is satisfied [4].

Let $\Omega \subset X$. We denote $l = \mu(\Omega)$. Assume that $p, q \in P_0([0, l])$. The variable exponent Lorentz spaces $L^{p(\cdot), p(\cdot)}(\Omega, \mu)$ are defined as the set of all (equivalence classes) measurable functions $f$ on $X$ such that $J_{p,q}(f) < \infty$, where

$$J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} \, dt. \quad (1.2)$$

We use the notation

$$\|f\|_{L^{p(\cdot), q(\cdot)}(\Omega, \mu)} = \inf \left\{ \lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \leq 1 \right\} \quad [2].$$

Let $p \in IP_0([0, l])$ and $q \in IP_1([0, l])$. If $l = \infty$, then the equality (1.2) is equivalent to the following sum:

$$\int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} \, dt + \int_1^\infty t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} \, dt \quad [2].$$

If $l < \infty$, then the equality (1.2) is equivalent to the integral $\int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} \, dt \quad [4]$. The space $L^{p(\cdot), q(\cdot)}(\Omega, \mu)$ is a normed vector space with norm

$$\|f\|_{L^{p(\cdot), q(\cdot)}(\Omega, \mu)} = \inf \left\{ \lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \leq 1 \right\}$$

such that $J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^**(t))^{q(t)} \, dt \quad [4].$

For $0 \leq p \leq q \leq \infty$, the inclusion $L^p(X) \subset L^q(X)$ is known. In [13], the inclusion $L^p(\mu) \subset L^q(\mu)$ was characterized by all positive measures whenever $0 < p \leq q \leq \infty$. Then Romero [10] improved some results of [13]. Lastly, the more general inclusion $L^p(\mu) \subset L^q(\nu)$ was considered by [8], where $\mu$ and $\nu$ are two measures on $(X, \Sigma)$. Moreover, in [5], Gürkanlı considered inclusion theorems of Lorentz spaces. Embeddings for discrete weighted Lebesgue spaces with variable exponents were studied by Nekvinda [9]. In [1], the inclusion $L^{p(\cdot)}(\mu) \subset L^{q(\cdot)}(\nu)$ was considered by Aydin and Gürkanlı. In [2], Bandaliev considered embeddings between
2. Main results

Let \((X, \Sigma, \mu)\) be a measure space. If two measures \(\mu\) and \(\nu\) are absolutely continuous with respect to each other \((\mu << \nu \text{ and } \nu << \mu)\) then we denote this by \(\mu \approx \nu\) [11].

Lemma 1 The inclusion \(L^{p_1,q_1}(X, \mu) \subset L^{p_2,q_2}(X, \nu)\) holds in the sense of equivalence classes if and only if \(\mu \approx \nu\) and \(L^{p_1,q_1}(X, \mu) \subset L^{p_2,q_2}(X, \nu)\) in the sense of individual functions.

Proof Assume that \(L^{p_1,q_1}(X, \mu) \subset L^{p_2,q_2}(X, \nu)\) holds in the sense of equivalence classes. Let \(f \in L^{p_1,q_1}(X, \mu)\) be any individual function. That means \(f \in L^{p_1,q_1}(X, \mu)\) in the sense of equivalence classes. Therefore, we have \(f \in L^{p_2,q_2}(X, \nu)\) in the sense of equivalence classes from the assumption.

Thus we obtain \(f \in L^{p_2,q_2}(X, \nu)\) in the sense of individual functions. Therefore, we find the inclusion \(L^{p_1,q_1}(X, \mu) \subset L^{p_2,q_2}(X, \nu)\) in the sense of individual functions. Let \(E \in S\) with \(\mu(E) = 0\). Then since \(\chi_E = 0\) \(\mu\)-almost everywhere \((a.e)\), we have

\[
J_{p_1,q_1} (\chi_E) = \int_0^t \frac{q_1(t)}{p_1(t)} - 1 (\chi_E^*(t))^{q_1(t)} dt = \int_0^t \frac{q_1(t)}{p_1(t)} - 1 (\chi_{[0, \mu(E)]}(t))^{q_1(t)} dt
\]

\[
= \int_0^{\mu(E)} t^{\frac{n(1)}{p_1(t)}} - 1 dt = 0
\]

and we write \(\chi_E \in L^{p_1,q_1}(X, \mu)\). Therefore, \(\chi_E\) is in the equivalence classes of \(0 \in L^{p_1,q_1}(X, \mu)\). Moreover, the equivalence classes of \(0\) (with respect to \(\mu\)) are also an element of \(L^{p_2,q_2}(X, \nu)\). Thus \(\chi_E\) is in the equivalence classes of \(0 \in L^{p_2,q_2}(X, \nu)\) with respect to \(\nu\). This implies \(\nu(E) = 0\). Therefore, \(\nu << \mu\). Similarly, \(\mu << \nu\) is proved. The proof of the other side is clear.

Throughout, we assume that \(p, q \in P_0([0, l])\) unless the contrary is stated.

Lemma 2 a) Let \(\mu(X) = \infty\), \(p, q \in IP_1([0, \infty]), q(\infty) > p(\infty)\) and \(q(0) < p(0)\). If \((f_n)_{n \in \mathbb{N}}\) conveergences to \(f\) in \(L^{p,q}(X, \mu)\) then \((f_n)_{n \in \mathbb{N}}\) conveergences to \(f\) in measure.

b) Let \(\mu(X) < \infty\) and \(p, q \in IP_1([0, \infty])\). If \((f_n)_{n \in \mathbb{N}}\) conveergences to \(f\) in \(L^{p,q}(X, \mu)\) then \((f_n)_{n \in \mathbb{N}}\) conveergences to \(f\) in measure.

Proof a) Assume that \((f_n)_{n \in \mathbb{N}}\) conveergences to \(f\) in \(L^{p,q}(X, \mu)\). Then we write

\[
J_{p,q}(f_n - f) \equiv \int_0^1 t^{\frac{n(q - p)}{mp}} - 1 (f_n - f)^s(t)^q dt + \int_0^\infty t^{\frac{n(q - p)}{mp}} - 1 (f_n - f)^s(t)^q dt \to 0
\]
for \( n \to \infty \). Since \( q(\infty) > p(\infty) \) and \( q(0) < p(0) \), we have
\[
\int_0^\infty (f_n - f)^* (t)^{q(t)} \, dt \leq J_{p,q}(f_n - f) \to 0.
\]
for \( n \to \infty \). Then \((f_n - f)^*\) converges to 0 in \( L^{q(\cdot)}([0, \infty])\). Thus we find that \((f_n - f)^*\) converges to 0 in measure (with respect to measure on \([0, \infty]\)) by [7]. Furthermore, since
\[
\lambda_{(f_n-f)^*}(\varepsilon) = \mu(\{t : (f_n - f)^* (t) > \varepsilon\}) = \mu(\{x : (f_n - f) (x) > \varepsilon\}) = \lambda_{(f_n-f)}(\varepsilon) \quad [6]
\]
for all \( \varepsilon > 0 \), \( f_n \) converges to \( f \) in measure.

b) Assume that \((f_n)_{n \in \mathbb{N}}\) converges to \( f \) in \( L^{p(\cdot),q(\cdot)}(X, \mu) \). Then since \( l = \mu(X) < \infty \),
\[
J_{p,q}(f_n - f) \equiv \int_0^1 t^{q(t)-1} (f_n - f)^* (t)^{q(t)} \, dt \to 0 \quad (2.1)
\]
holds for \( n \to \infty \). In addition, \( L^{p(\cdot),q(\cdot)}(X, \mu) \) is a Banach function space [4] and we have
\[
\int_X (f_n - f) (x) \, dx \leq C_X \|f_n - f\|_{L^{p(\cdot),q(\cdot)}(X, \mu)}. \quad (2.2)
\]

Therefore by using (2.1) and (2.2), we obtain \((f_n)_{n \in \mathbb{N}}\) converges to \( f \) in \( L^1(X) \). Thus \( f_n \) converges to \( f \) in measure.

**Theorem 1**

a) Let \( p_i, q_i \in IP_{1}([0, \infty]), \ (i = 1, 2) \), \( \mu(X) = \infty \), \( q_i(\infty) > p_i(\infty) \), and \( q_i(0) < p_i(0) \), \( (i = 1, 2) \). Then the inclusion
\[
L^{p_i(\cdot),q_i(\cdot)}(X, \mu) \subset L^{p(\cdot),q(\cdot)}(X, \nu)
\]
holds in the sense of equivalence classes if and only if \( \mu \approx \nu \) and there exists \( C > 0 \) such that
\[
\|f\|_{L^{p_i(\cdot),q_i(\cdot)}(X, \mu)} \leq C \|f\|_{L^{p(\cdot),q(\cdot)}(X, \nu)}
\]
for all \( f \in L^{p_i(\cdot),q_i(\cdot)}(X, \mu) \).

b) Let \( p_i, q_i \in IP_{1}([0, l]), \ (i = 1, 2) \) and \( l = \mu(X) < \infty \). Then the inclusion
\[
L^{p_i(\cdot),q_i(\cdot)}(X, \mu) \subset L^{p(\cdot),q(\cdot)}(X, \nu)
\]
holds in the sense of equivalence classes if and only if \( \mu \approx \nu \) and there exists \( C > 0 \) such that
\[
\|f\|_{L^{p_i(\cdot),q_i(\cdot)}(X, \mu)} \leq C \|f\|_{L^{p(\cdot),q(\cdot)}(X, \nu)}
\]
for all \( f \in L^{p_i(\cdot),q_i(\cdot)}(X, \mu) \).
Proof a) Suppose that \( L^{p_1,q_1}(X,\mu) \subset L^{p_2,q_2}(X,\nu) \) holds in the sense of equivalence classes. We define the unit operator \( I(f) = f \) from \( L^{p_1,q_1}(X,\mu) \) into \( L^{p_2,q_2}(X,\nu) \). Now we show that \( I \) is closed. Let \((f_n)_{n\in\mathbb{N}} \) be a sequence such that \( f_n \to f \) in \( L^{p_1,q_1}(X,\mu) \) and \( I(f_n) = f_n \to g \) in \( L^{p_2,q_2}(X,\nu) \). Thus, by Lemma 2, \((f_n)_{n\in\mathbb{N}} \) converges to \( f \) in measure (with respect to \( \mu \)). Hence there exists subsequence \((f_{n_k})_{n_k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}} \) such that \((f_{n_k})_{n_k\in\mathbb{N}} \) pointwise converges to \( f \), \( \mu \)-almost everywhere (a.e.). Moreover, since \((f_n)_{n\in\mathbb{N}} \) converges to \( g \) in \( L^{p_2,q_2}(X,\nu) \), it is easy to show that \((f_{n_k})_{n_k\in\mathbb{N}} \) converges to \( g \) in \( L^{p_2,q_2}(X,\nu) \). Then \((f_{n_k})_{n_k\in\mathbb{N}} \) converges to \( g \) in measure (with respect to \( \nu \)). Thus we find a subsequence \((f_{n_{ik}})_{n_{ik}\in\mathbb{N}} \subset (f_{n_k})_{n_k\in\mathbb{N}} \) such that \((f_{n_{ik}})_{n_{ik}\in\mathbb{N}} \) converges to \( g \) pointwise \( \nu \)-a.e. Let \( M \) be a set of the points such that \((f_{n_{ik}})_{n_{ik}\in\mathbb{N}} \) does not converge to \( g \) pointwise. Hence \( \nu(M) = 0 \). From the assumption \( L^{p_1,q_1}(X,\mu) \subset L^{p_2,q_2}(X,\nu) \) in the sense of equivalence classes and so we write \( \mu \approx \nu \) by Lemma 1. Thus \( \nu(M) = \mu(M) = 0 \). Hence \((f_{n_{ik}})_{n_{ik}\in\mathbb{N}} \) converges to \( g \) pointwise \( \mu \)-a.e. Consequently using the following inequality

\[
|f(x) - g(x)| \leq |f(x) - f_{n_{ik}}(x)| + |f_{n_{ik}}(x) - g(x)|,
\]

we have \( f = g \) \( \mu \)-a.e. and \( f = g \) \( \nu \)-a.e. That means \( I \) is closed. By the closed graph theorem, there exists \( C > 0 \) such that

\[
\|f\|_{L^{p_2,q_2}(X,\nu)} \leq C \|f\|_{L^{p_1,q_1}(X,\mu)}.
\]

The proof of the other direction is easy.

In this Theorem, (b) can be proved easily by using the technique of the proof in (a).

Lemma 3 a) If \( \nu(E) \leq \mu(E) \) for all \( E \in \Sigma \), then the inequality

\[
\|f\|_{L^{p(q)}(X,\nu)} \leq \|f\|_{L^{p(q)}(X,\mu)}
\]

holds for all \( f \in L^{p(q)}(X,\nu) \).

b) Let \( p \in IP_{0}([0,1]) \), \( 1 \leq q < \infty \). If there exists \( M > 0 \) such that \( \nu(E) \leq M\mu(E) \) for all \( E \in \Sigma \), then the inequality

\[
\|f\|_{L^{p(q)}(X,\nu)} \leq M \|f\|_{L^{p(q)}(X,\mu)}
\]

holds for all \( f \in L^{p(q)}(X,\mu) \).

Proof a) Let \( \nu(E) \leq \mu(E) \) for all \( E \in \Sigma \). From [5], we have \( f^{\ast\nu}(t) \leq f^{\ast\mu}(t) \) \( (f^{\ast\nu} \text{ and } f^{\ast\mu} \text{ are the rearrangements of } f \text{ with respect to the measures } \nu \text{ and } \mu \text{ respectively}) \) for all \( t \geq 0 \). This implies

\[
\int_{0}^{l} t^{\frac{q(t)}{p(t)}}^{-1} (f^{\ast\nu}(t))^{q(t)} dt \leq \int_{0}^{l} t^{\frac{q(t)}{p(t)}}^{-1} (f^{\ast\mu}(t))^{q(t)} dt.
\]

where \( l = \mu(X) \). Thus we have

\[
\|f\|_{L^{p(q)}(X,\nu)} \leq \|f\|_{L^{p(q)}(X,\mu)}.
\]
b) Let \( l = \mu (X) = \infty \). Assume that there exists \( M > 0 \) such that \( \nu (E) \leq M \mu (E) \) for all \( E \in \Sigma \). If we take \( k = M \mu \), then \( k \) is a measure. Then it is known that \( f^{*} \cdot k (t) = f^{*} \cdot \mu \left( \frac{t}{M} \right) \geq f^{*} \cdot \nu (t) \) by [5]. Therefore, if we set \( \frac{t}{M} = u \), then

\[
J_{p,q}^{\nu} (f) = \int_{0}^{\infty} t^{\frac{q}{p+q}} f^{*} (t)^{q} dt \leq J_{p,q}^{\mu} (f) = \int_{0}^{\infty} t^{\frac{q}{p+q}} f^{*} \cdot \mu \left( \frac{t}{M} \right)^{q} dt
\]

\[
= \int_{0}^{\infty} t^{\frac{q}{p+q}} f^{*} \cdot \mu \left( \frac{t}{M} \right)^{q} dt
\]

\[
\leq \int_{0}^{1} t^{\frac{q}{p+q}} f^{*} \cdot \mu \left( \frac{t}{M} \right)^{q} dt + \int_{1}^{\infty} t^{\frac{q}{p+q}} f^{*} \cdot \mu \left( \frac{t}{M} \right)^{q} dt
\]

\[
= M^{\frac{q}{p+q}} \int_{0}^{1} u^{\frac{q}{p+q}} f^{*} \cdot \mu (u)^{q} du + M^{\frac{q}{p+q}} \int_{1}^{\infty} u^{\frac{q}{p+q}} f^{*} \cdot \mu (u)^{q} du
\]

\[
\leq M_{0} \int_{0}^{\infty} u^{\frac{q}{p+q}} f^{*} \cdot \mu (u)^{q} du = M_{0} J_{p,q}^{\mu} (f)
\]

where \( M_{0} = \text{maks} \left\{ M^{\frac{q}{p+q}}, M_{\frac{q}{p+q}} \right\} \). Thus we have \( \| f \|_{L^{p(\cdot),q(\cdot)} (X, \nu)} ^{1} \leq M \| f \|_{L^{p(\cdot),q(\cdot)} (X, \mu)} ^{1} \).

Similarly the Lemma is proved for \( l = \mu (X) < \infty \). \( \square \)

**Lemma 4** Let \( p, q \in IP_{1} ([0, l]) \), and \( l = \mu (X) < \infty \).

a) If \( \mu \approx \nu \) and there exists \( M > 0 \) such that \( \nu (E) \leq M \mu (E) \) for all \( E \in \Sigma \) then the inclusion \( L^{1} (X, \mu) \subset L^{1} (X, \nu) \) holds.

b) If the inclusion \( L^{1} (X, \mu) \subset L^{1} (X, \nu) \) holds then the inclusion \( L^{p(\cdot),q(\cdot)} (X, \mu) \subset L^{p(\cdot),q(\cdot)} (X, \nu) \) holds.

**Proof**  a) It is known by [5].

b) Take any \( f \in L^{p(\cdot),q(\cdot)} (X, \mu) \) is given. Since \( \chi_{[0, \infty]} \left( \frac{q(t)}{p(t)} - 1 \right) f^{*} (t)^{q(t)} \in L^{1} (\mu) \) and \( L^{1} (\mu) \subset L^{1} (\nu) \). Thus we obtain \( \chi_{[0, \infty]} \left( \frac{q(t)}{p(t)} - 1 \right) f^{*} (t)^{q(t)} \in L^{1} (\nu) \). That means \( f \in L^{p(\cdot),q(\cdot)} (X, \nu) \). \( \square \)

**Theorem 2** Let \( p, q \in IP_{1} ([0, l]) \) and \( l = \mu (X) < \infty \). Then the inclusion \( L^{p(\cdot),q(\cdot)} (X, \mu) \subset L^{p(\cdot),q(\cdot)} (X, \nu) \) holds if and only if \( \mu \approx \nu \) and there exists \( M > 0 \) such that \( \nu (E) \leq M \mu (E) \) for all \( E \in \Sigma \).

**Proof** \( \Rightarrow \) By Theorem 1, there exists \( M > 0 \) such that

\[
\| f \|_{L^{p(\cdot),q(\cdot)} (X, \mu)} ^{1} \leq M \| f \|_{L^{p(\cdot),q(\cdot)} (X, \nu)} ^{1}
\]

(2.3)
for all $f \in L^{p(\cdot),q(\cdot)}(X, \mu)$. Moreover,

$$J_{p,q}^\nu(f) \cong \int_0^t \frac{dt}{t^{\frac{q_1(0)}{p_1(0)}}} \left((X_E)^*\nu\left(t\right)\right)^{\frac{\mu(E)}{p_1(0)}}dt = \int_0^{\nu(E)} \frac{dt}{t^{\frac{q_1(0)}{p_1(0)}}} \cdot \frac{p(0)}{q(0)} \cdot \nu(E)^{\frac{q_1(0)}{p_1(0)}}$$

holds. Similarly, we have

$$J_{p,q}^{\mu}(f) \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}.$$

Therefore, we write

$$\|X_E\|_{L^{p(\cdot),q(\cdot)}(X,\nu)}^{\frac{1}{p(\cdot),q(\cdot)}} \cong \frac{p(0)}{q(0)} \nu(E)^{\frac{q_1(0)}{p_1(0)}} \quad (2.4)$$

and

$$\|\chi_E\|_{L^{p(\cdot),q(\cdot)}(X,\mu)}^{\frac{1}{p(\cdot),q(\cdot)}} \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \quad (2.5)$$

Thus, from (2.3), (2.4), and (2.5), we have

$$\nu(E) \leq M \mu(E).$$

\textit{From Lemma 4, the proof is clear.} \hfill \Box

\textbf{Theorem 3} Let $p_i, q_i \in IP_i([0, l]), \ (i = 1, 2), l = \mu(X) < \infty$, and $q_1(0)p_2(0) > q_2(0)p_1(0)$. If $L^{p_1(\cdot),q_1(\cdot)}(X, \mu) \subset L^{p_2(\cdot),q_2(\cdot)}(X, \mu)$ then there exists a constant $m \geq 0$ such that $\mu(E) \geq m$ for every $\mu$-nonnull set $E \in \Sigma$.

\textbf{Proof} By Theorem 1, there exists $C > 0$ such that

$$\|f\|_{L^{p_2(\cdot),q_2(\cdot)}(X, \mu)} \leq C \|f\|_{L^{p_1(\cdot),q_1(\cdot)}(X, \mu)}$$

for all $f \in L^{p_1(\cdot),q_1(\cdot)}(X, \mu)$. Let $E \in \Sigma$ be a $\mu$-nonnull set. Since $\mu(E) < \infty$, we have

$$J_{p_1,q_1}^{\mu}(\chi_E) \cong \int_0^t \frac{dt}{t^{\frac{q_1(0)}{p_1(0)}}} \left((X_E)^*\left(t\right)\right)^{\frac{\mu(E)}{p_1(0)}}dt = \int_0^{\nu(E)} \frac{dt}{t^{\frac{q_1(0)}{p_1(0)}}} \cdot \frac{p(0)}{q(0)} \cdot \nu(E)^{\frac{q_1(0)}{p_1(0)}}.$$

and so

$$\|\chi_E\|_{L^{p_1(\cdot),q_1(\cdot)}(X,\nu)}^{\frac{1}{p_1(\cdot),q_1(\cdot)}} \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \quad \text{holds.}$$

Similarly, we have

$$\|\chi_E\|_{L^{p_2(\cdot),q_2(\cdot)}(X,\mu)}^{\frac{1}{p_2(\cdot),q_2(\cdot)}} \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}.$$

Then we write

$$\frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \leq C \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}$$

and

$$\frac{1}{C} \frac{q_1(0)p_2(0)}{p_1(0)q_2(0)} \leq \mu(E)^{\frac{q_1(0)}{p_1(0)} - \frac{q_2(0)}{p_2(0)}}.$$

If we set $m = \left(\frac{1}{C} \frac{q_1(0)p_2(0)}{p_1(0)q_2(0)}\right)^{\frac{q_1(0)}{p_1(0)}p_2(0)q_2(0)}$, we obtain $\mu(E) \geq m$ for every $\mu$-nonnull set $E \in \Sigma$. \hfill \Box
Theorem 4 Let $p, q \in IP_1([0, l])$ and $l = \mu(X) < \infty$.

a) If $q_2(.) \leq q_1(.)$, $q_1(0) \leq p_1(0)$, and $q_2(0) \geq p_2(0)$ then the inclusion

$$L^{p_1(.) - q_1(.)}(X, \mu) \subset L^{p_2(.) - q_2(.)}(X, \mu)$$

holds.

b) If $q(.) \leq p(.)$ and $q(0) = p(0)$, then the inclusion

$$L^{p(.) - p(.)}(X, \mu) \subset L^{p(.) - q(.)}(X, \mu)$$

holds.

c) If $p(.) \leq q(.)$ and $q(0) = p(0)$, then the inclusion

$$L^{p(.) - q(.)}(X, \mu) \subset L^{p(.) - p(.)}(X, \mu)$$

holds.

d) If $q(0) \geq p(0)$ then the inclusion

$$L^{q(.) - q(.)}(X, \mu) \subset L^{p(.) - q(.)}(X, \mu)$$

holds.

e) If $q(0) \leq p(0)$, then the inclusion

$$L^{p(.) - q(.)}(X, \mu) \subset L^{q(.) - q(.)}(X, \mu)$$

holds.

Proof a) Take any $f \in L^{p_1(.) - q_1(.)}(X, \mu)$. Then we have

$$\int_0^l f^*(t) q_1(t) dt \geq \frac{q_1(0)}{p_1(0)} \int_0^l f^*(t) q_1(t) dt = \int_0^l f^*(t) q_1(t) dt.$$ 

Therefore, we obtain $f^* \in L^{p_1(.)}(0, l)$. Moreover, since $q_2(.) \leq q_1(.)$, we write $L^{q_1(.)}(0, l) \subset L^{q_2(.)}(0, l)$ from [7]. That means $f^* \in L^{q_2(.)}(0, l)$. From this result, we have

$$J_{p_2, q_2}(f) \leq \frac{q_2(0)}{p_2(0)} \int_0^l f^*(t) q_2(t) dt < \infty.$$ 

Thus we find that $f \in L^{p_2(.) - q_2(.)}(X, \mu)$.

b) Take any $f \in L^{p(.) - p(.)}(X, \mu)$. That means $f^* \in L^{p(.)}(0, l)$. Again since $q(.) \leq p(.)$, we know that $L^{p(.)}(0, l) \subset L^{q(.)}(0, l)$ from [7]. Therefore, we have

$$J_{p, q}(f) \leq \frac{q(0)}{p(0)} \int_0^l f^*(t) q(t) dt = \int_0^l (f^*(t))^q(t) dt < \infty.$$
Thus we obtain \( f \in L^{p,q}(X, \mu) \).

c) This hypothesis is proved easily using the technique in (b).

d) Take any \( f \in L^{p,q}(X, \mu) \). Assume that \( l = \mu(X) \geq 1 \). Then since \( J_{q,q}(f) \equiv \frac{1}{l} \int_{0}^{l} (f^*(t))^{q(t)} dt < \infty \), and since \( q(0) \geq p(0) \), we have

\[
J_{p,q}(f) \leq \frac{1}{l} \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt + \int_{1}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt < \infty.
\]

Therefore, \( f \in L^{p,q}(X, \mu) \). Now let \( l < 1 \) and so we have

\[
J_{p,q}(f) \leq \frac{1}{l} \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt < \int_{0}^{l} (f^*(t))^{q(t)} dt < \infty.
\]

Thus similarly \( f \in L^{p,q}(X, \mu) \).

e) Take any \( f \in L^{p,q}(X, \mu) \). Assume that \( l = \mu(X) \geq 1 \). Then since \( J_{p,q}(f) \equiv \frac{1}{l} \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt < \infty \), we have

\[
\int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt \leq J_{p,q}(f) < \infty
\]

and

\[
\int_{1}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt \leq J_{p,q}(f) < \infty.
\]

In addition, since \( q(0) \leq p(0) \), we have

\[
J_{q,q}(f) \leq \int_{0}^{l} (f^*(t))^{q(t)} dt \leq \frac{1}{l} \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt + \int_{1}^{l} t^{q(0)/p-1} r^{-(q(0)/p-1)} (f^*(t))^{q(t)} dt
\]

\[
\leq \frac{1}{l} \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt + l^{-1} \int_{1}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt < \infty
\]

Therefore, \( f \in L^{p,q}(X, \mu) \). Now let \( l < 1 \) and we have

\[
J_{q,q}(f) \leq \int_{0}^{l} (f^*(t))^{q(t)} dt < \int_{0}^{l} t^{q(0)/p-1} (f^*(t))^{q(t)} dt < \infty.
\]
Thus similarly \( f \in L^{p(\cdot),q(\cdot)}(X,\mu) \).

\[ \Box \]

**Theorem 5** If \( \mu(X) < \infty \), \( 1 \leq q_2(\cdot) \leq q_1(\cdot) \), \( \frac{1}{p_1(\cdot)} + \frac{1}{q_2(\cdot)} = 1 \) and \( \frac{1}{p_2(\cdot)} + \frac{1}{q_1(\cdot)} = 1 \) then the inclusion
\[
L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \subset L^{p_2(\cdot),q_2(\cdot)}(X,\mu)
\]
holds.

**Proof** Take any \( f \in L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \). Then we have
\[
t \left( \frac{1}{p_2(\cdot)} - \frac{1}{q_2(\cdot)} \right) f^*(t) = t \left( \frac{1}{p_1(\cdot)} - \frac{1}{q_1(\cdot)} \right) f^*(t) t^{-1}
\]
Thus, we set
\[
t \left( \frac{1}{p_2(\cdot)} - \frac{1}{q_2(\cdot)} \right) f^*(t) \in L^{q_1(\cdot)}([0,\mu]) \subset L^{q_2(\cdot)}([0,\mu]).
\]
That means \( f \in L^{p_2(\cdot),q_2(\cdot)}(X,\mu) \).

\[ \Box \]

**Theorem 6** Let \( p_i, q_i \in IP_i((0,\mu]) \) \((i = 1, 2)\), \( \mu(X) < \infty \), \( q_2(\cdot) \geq q_1(\cdot) \), \( q_2(0) \geq p_2(0) \), and \( q_1(0) \leq p_1(0) \). If there exists a constant \( m > 0 \) such that \( \mu(E) \geq m \) for every \( \mu \)-nonnull set \( E \in \Sigma \) then the inclusion
\[
L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \subset L^{p_2(\cdot),q_2(\cdot)}(X,\mu)
\]
holds.

**Proof** Take any \( f \in L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \). Define the set \( E_n = \{ x \in X : |f(x)| > n \} \) for every \( n \in \mathbb{N} \). Since \( q_1(0) \leq p_1(0) \), we write \( L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \subset L^{q_1(\cdot),q_1(\cdot)}(X,\mu) \) from Theorem 4. Therefore, there exists \( C > 0 \) such that
\[
\|f\|_{L^{q_1(\cdot),q_1(\cdot)}(X,\mu)}^1 \leq C \|f\|_{L^{p_1(\cdot),q_1(\cdot)}(X,\mu)}^2 \quad (2.6)
\]
for all \( f \in L^{p_1(\cdot),q_1(\cdot)}(X,\mu) \). On the other hand, since \( |f(x)| > 1 \) for all \( x \in E_n \), we have \( |f^*(t)| > 1 \) for all \( t \in [0,\mu(E_n)] \). Thus if we set \( \frac{u}{2} = u \), then we have
\[
n^{q_1-} \mu(E_n) \leq \int_{E_n} |f|^{q_1-} d\mu = \int_X |f\chi_{E_n}|^{q_1-} d\mu = \int_0^{\infty} |(f\chi_{E_n})^*(t)|^{q_1-} dt
\]
\[
\leq \int_0^{\infty} |f^*| \left( \frac{t}{2} \right)^{q_1-} \chi_{[0,\mu(E_n)]} \left( \frac{t}{2} \right) dt = 2 \int_0^{\infty} |f^*| (u)^{q_1-} \chi_{[0,\mu(E_n)]} (u) du
\]
\[
= 2 \int_0^{\mu(E_n)} |f^*| (u)^{q_1-} du \leq 2 \int_0^{\mu(E_n)} |f^*| (u)^{q_1(u)} du \leq 2 \int_0^{\mu(E_n)} |f^*| (u)^{q_1(u)} du
\]
for every \( n \in \mathbb{N} \). Then using the inequality (2.6), we obtain
\[
n^{q_1-} \mu(E_n) \leq 2C \int_0^{\mu(E_n)} |f^*| (u)^{q_1(u)} du < \infty \quad (2.7)
\]

\[ 614 \]
for every $n \in \mathbb{N}$. By the hypothesis, either $\mu(E_n) = 0$ or $\mu(E_n) \geq m$. Since the sequence $(E_n)$ is nonincreasing and $\bigcap_{n=1}^{\infty} E_n = \varnothing$, we have $\mu(E_n) \to 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $|f(x)| \leq n_0$, $\mu - a.e.$ for all $x \in X$, and so we write $|f^*(t)| \leq n_0$, $\mu - a.e.$ for all $t \in [0, l]$. Therefore, we have

$$
\int_{0}^{l} |f^*(t)|^{q_2(t)} dt = \int_{0}^{l} |f^*(t)|^{q_2(t)-q_1(t)} |f^*(t)|^{q_1(t)} dt \leq \int_{0}^{l} n_0^{q_2(t)-q_1(t)} |f^*(t)|^{q_1(t)} dt.
$$

Therefore, we write

$$
\int_{0}^{l} |f^*(t)|^{q_2(t)} dt < n_0^{q_2-1} \int_{0}^{l} |f^*(t)|^{q_1(t)} dt < \infty. \quad (2.8)
$$

Thus we have $f \in L^{p^1(q), q_2} (X, \mu)$. That means $L^{p^1(q), q_1} (X, \mu) \subset L^{p^1(q), q_2} (X, \mu)$. Similarly, using the inequalities $(2.8)$, we write $L^{q_1, q_1} (X, \mu) \subset L^{q_1, q_2} (X, \mu)$. Lastly using $q_2(0) \geq p_2(0)$, we obtain

$$
L^{p^1(q), q_1} (X, \mu) \subset L^{q_1, q_1} (X, \mu) \subset L^{q_1, q_2} (X, \mu) \subset L^{p^2(q), q_2} (X, \mu)
$$

from Theorem 2.4.

\section*{Lemma 5 H"older inequality for variable exponent Lorentz spaces:}

Let $1 \leq q(\cdot) \leq q^+ < \infty$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, and $\frac{1}{q^+(\cdot)} + \frac{1}{q^-(\cdot)} = 1$. If $f \in L^{p(\cdot), q(\cdot)} (X, \mu)$ and $g \in L^{p'(\cdot), q'(\cdot)} (X, \mu)$ then $fg \in L^{1}(X, \mu)$ and there exists $C > 0$ such that

$$
\int_{X} |f(x)g(x)| \, d\mu(x) \leq C \|f\|_{L^{p(\cdot), q(\cdot)} (X, \mu)} \|g\|_{L^{p'(\cdot), q'(\cdot)} (X, \mu)}.
$$

\section*{Proof}

Let $f \in L^{p(\cdot), q(\cdot)} (X, \mu)$ and $g \in L^{p'(\cdot), q'(\cdot)} (X, \mu)$. Then we set $\frac{1}{2} = u$

$$
\int_{X} |f(x)g(x)| \, d\mu(x) = \int_{0}^{\infty} |fg^*(t)| \, dt \leq \int_{0}^{\infty} f^* \left( \frac{t}{2} \right) g^* \left( \frac{t}{2} \right) dt
$$

$$
= 2 \int_{0}^{\infty} f^*(u) g^*(u) \, du = 2 \int_{0}^{\infty} t^{1-1} f^*(u) g^*(u) \, du = 2 \int_{0}^{\infty} t \left( \frac{1}{p^{-(\cdot)}} + \frac{1}{q^{(\cdot)}} \right) t^{-\left( \frac{1}{p^{-(\cdot)}} + \frac{1}{q^{(\cdot)}} \right)} f^*(u) g^*(u) \, du.
$$

Furthermore, by using the Hölder inequality for variable exponent Lebesgue spaces in [7, 12], the inequalities $f^* \leq f^{**}$ and $g^* \leq g^{**}$, there exists $C_1 > 0$ such that

$$
\int_{X} |f(x)g(x)| \, d\mu(x) \leq 2 \int_{0}^{\infty} t \left( \frac{1}{p^{-(\cdot)}} - \frac{1}{q^{(\cdot)}} \right) f^*(u) t \left( \frac{1}{p^{(\cdot)}} - \frac{1}{q^{(\cdot)}} \right) g^*(u) \, du
$$

615
We also have $j$

On the other hand, we have $\|f\|_{L^p([0,1])} = C\|h\|_{L^p([0,1])}$, where $C = 2C_1$. □

**Theorem 7** a) Let $1 \leq q(\cdot) \leq q^+ < \infty$. If $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$ and $\frac{1}{q(t)} + \frac{1}{q(t)} = 1$ then the inclusion

$$L^p(q(\cdot), (X, \mu), L^q(\cdot), q(\cdot), (X, \mu)) \subset L^1(X, \mu)$$

holds.

b) Let $p \in IP_0([0,1]), q \in IP_1([0,1])$, $q(t) = \{c, d, x \in \Omega, x \notin \Omega, \text{and } c, d \geq 1 \text{ such that } \Omega \subset [0, \infty)$. If $q(0) > p(0)$ and $q(\infty) \geq p(\infty)$ then

$$L^1(X, \mu) \subset L^p(q(\cdot), (X, \mu), L^q(\cdot), q(\cdot), (X, \mu))$$

holds such that $X = [0, \infty]$ and $\mu(x) = dx$.

**Proof** a) Let $f \in L^p(q(\cdot), (X, \mu))$ and $h \in L^q(\cdot), q(\cdot), (X, \mu)$. From Lemma 5, there exists $C > 0$ such that

$$\int_X |f(x)| \mu(x) \leq C \|f\|_{L^p(q(\cdot), (X, \mu)} \|h\|_{L^q(\cdot), q(\cdot), (X, \mu)}.$$ 

Thus we obtain $f, h \in L^1(X, \mu)$. That means $L^p(q(\cdot), (X, \mu), L^q(\cdot), q(\cdot), (X, \mu)) \subset L^1(X, \mu)$.

b) Take any $g \in L^1(X, \mu)$. Define that

$$A_1 = \{x : 0 < |g(x)| < \infty\},$$

$$A_2 = \{x : |g(x)| = 0\},$$

and

$$(A_1 \cup A_2)^c = \{x : |g(x)| = \infty\}$$

such that $A_1 \cup A_2 \cup (A_1 \cup A_2)^c = [0, \infty]$. Now define that the functions

$$f(x) = \begin{cases} |g(x)|, & x \in A_1 \\ 0, & x \notin A_1 \cup A_2 \end{cases}$$

and

$$h(x) = \begin{cases} \frac{|g(x)|}{|g(x)|}, & x \in A_1 \\ 0, & x \notin A_1 \cup A_2 \end{cases}.$$ 

We also have $|g| = |fh|$. Since $g \in L^1(X, \mu)$, we know that $|g(x)| < \infty (a.e.)$. Thus we have $\mu((A_1 \cup A_2)^c) = 0$. On the other hand, we have

$$J_{p,q}(f) \leq \int_0^1 t^{\frac{q(0)}{q-1}} \left(\frac{f^+(t)}{t^p}\right) q(t) dt + \int_1^\infty t^{\frac{q(\infty)}{q-1}} \left(\frac{f^+(t)}{t^p}\right) q(t) dt.$$
\begin{align*}
\int_0^1 (f^*(t))^{q_1(t)} dt + \int_1^\infty (f^*(t))^{q_1(t)} dt \\
= \int_\Omega \left(g^*(t)^{\frac{2}{p}}\right)^c dt + \int_{\mathbb{R}^n \setminus \Omega} \left(g^*(t)^{\frac{2}{p}}\right)^d dt \\
= \int_X (g(x))^{q(x)} dx < \infty
\end{align*}

and similarly \( J_{p', q'}(h) < \infty \). Thus we obtain \( f \in L^{p(\cdot), q(\cdot)}(X, \mu) \) and \( h \in L^{p(\cdot), q'(\cdot)}(X, \mu) \). Therefore, from the inequality \(|g| = |fh|\), we find that \( g \in L^{p(\cdot), q(\cdot)}(X, \mu), L^{p(\cdot), q'(\cdot)}(X, \mu) \). Thus the proof is completed. \( \square \)

**Theorem 8** Let \( p_i \in IP_{\alpha}([0, l]), q_i \in IP_{\beta}([0, l]), (i = 1, 2), l = \mu(X) < \infty \). If \( L^{p_i(\cdot), q_i(\cdot)}(X, \mu) \subset L^{p_2(\cdot), q_2(\cdot)}(X, \mu) \) such that \( q_2 - q_1 < \frac{p_2(0) q_2(0)}{p_1(0) q_1(0)} \) then any collection of disjoint measurable sets of positive measure is a finite element.

**Proof** Assume that \( (E_n) \) is a sequence of disjoint measurable sets such that \( \mu(E_n) \neq 0 \) for infinite one \( n \). Thus, since \( \bigcup_{n=1}^\infty E_n \subset X \), we have \( \mu\left(\bigcup_{n=1}^\infty E_n\right) < \infty \). Therefore, we write \( \sum_{n=1}^\infty \mu(E_n) < \infty \). That means

\[ \lim_{n \to \infty} \mu(E_n) = 0. \]

Then there exists subsequence \( (E_{n_k})_{n_k} \) such that \( E = \bigcup_{k=1}^\infty E_{n_k} \), \( E_{n_k} \cap E_{n_j} (k \neq j) \), \( \mu(E) < \infty \), and \( \mu(E_{n_k}) = 2^{-kq_1^+ q_1(0)} \mu(E) \) for every \( k \in \mathbb{N} \). Define that

\[ f(x) = \sum_{k=1}^\infty (2^k k^{-2}) \chi_{E_{n_k}}(x). \]

Then since \( 2q_1^+ > 1 \)

\begin{align*}
J_{p_1, q_1}(f) &\geq \mu(E) \int_0^{\mu(E)} t^{\frac{q_1(0)}{p_1(0)} - 1} (f^*(t))^{q_1(t)} dt = \sum_{i=1}^\infty \mu(E_{n_i}) \int_0^{\mu(E_{n_i})} t^{\frac{q_1(0)}{p_1(0)} - 1} (f^*(t))^{q_1(t)} dt \\
&= \sum_{i=1}^\infty \mu(E_{n_i}) \int_0^{\mu(E_{n_i})} t^{\frac{q_1(0)}{p_1(0)} - 1} \left( \sum_{k=1}^\infty \left(2^k k^{-2}\right) \chi_{E_{n_k}}(t)^*\right)^{q_1(t)} dt \\
&\leq \int_0^{\mu(E_{n_2})} t^{\frac{q_1(0)}{p_1(0)} - 1} 2q_1^+ dt + \int_0^{(2^2 2^{-2})^{q_1^+ q_1(0)} t^{\frac{q_1(0)}{p_1(0)} - 1} dt} + \int_0^{(2^3 3^{-2})^{q_1^+ q_1(0)} t^{\frac{q_1(0)}{p_1(0)} - 1} dt} + \cdots
\end{align*}
\[
= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{q_1(0)}_{\mu(E_{n_1})^{q_1(0)}} + (2^{2^2 - 2})^{q_1^+} \mu(E_{n_2})^{q_1(0)}_{\mu(E_{n_2})^{q_1(0)}} + (2^{3^3 - 2})^{q_1^+} \mu(E_{n_3})^{q_1(0)}_{\mu(E_{n_3})^{q_1(0)}} \right\} \\
+ \frac{p_1(0)}{q_1(0)} \sum_{k=4}^{\infty} (2^k k^{-2})^{q_1^+} \mu(E_{n_k})^{q_1(0)}_{\mu(E_{n_k})^{q_1(0)}} \\
= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{q_1(0)}_{\mu(E_{n_1})^{q_1(0)}} + \mu(E_{n_2})^{q_1(0)}_{\mu(E_{n_2})^{q_1(0)}} + \left( \frac{8}{9} \right)^{q_1^+} \mu(E_{n_3})^{q_1(0)}_{\mu(E_{n_3})^{q_1(0)}} \right\} \\
+ \frac{p_1(0)}{q_1(0)} \mu(E)^{q_1(0)}_{\mu(E)^{q_1(0)}} \sum_{k=4}^{\infty} (2^k k^{-2})^{q_1^+} 2^{-kq_1^+} \mu(E)^{q_1(0)}_{\mu(E)^{q_1(0)}} \\
= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{q_1(0)}_{\mu(E_{n_1})^{q_1(0)}} + \mu(E_{n_2})^{q_1(0)}_{\mu(E_{n_2})^{q_1(0)}} + \left( \frac{8}{9} \right)^{q_1^+} \mu(E_{n_3})^{q_1(0)}_{\mu(E_{n_3})^{q_1(0)}} \right\} \\
+ \frac{p_1(0)}{q_1(0)} \mu(E)^{q_1(0)}_{\mu(E)^{q_1(0)}} \sum_{k=4}^{\infty} k^{-2q_1^+} < \infty \\
\]

holds. Thus, we have \( f \in L^{p_1(-),q_1(-)}(X,\mu) \). On the other hand, we find that

\[
J_{p_2,q_2}(f) \overset{\mu(E)}{=} \int_0^{q_2(0)} (f^*(t))^{q_2(t)} dt = \sum_{k=1}^{\infty} \int_0^{\infty} t^{q_2(0) - 1} (f^*(t))^{q_2(t)} dt \\
\geq \sum_{k=4}^{\infty} \mu(E_{n_k}) \left( \sum_{k=1}^{\infty} (2^k k^{-2}) \chi_{E_{n_k}} \right)^* (t)^{q_2(t)} dt \\
= \sum_{k=4}^{\infty} \int_0^{\infty} t^{q_2(0) - 1} (2^k k^{-2})^{q_2(t)} dt, \ (2^k k^{-2} > 1 \text{ for } k \geq 4) \\
\geq \sum_{k=4}^{\infty} \int_0^{\infty} t^{q_2(0) - 1} (2^k k^{-2})^{q_2(t)} dt = \frac{p_2(0)}{q_2(0)} \sum_{k=4}^{\infty} (2^k k^{-2})^{q_2^+} \mu(E_{n_k})^{q_2(0)}_{\mu(E_{n_k})^{q_2(0)}} \\
= \frac{p_2(0)}{q_2(0)} \mu(E)^{q_2(0)}_{\mu(E)^{q_2(0)}} \sum_{k=4}^{\infty} (2^k k^{-2})^{q_2^+} 2^{-kq_1^+} \left( \frac{p_1(0) q_2(0)}{q_1(0) q_2(0)} \right) \\
= \frac{p_2(0)}{q_2(0)} \mu(E)^{q_2(0)}_{\mu(E)^{q_2(0)}} \sum_{k=4}^{\infty} k^{-2q_2-2^k(q_2^+)-q_1^+} \left( \frac{p_1(0) q_2(0)}{q_1(0) q_2(0)} \right). \\
\]

If we say that \( b_k = k^{-2q_2-2^k(q_2^+)-q_1^+} \left( \frac{p_1(0) q_2(0)}{q_1(0) q_2(0)} \right) \), then we have \( \lim_{k \to \infty} b_k = 0 \). Thus, we find that \( f \notin L^{p_2(-),q_2(-)}(X,\mu) \). However, from the assumption, we must obtain \( f \in L^{p_2(-),q_2(-)}(X,\mu) \). Therefore, we find that any collection of disjoint measurable sets of positive measure is a finite element. \( \square \)
References


