On the finite $p$-groups with unique cyclic subgroup of given order

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Abstract: In this paper, we prove that if $G$ is nonabelian and $|G| > p^4$, then $G$ has a unique cyclic subgroup of order $p^m$ with $m \geq 3$ if and only if $G$ has a unique abelian subgroup of order $p^3$ if and only if $G$ is a 2-group of maximal class.

Key words: $p$-group of maximal class, extra-special $p$-group

1. Introduction

All groups considered in this paper are finite $p$-groups. The terminology and the notation in this paper are standard. The Frattini subgroup, the commutator subgroup, and the center of a group $G$ will be denoted by $\Phi(G)$, $G'$, and $Z(G)$ respectively. We use $c(G)$ and $G_i$ to denote the nilpotent class and the $i$th term of the lower central series of $G$, respectively. The number of subgroups of order $p^m$, abelian subgroups of order $p^m$, and cyclic subgroups of order $p^m$ are denoted by $s_m(G)$, $a_m(G)$, and $c_m(G)$, respectively. For a subgroup $H$ in $G$, the centralizer of $H$ in $G$ is denoted by $C_G(H)$.

There is much interest in investigating the structure of a group whenever the number of some kind of subgroups is given. For example, finite $p$-groups with exactly one minimal nonabelian subgroup of given structure of order $p^3$ are classified by [5]. In [3], finite $p$-groups with exactly one minimal nonabelian subgroup of index $p$ are investigated. In this paper, we are interested in the finite $p$-groups with unique cyclic subgroup of given order.

In [1] and [2], the authors proved that

**Theorem 1** ([1]) Suppose that a 2-group $G$ is neither cyclic nor of maximal class. If $n > 1$, then $c_n(G)$ is even.

**Theorem 2** ([2]) Let $G$ be a noncyclic $p$-group, $p > 2$, and $n > 0$. If $n > 1$, then $p$ divides $c_n(G)$.

By the above two theorems, we see that finite $p$-groups with unique cyclic subgroup of order $p^m$ are cyclic groups or 2-groups of maximal class. In this paper, we give a direct elementary proof. Moreover, we proved the following theorem:

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Theorem 3 Let $G$ be a nonabelian $p$-group and $|G| > p^4$. Then $c_m(G) = 1$ if and only if $a_3(G) = 1$ if and only if $G$ is 2-group of maximal class, where $m \geq 3$.

2. Preliminaries
We gathered all the results used in what follows.

Lemma 4 ([4] Theorem 2.2.13) Let $G$ be a finite $p$-group with unique subgroup of order $p$. Then
(i) $G$ is cyclic if $p > 2$;
(ii) $G$ is a cyclic group or generalized quaternion group when $p = 2$.

Lemma 5 ([4] Exercise 2.2.3) Let $G$ be a nonabelian $p$-group. Then the number of abelian maximal subgroups of $G$ is 0, 1, or $1 + p$.

Lemma 6 ([4] Theorem 2.7.1) Let $G$ be a $p$-group with $|G| = p^n$ and $1 < m < n$. If $s_m(G) = 1$, then $G$ is cyclic.

Lemma 7 ([4] Theorem 2.7.2) Let $G$ be a $p$-group with $|G| = p^n$ and $1 < m \leq n$. If $s_m(G) = c_m(G)$, then
(1) $G$ is cyclic if $p^m \neq 4$;
(2) $G$ is a cyclic group or generalized quaternion group if $p^m = 4$.

Lemma 8 ([4] Theorem 2.5.2) Let $G$ be a $p$-group of maximal class with $|G| = p^n$. Then
(1) $G_i$ is the unique normal subgroup of order $p^{n-i}$;
(2) If $p > 2$ and $n > 3$, then $G$ does not have a cyclic normal subgroup of order $p^2$.

Lemma 9 ([4] Theorem 2.5.5) $G$ is a 2-group of maximal class if and only if $|G : G'| = 4$.

Lemma 10 ([4] Theorem 2.5.6) Let $G$ be a nonabelian $p$-group with $p > 2$. If $G$ has an abelian maximal subgroup, then $G$ is of maximal class if and only if $|G : G'| = p^2$.

Lemma 11 ([4] Theorem 2.5.7) Let $G$ be a nonabelian $p$-group. If $G$ has a subgroup $A$ of order $p^2$ such that $C_G(A) = A$, then $G$ is of maximal class.

Lemma 12 ([4] Theorem 7.1.6) Let $G$ be an extra-special $p$-group. Then $|G| = p^{2m+1}$ for some integer $m$.

3. Finite $p$-groups with $c_m(G) = 1$
Firstly, we have the following Lemmas.

Lemma 13 Let $G$ be a metacyclic $p$-group with $p > 2$ and $H \leq G$. Then $H$ is abelian if $|H| \leq |G/G'|$ and $H$ is nonabelian if $|H| > |G/G'|$.

Proof Suppose $G = \langle a \rangle \langle b \rangle$ and $G/\langle a \rangle \cong \langle b \rangle$. For any subgroup $H$ of $G$, assume that $H/\langle a \rangle \cong H/\langle a \rangle = \langle b^p \rangle$ and $H \cap \langle a \rangle = \langle a^p \rangle$. Thus, $|H| = |G|/p^{i+j}$ and $H = \langle a^p, b^p \rangle$. Since $p > 2$, we see that $[a^p, b^p] = 1$ if and only if $[a, b]^{p^{i+j}} = 1$. Hence, $H' = 1$ if and only if $p^{i+j} \geq |G'|$. By $|H| = |G|/p^{i+j}$, we see $p^{i+j} = |G|/|H|$. Then $H$ is abelian if and only if $|H| \leq |G/G'|$. \hfill $\Box$
Lemma 14 Let $G$ be a $p$-group with $|G| \geq p^4$. Then $G$ has an abelian normal subgroup of order $p^3$.

Proof Take a subgroup $N$ of order $p^2$ that is normal in $G$. Since $\langle N/C \rangle$, $G/C_G(N) \leq \text{Aut}(N)$. Thus, $|G/C_G(N)|p$, $|C_G(N)| \geq p^3$. By $C_G(N) \leq G$, we see that there exists a normal subgroup $M$ such that $N \leq M \leq C_G(N)$ and $|M| = p^3$. Then $M = \langle N, c \rangle$, where $c \in C_G(N)$. Thus, $M$ is abelian. Therefore, $M$ is desired.

Proposition 15 Let $G$ be a finite $p$-group and $n > 3$. If $c_n(G) = 1$, then $c_3(G) = 1$.

Proof Suppose, by way of contradiction, that $C_{p^n} \cong \langle a \rangle \leq G$, $C_{p^3} \cong \langle x \rangle \leq G$, and $x \notin \langle a \rangle$. Since $\langle a \rangle \langle \text{char}(a) \rangle \leq G$, $\langle a \rangle \langle y \rangle \leq \langle a \rangle \langle x \rangle$. By $\langle a \rangle \langle y \rangle \leq \langle a \rangle \langle x \rangle$ and $o(a)/o(\langle a \rangle) = p$, we see $\langle a, x \rangle \in \langle a \rangle$. Thus, we may assume $[a, x] = a^p$. If $p > 2$, then

$$(ax^{-1})^p = a^p[a, x][a, x][a, x] \ldots x^{-p} = a^{p^3}$$

with $(v, p) = 1$. Hence, $C_{p^n} \cong \langle ax^{-1} \rangle \neq \langle a \rangle$, a contradiction. Now, assume $p = 2$. If $x^2 \notin \langle a \rangle$, then it is easy to see $C_{2^n} \cong \langle ax^{-1} \rangle \neq \langle a \rangle$. When $2i$, there exists $\langle ax^{-1} \rangle$ such that $C_{2^n} \cong \langle ax^{-1} \rangle \neq \langle a \rangle$. For $x^2 \in \langle a \rangle$ and $(2, i) = 1$, setting $x^2 = a^{2j_{n-2}}$ with $(j, 2) = 1$, $1 = [a^{2j_{n-2}}, x] = [a, x]^{2j_{n-2}} = a^{ij_{n-1}}$. Thus, $a^{2n-1} = 1$, a contradiction. The proof is complete.

Proposition 16 Let $G$ be a nonabelian $p$-group with $|G| \geq p^4$. If $c_3(G) = 1$, then $a_3(G) = 1$.

Proof Assume the contrary; there exists a subgroup $N$ such that $N \cong C_p \times C_p$ or $C_p \times C_p \times C_p$. Suppose the unique cyclic subgroup of order $p^3$ is $M = \langle a \rangle$. Now we divide our analysis into two cases: (1) $N \cong C_{p^2} \times C_p$ and (2) $N \cong C_p \times C_p \times C_p$.

Case 1: $C_{p^2} \times C_p \cong N = \langle x \rangle \times \langle y \rangle$.

Since $\langle a \rangle \langle \text{char}(a) \rangle \leq G$, we may assume that $[a, x] = a^{ip}$ and $[a, y] = a^{jp}$ for some integers $i$ and $j$.

(1.1) $x \in \langle a \rangle$. We see $y \notin \langle a \rangle$ from $y \notin \langle x \rangle$. If $p|j$, then $(ay)^{-1} = a^p[y, a] = a^{pv}$ with $(v, p) = 1$. Thus, $C_{p^3} \cong \langle ay^{-1} \rangle \neq \langle a \rangle$. Therefore, $(p, j) = 1$. By $x \in \langle a \rangle$, we may assume $x = a^{ip}$ with $(r, p) = 1$. Then

$$1 = [x, y] = [a^{pr}, y] = [a, y]^{pr} = a^{p^2jr} \neq 1,$$

a contradiction.

(1.2) Let $x \notin \langle a \rangle$. Since $N = \langle x \rangle \times \langle y \rangle = \langle x^{-1} \rangle \times \langle y \rangle$, we may assume that $x^{-1}y \notin \langle a \rangle$ by (1.1). If $x^p \notin \langle a \rangle$, then $C_{p^3} \cong \langle ay^{-p} \rangle \neq \langle a \rangle$. Therefore, $x^p = a^{kp^2}$ with $(k, p) = 1$. It is easy to see $(ij, p) = 1$ from $c_3(G) = 1$. If $p > 2$, then

$$1 = [a, x^p] = [a, x]^{p[2]} = a^{p^2},$$

which contradicts $o(a) = p^3$. When $p = 2$,

$$(ax^{-1})^2 = a^2[a, x]x = a^2[a, y][a, x][a, x][a, x]x = a^2a^{2(i+j)}a^{4ij}a^{4k} = a^{2v}$$

with $(v, 2) = 1$. Thus, $C_{p^3} \cong \langle ax^{-1} \rangle \neq \langle a \rangle$, a contradiction.
Case 2: \( C_p \times C_p \times C_p \cong N = \langle x \rangle \times \langle y \rangle \times \langle z \rangle \).

Since \(|M \cap N| \leq p\), we may assume that \( y, z \notin \langle a \rangle \) and \( yz \notin \langle a \rangle \). By \( \langle a^p \rangle \leq MN\), \( [a, y] = a^{jp}\) and \([a, z] = a^{kp}\). It is easy to see that \( (jk, p) = 1\). If \( p > 2\), then \( C_{p^3} \cong \langle ay^{-1} \rangle \neq \langle a \rangle \). Therefore, \( p = 2\).

However, there exists \( ayz\) such that \( (ayz)^2 = a^3[a, yz] = a^2a^{2(k+j)}a^{4kj}\). Hence, \( (ayz) \cong \langle a \rangle\), a contradiction. \( \square \)

Proposition 17 Let \( G \) be a \( p \)-group with \( |G| \geq p^4\). Then \( a_3(G) = 1 \) and \( c_3(G) = 1 \) if and only if \( G \) is a cyclic group or 2-group of maximal class.

Proof If \( G \) is abelian, then \( G \) is cyclic from Lemma 6. Now assume \( G \) is nonabelian. If there exists a subgroup \( A \cong C_p \times C_p\), then \( C_G(A) = A \) from \( a_3(G) = 1 \) and \( c_3(G) = 1\). Thus, we see \( G \) is a \( p \)-group of maximal class by Lemma 11. If all the subgroups of order \( p^2\) are cyclic, then it follows from Lemma 7 that \( G \) is a generalized quaternion group, which is also of maximal class. Hence, \( G \) is a 2-group of maximal class from Lemma 8.

Since \( G \) is a 2-group of maximal class, it is easy to check that \( c_3(G) = 1 \) and \( a_3(G) = 1\). \( \square \)

Proposition 18 Let \( G \) be a \( p \)-group with \( |G| \geq p^4\). Then \( a_3(G) = 1 \) and \( c_3(G) = 0 \) if and only if \( G \) is of maximal class of order \( p^3 \) with \( p > 2\).

Proof If \( G \) is of maximal class of order \( p^4\), then \( d(G) = 2\). Therefore, the number of maximal subgroups is \( 1 + p\). By Lemma 14, we see that \( a_3(G) \geq 1\). It follows that \( a_3(G) = 1 \) or \( 1 + p \) from Lemma 5. If \( a_3(G) = 1 + p\), then \( G \) is minimal nonabelian and \( c(G) = 2\), a contradiction. Therefore, \( a_3(G) = 1 \) and then \( c_3(G) = 0 \) or \( 1\). If \( c_3(G) = 1\), then there exists a cyclic normal subgroup of order \( p^2\), which contradicts Lemma 8. Thus, \( c_3(G) = 0\).

Conversely, we see that \( G \) is nonabelian by Lemma 6. First we prove that the groups of order \( p^4\) satisfying \( a_3(G) = 1 \) and \( c_3(G) = 0 \) are \( p \)-groups of maximal class with \( p > 2\). In this case \( |Z(G)| = p\). If not, \( |Z(G)| = p^2\). We see \( G/Z(G) \cong C_p \times C_p \) from \( G \) is nonabelian. Then the number of abelian subgroups of order \( p^3\) containing \( Z(G) \) is \( 1 + p\), a contradiction. By Lemmas 9 and 10, we need to prove \( |G'| = p^2\). Assume that \( |G'| = p\). If \( d(G) = 2\), then \( G \) is a minimal nonabelian \( p \)-group. Hence, \( |Z(G)| = p^2\), which is impossible. Therefore, \( d(G) = 3 \) and \( G' = \Phi(G)\); therefore, \( G \) is an extra-special \( p \)-group. Again, we have a contradiction, Lemma 12, because \( |G| = p^4\). Thus, \( |G'| = p^2\) and \( G \) is of maximal class. Now, if \( G \) is a 2-group of maximal class, then the abelian subgroup of order \( p^3\) is cyclic. Therefore, \( p > 2\).

Next, noting that the property is inherited by subgroups, we only need to prove that any group of order \( p^5\) (\( p > 2\)) does not satisfy \( a_3(G) = 1 \) and \( c_3(G) = 0\). If there exists a group \( G \) of order \( p^5\) that satisfies the property, then for each maximal subgroup \( M \) of \( G\), \( M \) has an abelian subgroup of order \( p^3\) by Lemma 14. Thus, \( M \) satisfies \( a_3(G) = 1 \) and \( c_3(G) = 0\). Thus, \( M \) is of class 3 by the above paragraph. Therefore, \( c(G) = 3 \) or \( 4\).

Case (i) \( c(G) = 3\). If \( Z(G) \geq p^2\), then there exists \( A \) such that \( |A| = p^2\) and \( A \leq Z(G)\). By the hypothesis, \( G/A \) has the unique subgroup of order \( p\), and \( p > 2\). Thus, \( G/A \) is cyclic since Lemma 4. Therefore, \( G \) is abelian, a contradiction, and so \( |Z(G)| = p\), \( |G_3| = p\). By Lemma 14, we see \( d(G) = 2\). Write \( \bar{G} = G/G_3\). Then \(|(\bar{G})'| = p \) or \( p^2\). If \(|(\bar{G})'| = p^2\), then \( \bar{G} \) is of maximal class by Lemma 10, which contradicts
$c(G) = 3$. Thus, $|(G)'| = p$. If $G$ is metacyclic, then $G$ is metacyclic. We may get a contradiction from Lemma 13 and $a_3(G) = 1$. Therefore,

$$G \cong M_p(2, 1, 1) = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle.$$ 

Assume $G_3 = \langle x \rangle \cong C_p$. Since $c_3(G) = 0$, $a^p = 1$. Thus, $\langle a, x \rangle \cong C_p \times C_p$. By $Z(G) = G_3$, we see $a^p \not\in Z(G)$. Since $[a^p, c] = [a, c]^p[a, c]^{(z)} = 1$, $[a^p, b] = c^p \neq 1$. Thus, $\langle c, a^p \rangle \cong C_p^2 \times C_p$. However, $\langle c, a^p \rangle \neq \langle a, x \rangle$, a contradiction.

Case (ii) $c(G) = 4$. We see $G_3 \cong C_p \times C_p$ from Lemma 8. Assume that $G_4 = \langle z \rangle$ and $G_3 = \langle z \rangle \times \langle y \rangle$. It is easy to see that $G/G_3 = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle$, and $G = \langle a, b \rangle$. $G' = \langle c, z, y \rangle$ is the unique abelian subgroup of order $p^3$. Since $[(a, b), (z, y)] = G_4$, we have $[a, y] = z^i$, $[b, y] = z^j$ and at least one of $i$ and $j$ cannot be divided exactly by $p$. Then $\langle a^{-i}b^j, y, z \rangle$ is another abelian subgroup of order $p^3$ of $G$, a contradiction. The proof is complete. □

By the above propositions, we easily get the following theorem.

**Theorem 19** Let $G$ be a nonabelian $p$-group with $|G| > p^4$. Then the following conclusions are equivalent:

1. $c_m(G) = 1$ where $m \geq 3$
2. $a_3(G) = 1$
3. $G$ is a 2-group of maximal class.

**Proof** If (1), then (2) by Propositions 15 and 16. When (2) holds, we see (3) by Propositions 17 and 18. If $G$ is a 2-group of maximal class, then $G$ is isomorphic to one of the following three types of groups by Theorem 2.5.3 in [4]:

(a) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1} \rangle$, $n \geq 3$;
(b) $\langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1} \rangle$, $n \geq 3$;
(c) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1+2^{n-2}} \rangle$, $n \geq 4$.

By calculation, we see that cyclic subgroups of order $\geq 2^3$ are in $\langle a \rangle$. Therefore, $c_m(G) = 1$ where $m \geq 3$. □

**Theorem 20** Let $G$ be a finite $p$-group. Then $c_2(G) = 1$ if and only if $G$ is a cyclic group or dihedral group.

**Proof** If $G$ is abelian, then $G$ is cyclic. Assume that $G$ is nonabelian and $M$ is the unique cyclic subgroup of order $p^2$. If $C_G(M) = M$, then $G$ is of maximal class by Lemma 11. For $C_G(M) > M$, we see $C_G(M)$ is cyclic from Lemma 6. Since any cyclic subgroup of order $p^n$ contains $M$ and lies in $C_G(M)$, we have $a_3(G) = 1$. Therefore, $G$ is of maximal class. By Lemma 8, $G$ is a 2-group. It is easy to check that only the dihedral group in 2-groups of maximal class satisfies $c_2(G) = 1$. Conversely, the conclusion is obvious. □

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