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Popoviciu type inequalities via Green function and Taylor polynomial

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Abstract: The well-known Taylor polynomial is used to construct the identities coming from Popoviciu type inequalities for convex functions via the Green function. The bounds for the new identities are found using the Čebyšev functional to develop the Grüss and Ostrowski type inequalities. Further, more exponential convexity together with Cauchy means is presented for linear functionals associated with the obtained inequalities.

Key words: Popoviciu inequality, Taylor formula, Green function, Čebyšev functional, Grüss inequality, Ostrowski inequality, exponential convexity, Cauchy mean

1. Introduction and preliminary results

The theory of convex functions has experienced a rapid development. This can be attributed to several causes: first, so many areas in modern analysis directly or indirectly involve the application of convex functions; second, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [10]). Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [10, p. 14].

Definition 1 *The mth-order divided difference of a function $f : [a,b] \to \mathbb{R}$ at mutually distinct points $x_0, ..., x_m \in [a,b]$ is defined recursively by*

$$[x_i; f] = f(x_i), \quad i = 0, \dots, m,$$
$$[x_0, \dots, x_m; f] = \frac{[x_1, \dots, x_m; f] - [x_0, \dots, x_{m-1}; f]}{x_m - x_0}. \tag{1}$$

It is easy to see that (1) is equivalent to

$$[x_0, \dots, x_m; f] = \sum_{i=0}^{m} \frac{f(x_i)}{q'(x_i)}, \quad \textit{where } q(x) = \prod_{j=0}^{m} (x - x_j).$$

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The following definition of a real valued convex function is characterized by mth-order divided difference (see [10, p. 15]).

Definition 2 *A function $f:[a,b]\to\mathbb{R}$ is said to be m-convex $(m\geq 0)$ if and only if for all choices of $(m+1)$ distinct points $x_0,\ldots,x_m\in[a,b]$, $[x_0,\ldots,x_m;f]\geq 0$ holds.*
If this inequality is reversed, then f is said to be m-concave. If the inequality is strict, then f is said to be a strictly m-convex (m-concave) function.

Remark 1.1 *Note that 0-convex functions are nonnegative functions, 1-convex functions are increasing functions, and 2-convex functions are simply the convex functions.*

The following theorem gives an important criterion to examine the m-convexity of a function f (see [10, p. 16]).

Theorem 1.2 *If $f^{(m)}$ exists, then f is m-convex if and only if $f^{(m)}\geq 0$.*

In 1965, Popoviciu introduced a characterization of convex functions [11]. In 1976, Vasić and Stanković [12] (see also [10, p. 173]) gave the weighted version. In recent years that inequality of Popoviciu was studied in [3, 6, 7, 8, 9].

Theorem 1.3 *Let $n,k\in\mathbb{N}$, $n\geq 3$, $2\leq k\leq n-1$, $[\alpha,\beta]\subset\mathbb{R}$, $\mathbf{x}=(x_1,...,x_n)\in[\alpha,\beta]^n$, $\mathbf{p}=(p_1,...,p_n)$ be a positive n-tuple such that $\sum_{i=1}^n p_i=1$. Also let $f:[\alpha,\beta]\to\mathbb{R}$ be a convex function. Then*

$$p_{k,n}(\mathbf{x},\mathbf{p};f)\leq\frac{n-k}{n-1}p_{1,n}(\mathbf{x},\mathbf{p};f)+\frac{k-1}{n-1}p_{n,n}(\mathbf{x},\mathbf{p};f),\tag{2}$$

where

$$p_{k,n}(\mathbf{x},\mathbf{p};f)=p_{k,n}(\mathbf{x},\mathbf{p};f(x)):=\frac{1}{C_{k-1}^{n-1}}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(\sum_{j=1}^{k}p_{i_j}\right)f\left(\frac{\sum\limits_{j=1}^{k}p_{i_j}x_{i_j}}{\sum\limits_{j=1}^{k}p_{i_j}}\right)$$

is the linear functional with respect to f.

By inequality (2), we write

$$\Upsilon(\mathbf{x},\mathbf{p};f):=\frac{n-k}{n-1}p_{1,n}(\mathbf{x},\mathbf{p};f)+\frac{k-1}{n-1}p_{n,n}(\mathbf{x},\mathbf{p};f)-p_{k,n}(\mathbf{x},\mathbf{p};f).\tag{3}$$

Remark 1.4 *It is important to note that under the assumptions of Theorem 1.3, if the function f is convex then $\Upsilon(\mathbf{x},\mathbf{p};f)\geq 0$ and $\Upsilon(\mathbf{x},\mathbf{p};f)=0$ for $f(x)=x$ or f is a constant function.*

Consider the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha\leq s\leq t;\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\leq s\leq\beta.\end{cases}\tag{4}$$

The function G is convex in s, it is symmetric, and so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi : [\alpha, \beta] \to \mathbb{R}$, $\phi \in C^2([\alpha, \beta])$, we can easily show by integrating by parts that the following is valid:

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_\alpha^\beta G(x, s)\phi''(s)ds, \tag{5}$$

where the function G is defined as above in (4) [13].

The well-known Taylor formula is as follows:

Let m be a positive integer and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(m-1)}$ is absolutely continuous, and then for all $x \in [\alpha, \beta]$, the Taylor formula at point $c \in [\alpha, \beta]$ is

$$\phi(x) = T_{m-1}(\phi; c, x) + R_{m-1}(\phi; c, x) \tag{6}$$

where $T_{m-1}(\phi; c, x)$ is a Taylor polynomial of degree $m - 1$, i.e.

$$T_{m-1}(\phi; c, x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(c)}{k!}(x - c)^k,$$

and the remainder is given by

$$R_{m-1}(\phi; c, x) = \frac{1}{(m-1)!}\int_c^x \phi^{(m)}(t)(x - t)^{m-1}dt.$$

The mean value theorems and exponential convexity of the linear functional $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ are given in [6] for a positive n-tuple $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ in [6]. In [7] (see also [3]), the results related to $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ are generalized for real n-tuple $\mathbf{p}$ with the help of the Green function and m-exponential convexity is proved in a more general setting.

In Section 2 of this paper, we use Taylor's formula and the Green function to generalize the Popoviciu inequality. In Section 3, the Čebyšev functional is used to find the bounds for new identities. Grüss and Ostrowski type inequalities related to generalized Popoviciu inequalities are constructed. In Section 4, higher order convexity is used to produce exponential convexity of positive linear functionals coming from Section 2. The last section is devoted to the respective Cauchy means. We employ a similar method as adopted in [5] for Steffensen's inequality.

2. Generalization of the Popoviciu inequality

Motivated by identity (3), we construct the following new identities with the help of Taylor's formula.

Theorem 2.1 *Let n, k, m be positive integers such that $n \geq 3$, $2 \leq k \leq n - 1$, $\mathbf{x} = (x_1, ..., x_n) \in [\alpha, \beta]^n$ for real interval $[\alpha, \beta]$, and let $\mathbf{p} = (p_1, ..., p_n)$ be a positive n-tuple such that $\sum_{i=1}^n p_i = 1$. Consider $\phi : [\alpha, \beta] \to \mathbb{R}$*

to be a function such that $\phi^{(m-1)}$ *is absolutely continuous. Then we have the following identities:*

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) = \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!} \right) ds + \frac{1}{(m-3)!} \int_{\alpha}^{\beta} \phi^{(m)}(t) \left(\int_{t}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{m-3} ds \right) dt, \quad (7)$$

and

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) = \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!} \right) ds - \frac{1}{(m-3)!} \int_{\alpha}^{\beta} \phi^{(m)}(t) \left(\int_{\alpha}^{t} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{m-3} ds \right) dt. \quad (8)$$

Proof Using (5) in (3) and by linearity of $\Upsilon(\mathbf{x},\mathbf{p};\cdot)$ we get

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) = \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))\phi''(s) ds. \quad (9)$$

Using Taylor's formula (6) on the function ϕ'' at the point α and replacing m by $m-2$ $(m \geq 3)$ or differentiating (6) twice and taking $c = \alpha$, we get

$$\phi''(s) = \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{(z-2)!}(s-\alpha)^{z-2} + \frac{1}{(m-3)!} \int_{\alpha}^{s} \phi^{(m)}(t)(s-t)^{m-3} dt, \quad (10)$$

and for $c = \beta$, we get

$$\phi''(s) = \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)}{(z-2)!}(s-\beta)^{z-2} - \frac{1}{(m-3)!} \int_{s}^{\beta} \phi^{(m)}(t)(s-t)^{m-3} dt. \quad (11)$$

Now, using (10) in (9), we get

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) = \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{(z-2)!} \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-\alpha)^{z-2} ds + \frac{1}{(m-3)!} \int_{\alpha}^{\beta} \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \left(\int_{\alpha}^{s} \phi^{(m)}(t)(s-t)^{m-3} dt \right) ds,$$

and then applying Fubini's theorem on the last term, we get (7).

Similarly, using (11) in (9) and applying Fubini's theorem, we get (8). □

In the following theorem we obtain generalizations of Popoviciu's inequality for m-convex functions.

Theorem 2.2 *Let all the assumptions of Theorem 2.1 be satisfied.*

(i) If ϕ is an m-convex function and

$$\int_t^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{m-3}ds \geq 0, \quad t \in [\alpha,\beta], \tag{12}$$

then

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!}\right) ds. \tag{13}$$

(ii) If ϕ is an m-convex function and

$$\int_\alpha^t \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{m-3}ds \leq 0, \quad t \in [\alpha,\beta], \tag{14}$$

then

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!}\right) ds. \tag{15}$$

Proof Since $\phi^{(m-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\phi^{(m)}(x)$ exists almost everywhere. As ϕ is m-convex, applying Theorem 1.2, we have $\phi^{(m)}(x) \geq 0$ for all $x \in [\alpha,\beta]$. Hence, we can apply Theorem 2.1 to obtain (13) and (15), respectively. □

Corollary 2.3 *Let all the assumptions of Theorem 2.1 be satisfied for the m-convex function ($m \geq 3$) and in addition let $\mathbf{p} = (p_1, ..., p_m)$ be a positive m-tuple such that $\sum_{i=1}^m p_i = 1$. Then:*

(i) (13) is valid for $m = 3, 4, \ldots$. Moreover, if

$$\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!} \geq 0, \tag{16}$$

then

$$\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) \geq 0. \tag{17}$$

(ii) If m is even, then (15) holds. Moreover, if

$$\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!} \geq 0, \tag{18}$$

then (17) is valid, too.

Proof By using Theorem 2.2 and Remark 1.4. □

3. Bounds for identities related to generalization of the Popoviciu inequality

For two Lebesgue integrable functions $f, h : [\alpha, \beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

In [2], the authors proved the following theorems:

Theorem 3.1 *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha, \beta] \to \mathbb{R}$ be an absolutely continuous function with $(. - \alpha)(\beta - .)[h']^2 \in L[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{19}$$

The constant $\frac{1}{\sqrt{2}}$ in (19) is the best possible.

Theorem 3.2 *Assume that $h : [\alpha, \beta] \to \mathbb{R}$ is monotonic nondecreasing on $[\alpha, \beta]$ and $f : [\alpha, \beta] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leq \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dh(x). \tag{20}$$

The constant $\frac{1}{2}$ in (20) is the best possible.

In the sequel, we consider the above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\mathfrak{R}(t) = \int_t^\beta \Upsilon(\mathbf{x}, \mathbf{p}; G(x,s))(s-t)^{m-3}ds, \ \ t \in [\alpha, \beta], \tag{21}$$

$$\hat{\mathfrak{R}}(t) = \int_\alpha^t \Upsilon(\mathbf{x}, \mathbf{p}; G(x,s))(s-t)^{m-3}ds, \ \ t \in [\alpha, \beta]. \tag{22}$$

Consider the Čebyšev functionals $\Delta(\mathfrak{R}, \mathfrak{R})$ and $\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ given by:

$$\Delta(\mathfrak{R}, \mathfrak{R}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt\right)^2, \tag{23}$$

$$\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{R}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{R}}(t)dt\right)^2. \tag{24}$$

Theorem 3.3 *Let $m \geq 3$ be a positive integer, let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(m)}$ is absolutely continuous with $(. - \alpha)(\beta - .)[\phi^{(m+1)}]^2 \in L[\alpha, \beta]$, and let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_n) \in [\alpha, \beta]^n$, $\mathbf{p} = (p_1, ..., p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < ... < i_k \leq n$ and $\sum_{i=1}^{n} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha, \beta]$ for any $1 \leq i_1 < ... < i_k \leq n$ and $\mathfrak{R}$, $\hat{\mathfrak{R}}$ be defined by (21), (22), respectively . Then*

(i)

$$\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) = \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!} \right) ds + \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\beta - \alpha)(m-3)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt + \mathfrak{K}_m^1(\alpha, \beta; \phi), \quad (25)$$

where the remainder $\mathfrak{K}_m^1(\alpha, \beta; \phi)$ satisfies the bound

$$|\mathfrak{K}_m^1(\alpha, \beta; \phi)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(m-3)!} [\Delta(\mathfrak{R}, \mathfrak{R})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta - t)[\phi^{(m+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \quad (26)$$

(ii)

$$\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) = \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!} \right) ds + \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\alpha - \beta)(m-3)!} \int_{\alpha}^{\beta} \hat{\mathfrak{R}}(t) dt - \mathfrak{K}_m^2(\alpha, \beta; \phi), \quad (27)$$

where the remainder $\mathfrak{K}_m^2(\alpha, \beta; \phi)$ satisfies the bound

$$|\mathfrak{K}_m^2(\alpha, \beta; \phi)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(m-3)!} [\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta - t)[\phi^{(m+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \quad (28)$$

Proof *(i)* If we apply Theorem 3.1 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(m)}$, we get

$$\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) \phi^{(m)}(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt . \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(m)}(t) dt \right| \leq \frac{1}{\sqrt{2}} [\Delta(\mathfrak{R}, \mathfrak{R})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta - t)[\phi^{(m+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

Hence, we have

$$\frac{1}{(m-3)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) \phi^{(m)}(t) dt = \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\beta - \alpha)(m-3)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt + \mathfrak{K}_m^1(\alpha, \beta; \phi),$$

where the remainder $\mathfrak{K}_m^1(\alpha,\beta;\phi)$ satisfies the estimation (26). Now from identity (7), we obtain (25).
(ii) Similar to the above part. □

The following Grüss type inequalities can be obtained by using Theorem 3.2.

Theorem 3.4 *Let $m \geq 3$ be a positive integer, let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(m)}$ is an absolutely continuous function and $\phi^{(m+1)} \geq 0$ on $[\alpha,\beta]$, and let the functions $\mathfrak{R}$, $\hat{\mathfrak{R}}$ be defined by (21), (22), respectively. Then we have:*
(i) the representation (25) and the remainder $\mathfrak{K}_m^1(\alpha,\beta;\phi)$ satisfies the estimation

$$|\mathfrak{K}_m^1(\alpha,\beta;\phi)| \leq \frac{\beta-\alpha}{(m-3)!}||\mathfrak{R}'||_\infty\left[\frac{\phi^{(m-1)}(\beta)+\phi^{(m-1)}(\alpha)}{2} - \frac{\phi^{(m-2)}(\beta)-\phi^{(m-2)}(\alpha)}{\beta-\alpha}\right]. \tag{29}$$

(ii) The representation (27) and the remainder $\mathfrak{K}_m^2(\alpha,\beta;\phi)$ satisfies the estimation

$$|\mathfrak{K}_m^2(\alpha,\beta;\phi)| \leq \frac{\beta-\alpha}{(m-3)!}||\hat{\mathfrak{R}}'||_\infty\left[\frac{\phi^{(m-1)}(\beta)+\phi^{(m-1)}(\alpha)}{2} - \frac{\phi^{(m-2)}(\beta)-\phi^{(m-2)}(\alpha)}{\beta-\alpha}\right]. \tag{30}$$

Proof (i) Applying Theorem 3.2 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(m)}$, we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\phi^{(m)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(m)}(t)dt\right|$$
$$\leq \frac{1}{2(\beta-\alpha)}||\mathfrak{R}'||_\infty \int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(m+1)}(t)dt. \tag{31}$$

Since

$$\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(m+1)}(t)dt = \int_\alpha^\beta [2t-(\alpha+\beta)]\phi^{(m)}(t)dt$$
$$= (\beta-\alpha)\big[\phi^{(m-1)}(\beta)+\phi^{(m-1)}(\alpha)\big] - 2\big(\phi^{(m-2)}(\beta)-\phi^{(m-2)}(\alpha)\big),$$

therefore, using identity (7) and the inequality (31), we deduce (29).
(ii) Similar to the above proof. □

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5 *Suppose all the assumptions of Theorem 2.1 hold. Moreover, assume that (p,q) is a pair of conjugate exponents; that is, $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $|\phi^{(m)}|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $m \geq 3$. Then we have*
(i)

$$\left|\Upsilon(\mathbf{x},\mathbf{p};\phi(x)) - \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))\left(\sum_{z=2}^{m-1}\frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!}\right)ds\right|$$
$$\leq \frac{1}{(m-3)!}||\phi^{(m)}||_p||\int_t^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{m-3}ds||_q. \tag{32}$$

The constant on the R.H.S. of (32) is sharp for $1 < p \leq \infty$ and the optimal for $p = 1$.
(ii)

$$\left| \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!} \right) ds \right|$$
$$\leq \frac{1}{(m-3)!} ||\phi^{(m)}||_p || \int_{\alpha}^{t} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s))(s-t)^{m-3} ds||_q. \quad (33)$$

The constant on the R.H.S. of (33) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof (i) Let us denote

$$\mathfrak{V} = \frac{1}{(m-3)!} \int_{t}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s))(s-t)^{m-3} ds, \ \ t \in [\alpha, \beta].$$

Using identity (7) and applying Hölder's inequality, we obtain

$$\left| \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x, s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!} \right) ds \right|$$
$$= \left| \int_{\alpha}^{\beta} \mathfrak{V}(t) \phi^{(m)}(t) dt \right| \leq ||\phi^{(m)}||_p ||\mathfrak{V}(t)||_q,$$

where

$$||f||_q := \begin{cases} \left(\int_{\alpha}^{\beta} |f(t)|^q dt \right)^{\frac{1}{q}}; & 1 \leq q < \infty, \\ \sup_{t \in [\alpha, \beta]} |f(t)|; & q = \infty. \end{cases}$$

For the proof of the sharpness of the constant $||\mathfrak{V}(t)||_q$, let us define the function ϕ for which the equality in (32) is obtained.
For $1 < p \leq \infty$ take ϕ to be such that

$$\phi^{(m)}(t) = \mathrm{sgn} \mathfrak{V}(t) |\mathfrak{V}(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(m)}(t) = \mathrm{sgn} \mathfrak{V}(t)$.
For $p = 1$, we prove that

$$\left| \int_{\alpha}^{\beta} \mathfrak{V}(t) \phi^{(m)}(t) dt \right| \leq \max_{t \in [\alpha, \beta]} |\mathfrak{V}(t)| \left(\int_{\alpha}^{\beta} \phi^{(m)}(t) dt \right) \quad (34)$$

is the best possible inequality. Suppose that $|\mathfrak{V}(t)|$ attains its maximum at $t_0 \in [\alpha, \beta]$. To start with, first we assume that $\mathfrak{V}(t_0) > 0$. For δ small enough we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} 0\,, & \alpha \le t \le t_0\,, \\ \frac{1}{\delta m!}(t-t_0)^m\,, & t_o \le t \le t_0+\delta\,, \\ \frac{1}{m!}(t-t_0)^{m-1}\,, & t_0+\delta \le t \le \beta\,. \end{cases}$$

Then for δ small enough:

$$\left|\int_\alpha^\beta \mathfrak{V}(t)\phi^{(m)}(t)dt\right| = \left|\int_{t_0}^{t_0+\delta} \mathfrak{V}(t)\frac{1}{\delta}dt\right| = \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{V}(t)dt.$$

Now, from inequality (34), we have

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{V}(t)dt \le \mathfrak{V}(t_0)\int_{t_0}^{t_0+\delta}\frac{1}{\delta}dt = \mathfrak{V}(t_0).$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\mathfrak{V}(t)dt = \mathfrak{V}(t_0),$$

the statement follows. For the case when $\mathfrak{V}(t_0) < 0$, we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} \frac{1}{m!}(t-t_0-\delta)^{m-1}\,, & \alpha \le t \le t_0\,, \\ \frac{-1}{\delta m!}(t-t_0-\delta)^m\,, & t_o \le t \le t_0+\delta\,, \\ 0\,, & t_0+\delta \le t \le \beta\,, \end{cases}$$

and rest of the proof is the same as above.
(ii) Similar to first part. □

4. Mean value theorems and m-exponential convexity

We recall some definitions and basic results from [1, 4, 5] that are required in the sequel.

Definition 3 *A function $\phi : I \to \mathbb{R}$ is m-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^m \xi_i\xi_j\,\phi\left(\frac{x_i+x_j}{2}\right) \ge 0$$

holds for all choices $\xi_1, \ldots, \xi_m \in \mathbb{R}$ and all choices $x_1, \ldots, x_m \in I$. A function $\phi : I \to \mathbb{R}$ is m-exponentially convex if it is $m-$ exponentially convex in the Jensen sense and continuous on I.

Definition 4 *A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is m-exponentially convex in the Jensen sense for all $m \in \mathbb{N}$.*

A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1 *If $\phi : I \to \mathbb{R}$ is m-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$ is a positive semidefinite matrix for all $n \in \mathbb{N}, n \leq m$. Particularly,*

$$\det\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0$$

for all $n \in \mathbb{N}$, $n = 1, 2, ..., m$.

Remark 4.2 It is known that $\phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Remark 4.3 *By virtue of Theorem 2.2, we define the positive linear functionals with respect to m-convex function ϕ as follows:*

$$\Gamma_1(\phi) := \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)(s-\alpha)^{z-2}}{(z-2)!}\right) ds \geq 0, \tag{35}$$

$$\Gamma_2(\phi) := \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \int_{\alpha}^{\beta} \Upsilon(\mathbf{x}, \mathbf{p}; G(x,s)) \left(\sum_{z=2}^{m-1} \frac{\phi^{(z)}(\beta)(s-\beta)^{z-2}}{(z-2)!}\right) ds \geq 0. \tag{36}$$

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 4.4 *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi \in C^m[\alpha, \beta]$. If the inequalities in (12) $(i = 1)$ and (14) $(i = 2)$ hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\Gamma_i(\phi) = \phi^{(m)}(\xi_i)\Gamma_i(\varphi), \quad i = 1, 2 \tag{37}$$

where $\varphi(x) = \frac{x^m}{m!}$ and $\Gamma_i(\cdot)$ $(i = 1, 2)$ are defined by (35) and (36).

Proof Similar to the proof of Theorem 4.1 in [5]. □

Theorem 4.5 *Let $\phi, \psi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi, \psi \in C^m[\alpha, \beta]$. If the inequalities in (12) $(i = 1)$ and (14) $(i = 2)$ hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\frac{\Gamma_i(\phi)}{\Gamma_i(\psi)} = \frac{\phi^{(m)}(\xi_i)}{\psi^{(m)}(\xi_i)}, \quad i = 1, 2 \tag{38}$$

provided that the denominators are nonzero and $\Gamma_i(\cdot)$ $(i = 1, 2)$ are defined by (35) and (36).

Proof Similar to the proof of Corollary 4.2 in [5]. □

Theorem 4.5 enables us to define Cauchy means, because if

$$\xi_i = \left(\frac{\phi^{(m)}}{\psi^{(m)}}\right)^{-1}\left(\frac{\Gamma_i(\phi)}{\Gamma_i(\psi)}\right), \quad i = 1, 2$$

it means that ξ_i $(i = 1, 2)$ are means for given functions ϕ and ψ.

Next we construct the nontrivial examples of m-exponentially and exponentially convex functions from positive linear functionals $\Gamma_i(\cdot)$ $(i = 1, 2)$. In the sequel I and J are intervals in $\mathbb{R}$.

Theorem 4.6 *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is m-exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_m \in I$. Then for the linear functionals $\Gamma_i(\phi_t)$ $(i = 1, 2)$ as defined by (35) and (36), the following statements are valid:*

(i) The function $t \to \Gamma_i(\phi_t)$ is m-exponentially convex in the Jensen sense on J and the matrix $[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n$ is positive semidefinite for all $n \in \mathbb{N}, n \le m$, $t_1, .., t_n \in J$. Particularly,

$$\det[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n \ge 0 \text{ for all } n \in \mathbb{N},\ n = 1, 2, ..., m.$$

(ii) If the function $t \to \Gamma_i(\phi_t)$ is continuous on J, then it is m-exponentially convex on J.

Proof (i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1, \dots, m$, we define the function

$$h(x) = \sum_{j,l=1}^{m} \xi_j \xi_l \phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is m-exponentially convex in the Jensen sense, we have

$$[x_0, \dots, x_m, h] = \sum_{j,l=1}^{m} \xi_j \xi_l [x_0, \dots, x_m; \phi_{\frac{t_j+t_l}{2}}] \ge 0,$$

which in turn implies that h is an m-convex function on J, and therefore from Remark 4.3 we have $\Gamma_i(h) \ge 0$, $i = 1, 2$. The linearity of $\Gamma_i(\cdot)$ gives

$$\sum_{j,l=1}^{m} \xi_j \xi_l \Gamma_i(\phi_{\frac{t_j+t_l}{2}}) \ge 0.$$

We conclude that the function $t \mapsto \Gamma_i(\phi_t)$ is m-exponentially convex on J in the Jensen sense.
The remaining part follows from Proposition 4.1.

(ii) If the function $t \to \Gamma_i(\phi_t)$ is continuous on J, then it is m-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 4.7 *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_m \in I$. Then for the linear functionals $\Gamma_i(\phi_t)$ $(i = 1, 2)$ as defined by (35) and (36), the following statements hold:*

(i) The function $t \to \Gamma_i(\phi_t)$ *is exponentially convex in the Jensen sense on* J *and the matrix* $[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n$ *is positive semidefinite for all* $n \in \mathbb{N}, n \le m$, $t_1, .., t_n \in J$. *Particularly,*

$$\det[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n \ge 0 \text{ for all } n \in \mathbb{N},\ n = 1, 2, ..., m.$$

(ii) If the function $t \to \Gamma_i(\phi_t)$ *is continuous on* J, *then it is exponentially convex on* J.

Corollary 4.8 *Let* $\Omega = \{\phi_t : t \in J\}$, *where* J *is an interval in* $\mathbb{R}$, *be a family of functions defined on an interval* I *in* $\mathbb{R}$, *such that the function* $t \mapsto [x_0, \ldots, x_m; \phi_t]$ *is* 2*-exponentially convex in the Jensen sense on* J *for every* $(m+1)$ *mutually different points* $x_0, \ldots, x_m \in I$. *Let* $\Gamma_i(\cdot)$, $i = 1, 2$ *be linear functionals defined by* (35) *and* (36). *Then the following statements hold:*

(i) If the function $t \mapsto \Gamma_i(\phi_t)$ *is continuous on* J, *then it is a* 2*-exponentially convex function on* J. *If* $t \mapsto \Gamma_i(\phi_t)$ *is additionally strictly positive, then it is also log-convex on* J. *Furthermore, the following inequality holds true:*

$$[\Gamma_i(\phi_s)]^{t-r} \le [\Gamma_i(\phi_r)]^{t-s} [\Gamma_i(\phi_t)]^{s-r}, \quad i = 1, 2$$

for every choice $r, s, t \in J$, *such that* $r < s < t$.

(ii) If the function $t \mapsto \Gamma_i(\phi_t)$ *is strictly positive and differentiable on J, then for every* $p, q, u, v \in J$, *such that* $p \le u$ *and* $q \le v$, *we have*

$$\mu_{p,q}(\Gamma_i, \Omega) \le \mu_{u,v}(\Gamma_i, \Omega), \tag{39}$$

where

$$\mu_{p,q}(\Gamma_i, \Omega) = \begin{cases} \left(\frac{\Gamma_i(\phi_p)}{\Gamma_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\frac{\frac{d}{dp}\Gamma_i(\phi_p)}{\Gamma_i(\phi_p)}\right), & p = q, \end{cases} \tag{40}$$

for $\phi_p, \phi_q \in \Omega$.

Proof

(i) This is an immediate consequence of Theorem 4.6 and Remark 4.2.

(ii) Since $p \mapsto \Gamma_i(\phi_t)$ is positive and continuous, by (i) we have that $t \mapsto \Gamma_i(\phi_t)$ is log-convex on J; that is, the function $t \mapsto \log \Gamma_i(\phi_t)$ is convex on J. Hence, we get

$$\frac{\log \Gamma_i(\phi_p) - \log \Gamma_i(\phi_q)}{p-q} \le \frac{\log \Gamma_i(\phi_u) - \log \Gamma_i(\phi_v)}{u-v}, \tag{41}$$

for $p \le u, q \le v, p \ne q, u \ne v$. Thus, we conclude that

$$\mu_{p,q}(\Gamma_i, \Omega) \le \mu_{u,v}(\Gamma_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (41) as limit cases.

□

5. Applications to Cauchy means

In this section, we present some families of functions that fulfill the conditions of Theorem 4.6, Corollary 4.7, and Corollary 4.8. This enables us to construct large families of functions that are exponentially convex. An explicit form of this function is obtained after we calculate the explicit action of functionals on a given family.

Example 5.1 *Let us consider a family of functions*

$$\Omega_1 = \{\phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^m}, & t \neq 0, \\ \frac{x^m}{m!}, & t = 0. \end{cases}$$

Since $\frac{d^m \phi_t}{dx^m}(x) = e^{tx} > 0$*, the function* ϕ_t *is* m*-convex on* $\mathbb{R}$ *for every* $t \in \mathbb{R}$ *and* $t \mapsto \frac{d^m \phi_t}{dx^m}(x)$ *is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that* $t \mapsto [x_0, \ldots, x_m; \phi_t]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7, we conclude that* $t \mapsto \Gamma_i(\phi_t)$ $(i = 1, 2)$ *are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping* $t \mapsto \phi_t$ *is not continuous for* $t = 0$*), so it is exponentially convex. For this family of functions,* $\mu_{t,q}(\Gamma_i, \Omega_1)$ $(i = 1, 2)$*, from (40), becomes*

$$\mu_{t,q}(\Gamma_i, \Omega_1) = \begin{cases} \left(\frac{\Gamma_i(\phi_t)}{\Gamma_i(\phi_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left(\frac{\Gamma_i(id \cdot \phi_t)}{\Gamma_i(\phi_t)} - \frac{m}{t}\right), & t = q \neq 0, \\ \exp\left(\frac{1}{m+1}\frac{\Gamma_i(id \cdot \phi_0)}{\Gamma_i(\phi_0)}\right), & t = q = 0, \end{cases}$$

where "id" is the identity function. By Corollary 4.8 $\mu_{t,q}(\Gamma_i, \Omega_1)$ $(i = 1, 2)$ *are monotone functions in parameters* t *and* q*.*

Since

$$\left(\frac{\frac{d^m f_t}{dx^m}}{\frac{d^m f_q}{dx^m}}\right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 4.5 it follows that:

$$M_{t,q}(\Gamma_i, \Omega_1) = \log \mu_{t,q}(\Gamma_i, \Omega_1), \quad i = 1, 2$$

satisfies

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_1) \leq \beta, \quad i = 1, 2.$$

Hence, $M_{t,q}(\Gamma_i, \Omega_1)$ $(i = 1, 2)$ *are monotonic means.*

Example 5.2 *Let us consider a family of functions*

$$\Omega_2 = \{g_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-m+1)}, & t \notin \{0, 1, \ldots, m-1\}, \\ \frac{x^j \log x}{(-1)^{m-1-j} j!(m-1-j)!}, & t = j \in \{0, 1, \ldots, m-1\}. \end{cases}$$

Since $\frac{d^m g_t}{dx^m}(x) = x^{t-m} > 0$, *the function* g_t *is* m*-convex for* $x > 0$ *and* $t \mapsto \frac{d^m g_t}{dx^m}(x)$ *is exponentially convex by definition. Arguing as in Example 5.1, we get that the mappings* $t \mapsto \Gamma_i(g_t)$ $(i = 1, 2)$ *are exponentially convex. Hence, for this family of functions* $\mu_{p,q}(\Gamma_i, \Omega_2)$ $(i = 1, 2)$*, from (40), are equal to*

$$\mu_{t,q}(\Gamma_i, \Omega_2) = \begin{cases} \left(\frac{\Gamma_i(g_t)}{\Gamma_i(g_q)}\right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp\left((-1)^{m-1}(m-1)!\frac{\Gamma_i(g_0 g_t)}{\Gamma_i(g_t)} + \sum_{k=0}^{m-1} \frac{1}{k-t}\right), & t = q \notin \{0, 1, \ldots, m-1\}, \\ \exp\left((-1)^{m-1}(m-1)!\frac{\Gamma_i(g_0 g_t)}{2\Gamma_i(g_t)} + \sum_{\substack{k=0 \\ k \neq t}}^{m-1} \frac{1}{k-t}\right), & t = q \in \{0, 1, \ldots, m-1\}. \end{cases}$$

Again, using Theorem 4.5, we conclude that

$$\alpha \leq \left(\frac{\Gamma_i(g_t)}{\Gamma_i(g_q)}\right)^{\frac{1}{t-q}} \leq \beta, \quad i = 1, 2. \tag{42}$$

Hence, $\mu_{t,q}(\Gamma_i, \Omega_2)$ $(i = 1, 2)$ *are means and their monotonicity is followed by (39).*

Example 5.3 *Let*

$$\Omega_3 = \{\zeta_t : (0, \infty) \to \mathbb{R} : t \in (0, \infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^m}, & t \neq 1; \\ \frac{x^m}{(m)!}, & t = 1. \end{cases}$$

Since $\frac{d^m \zeta_t}{dx^m}(x) = t^{-x}$ *is the Laplace transform of a nonnegative function (see [13]) it is exponentially convex. Obviously* ζ_t *are* m*-convex functions for every* $t > 0$.
For this family of functions, $\mu_{t,q}(\Gamma_i, \Omega_3)$ $(i = 1, 2)$*, in this case for* $[\alpha, \beta] \subset \mathbb{R}^+$*, from (40) becomes*

$$\mu_{t,q}(\Gamma_i, \Omega_3) = \begin{cases} \left(\frac{\Gamma_i(\zeta_t)}{\Gamma_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Gamma_i(id.\zeta_t)}{t\Gamma_i(\zeta_t)} - \frac{m}{t \log t}\right), & t = q \neq 1; \\ exp\left(-\frac{1}{m+1}\frac{\Gamma_i(id.\zeta_1)}{\Gamma_i(\zeta_1)}\right), & t = q = 1, \end{cases}$$

where id *is the identity function. By Corollary 4.8* $\mu_{p,q}(\Gamma_i, \Omega_3)$ $(i = 1, 2)$ *are monotone functions in parameters* t *and* q.
Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma_i, \Omega_3) = -L(t,q) log \mu_{t,q}(\Gamma_i, \Omega_3); \quad i = 1, 2$$

satisfy

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_3) \leq \beta; \quad i = 1, 2.$$

This shows that $M_{t,q}(\Gamma_i, \Omega_3)$ $(i = 1, 2)$ *are means. Because of inequality (39), these means are monotonic.* $L(t, q)$ *is a logarithmic mean defined by*

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & t \neq q; \\ t, & t = q. \end{cases}$$

Example 5.4 *Let*

$$\Omega_4 = \{\gamma_t : (0,\infty) \to \mathbb{R} : t \in (0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^m}.$$

Since $\frac{d^m \gamma_t}{dx^m}(x) = e^{-x\sqrt{t}}$ *is the Laplace transform of a nonnegative function (see [13]) it is exponentially convex. Obviously* γ_t *are* m*-convex function for every* $t > 0$.
For this family of functions, $\mu_{t,q}(\Gamma_i, \Omega_4)$ $(i = 1, 2)$*, in this case for* $[\alpha, \beta] \subset \mathbb{R}^+$*, from (40) becomes*

$$\mu_{t,q}(\Gamma_i, \Omega_4) = \begin{cases} \left(\frac{\Gamma_i(\gamma_t)}{\Gamma_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Gamma_i(id.\gamma_t)}{2\sqrt{t}\Gamma_i(\gamma_t)} - \frac{m}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 4.8, these are monotone functions in parameters t *and* q.
Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma_i, \Omega_4) = -\left(\sqrt{t} + \sqrt{q}\right) ln\mu_{t,q}(\Gamma_i, \Omega_4); \quad i = 1, 2$$

satisfy

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_4) \leq \beta; \quad i = 1, 2.$$

This shows that $M_{t,q}(\Gamma_i, \Omega_4)$ $(i = 1, 2)$ *are means. Because of the above inequality (39), these means are monotonic.*

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