Abundance of \(E\)-order-preserving transformation semigroups

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Abstract: Let \(T_X\) be the full transformation semigroup on a finite totally ordered set \(X = \{1 < 2 < \ldots < n\} (n \geq 3)\) and \(E\) be a nontrivial equivalence relation on \(X\). In this paper, we consider a subsemigroup of \(T_X\) defined by

\[
EOP_X = \{ f \in T_X : \forall x, y \in X, (x, y) \in E, x \leq y \Rightarrow (f(x), f(y)) \in E, f(x) \leq f(y) \}
\]

and present a necessary and sufficient condition under which the semigroup \(EOP_X\) is abundant.

Key words: Transformation semigroup, \(L^*\)-relation, \(R^*\)-relation, idempotent, abundance

1. Introduction

Let \(S\) be a semigroup. We say that \(a, b \in S\) are \(L^*\)-related in \(S\) if they are \(L\)-related in a semigroup \(T\) such that \(S\) is a subsemigroup of \(T\) and write \((a, b) \in L^*\). The relation \(R^*\) is defined in the dual way. The equivalence relations \(L^*\) and \(R^*\) have been intensely studied in semigroup theory and have been used to define some important classes of semigroups. For instance, Fountain [3] pointed out that a semigroup \(S\) has the property that for every \(a \in S\) the right ideal \(aS^1\) is projective (as an \(S\)-act) if and only if every \(L^*\)-class of \(S\) contains an idempotent. We call such semigroups right abundant. Left abundant semigroups are defined dually. A semigroup is abundant if it is both left and right abundant; see Fountain [4]. The property of being abundant can be considered as a wide generalization of regularity. (Recall that in a regular semigroup \(L^* = L\) and \(R^* = R\).

Many papers have been written describing the abundances of various transformation semigroups on the nonempty set \(X\) (see [1, 8–12]). For example, Umar [11] observed that the semigroup \(S_n^*\) of nonbijective, order-decreasing transformations on a finite totally ordered set \(X = \{1 < 2 < \ldots < n\}\) is abundant but not regular. Let \(T_X\) be the full transformation semigroup on a set \(X\) and \(E\) be an arbitrary equivalence relation on \(X\). Araujo and Konieczny [1] proved that the semigroup

\[
T_E(X, R) = \{ f \in T_X : f(R) \subseteq R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (f(x), f(y)) \in E \},
\]

where \(R\) is a cross-section of the partition \(X/E\) of \(X\) induced by \(E\), is abundant if and only if it is regular. Pei and Zhou [8] gave a condition under which the semigroup

\[
T_E(X) = \{ f \in T_X : \forall x, y \in X, (x, y) \in E \Rightarrow (f(x), f(y)) \in E \}
\]

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is abundant. Sun [9] proved that the semigroup
\[ T(X,Y) = \{ f \in T_X : f(X) \subseteq Y \} \ (Y \subseteq X) \]
is left abundant but not right abundant if \(|Y| \geq 2\) and \(Y \neq X\). Sun and Wang [10] showed that the semigroup
\[ T_3(X) = \{ f \in T_X : \forall x, y \in X, (f(x), f(y)) \in E \Rightarrow (x, y) \in E \} \]
is also left abundant but not right abundant if the partition \(X/E\) of \(X\) is infinite.

Given an arbitrary equivalence relation \(E\) on a finite totally ordered set \(X = \{1 < 2 < \ldots < n\}\), the authors [6] introduced a new family of the subsemigroup of \(T_X\) defined by
\[ EOP_X = \{ f \in T_X : \forall x, y \in X, (x, y) \in E, x \leq y \Rightarrow (f(x), f(y)) \in E, f(x) \leq f(y) \}, \]
which is called an \(E\)-order-preserving transformation semigroup, and investigated the properties for \(EOP_X\), such as Green’s relations and the natural partial order on the semigroup \(EOP_X\) in [6] and [7], respectively. In particular, the regularity of the semigroup \(EOP_X\) was described as follows.

**Lemma 1.1** ([6]) The \(E\)-order-preserving transformation semigroup \(EOP_X\) is regular if and only if either \(E = X \times X\) or \(E = \{(x, x) : x \in X\}\).

In this paper our aim is to investigate the abundance of the semigroup \(EOP_X\). Note that if \(E = X \times X\) or \(E = \{(x, x) : x \in X\}\) then \(EOP_X\) is abundant. Thus, for the remainder of the paper, we assume that \(E\) is nontrivial on the finite totally ordered set \(X = \{1 < 2 < \ldots < n\}\) \((n \geq 3)\); that is, both \(E \neq X \times X\) and \(E \neq \{(x, x) : x \in X\}\). Under the assumption, we first characterize the relations \(L^*\) and \(R^*\) on the semigroup \(EOP_X\) and then present a necessary and sufficient condition under which the semigroup \(EOP_X\) is abundant. Throughout this paper, we apply transformations on the left so that for \(f, g \in EOP_X\), their product \(fg\) is the transformation obtained by performing first \(g\) and then \(f\).

### 2. The main result
The following lemma gives a characterization of \(L^*\) and \(R^*\) that can be found, for instance, in [5, Sect. X.1].

**Lemma 2.1** Let \(S\) be a semigroup. Then
\[ L^* = \{(a, b) \in S \times S : (\forall s, t \in S^1) as = at \iff bs = bt\} \]
and
\[ R^* = \{(a, b) \in S \times S : (\forall s, t \in S^1) sa = ta \iff sb = tb\}. \]

We begin with the \(L^*\)-relation.

**Lemma 2.2** Let \(f, g \in EOP_X\). Then \((f, g) \in L^*\) if and only if \(kerf = kerg\).

**Proof** For the ‘if’ part, suppose that \(kerf = kerg\), and then \(f\) and \(g\) are known to be \(L\)-related in the full transformation semigroup \(T_X\); see, for instance, [2, Sect. 2.2]. Hence, \(f\) and \(g\) are \(L^*\)-related in \(EOP_X\).

\(^1\)In order to prevent any chance of confusion, recall that in [2] transformations are written on the right of their arguments, while the description of Green’s relations in [2, Section 2.2] should be left-right dualized to be applied in the present paper’s setting.
For the ‘only if’ part, suppose that \((f, g) \in \mathcal{L}^*\). For \(x \in X\), let \(\langle x \rangle\) be the constant transformation with the range \(\{x\}\); this transformation clearly belongs to \(EOP_X\). Take \((x, y) \in \ker f\) for \(x, y \in X\). Then \(f(x) = \{f(x)\} = \{f(y)\} = \{f(y)\}\). Applying the characterization of \(\mathcal{L}^*\) from Lemma 2.1, we have \(g(x) = g(y)\). This means \(g(x) = g(y)\) and \((x, y) \in \ker g\). Thus, \(\ker f \subseteq \ker g\) and by symmetry \(\ker g \subseteq \ker f\). Hence, \(\ker f = \ker g\).

In what follows we consider the \(\mathcal{R}^\ast\)-relation.

**Lemma 2.3** Let \(f, g \in EOP_X\). Then \((f, g) \in \mathcal{R}^\ast\) if and only if \(f(X) = g(X)\).

**Proof** For the ‘if’ part, suppose that \(f(X) = g(X)\), and then \(f\) and \(g\) are known to be \(\mathcal{R}\)-related in the full transformation semigroup \(T_X\). Hence, \(f\) and \(g\) are \(\mathcal{R}^\ast\)-related in \(EOP_X\).

For the ‘only if’ part, suppose that \((f, g) \in \mathcal{R}^\ast\) and \(a \notin f(X)\). Let

\[
A = \{A \in X/E : A \cap f(X) \neq \emptyset\}.
\]

For each \(A \in A\), let \(A \cap f(X) = \{a_1 < a_2 < \ldots < a_s\}\). Write \(a_0 = \min A\) and \(a_s = \max A\). Define \(h_s : A \to A\) by

\[
h_s(x) = \begin{cases} 
a_1 & \text{if } x \in [a_0, a_1] 
\quad \quad a_t & \text{if } x \in (a_{t-1}, a_t)(2 \leq t \leq s) 
\quad \quad a_s & \text{if } x \in (a_s, a_s].
\end{cases}
\]

Clearly, \(h_s(A) = \{a_1, a_2, \ldots, a_s\} = A \cap f(X)\). Now we define \(h : X \to X\). There are two cases to consider.

Case 1. \(\overline{a} \notin A\) where \(\overline{a}\) is the \(E\)-class containing \(a\). Fix \(A_0 \in A\) and \(b \in A_0 \cap f(X)\). For each \(A \in X/E\), define \(h : X \to X\) by

\[
h(x) = \begin{cases} 
h_s(x) & \text{if } x \in A \text{ where } A \in A 
\quad \quad x & \text{if } x \in A \text{ where } A \notin A \text{ and } A \neq \overline{a} 
b & \text{if } x \in \overline{a}.
\end{cases}
\]

Case 2. \(\overline{a} \in A\). For each \(A \in X/E\), define \(h : X \to X\) by

\[
h(x) = \begin{cases} 
h_s(x) & \text{if } x \in A \text{ where } A \in \mathcal{A} 
\quad \quad x & \text{if } x \in A \text{ where } A \notin \mathcal{A} \text{ and } A \notin \overline{A}.
\end{cases}
\]

It is routine to show \(h \in EOP_X\), \(f \neq \text{id}_X\), and \(hf = \text{id}_X f\), where \(\text{id}_X\) is the identity transformation on \(X\). We assert that \(a \notin g(X)\). Indeed, if \(g(x') = a\) for some \(x' \in X\), then applying the characterization of \(\mathcal{R}^\ast\) from Lemma 2.1, we have \(hg = \text{id}_X g\) and \(hg(x') = \text{id}_X g(x')\). If \(\overline{a} \notin A\), then

\[
b = h(\overline{a}) = hg(x') = \text{id}_X g(x') = a,
\]

a contradiction. If \(\overline{a} \in A\), then

\[
h_s g(x') = hg(x') = \text{id}_X g(x') = a \in f(X),
\]

a contradiction. It follows readily that \(a \notin g(X)\). This means that \(g(X) \subseteq f(X)\). By symmetry, \(f(X) \subseteq g(X)\). Consequently, \(f(X) = g(X)\), as required.

Let \(Y, Z \subseteq X\) and \(Y \cap Z = \emptyset\). \(Y < Z\) means that \(y < z\) for any \(y \in Y\) and \(z \in Z\).
Lemma 2.4 Let \( f \in EOP_X \). Then \((f, e) \in R^*\) for some idempotent \( e \in EOP_X \). Consequently, the semigroup \( EOP_X \) is left abundant.

**Proof** Assume that

\[
\{ A \in X/E : A \cap f(X) \neq \emptyset \} = \{ A_1 < A_2 < \ldots < A_t \}.
\]

For each \( A_i (1 \leq i \leq t) \), let \( A_i \cap f(X) = \{ a_{i1} < a_{i2} < \ldots < a_{is} \} \). Write \( a_{i0} = \min A_i \) and \( a_{is} = \max A_i \) and then define \( e_i : A_i \to A_i \) by

\[
e_i(x) = \begin{cases} a_{i1} & \text{if } x \in [a_{i0}, a_{i1}] \\ a_{il} & \text{if } x \in (a_{il-1}, a_{il}) (2 \leq l \leq s) \\ a_{is} & \text{if } x \in (a_{is}, a_{is}].
\end{cases}
\]

For every \( A \in X/E \), define \( e : X \to X \) by

\[
e(x) = \begin{cases} e_i(x) & \text{if } x \in A_i \ (1 \leq i \leq t) \\ a_{i1} & \text{if } x \in A \text{ where } \overline{A} < A_1 \\ a_{i1} & \text{if } x \in A \text{ where } A_{i-1} < A < A_i \ (2 \leq i \leq t) \\ a_{ts} & \text{if } x \in A \text{ where } A_i < \overline{A} \leq \overline{A}.
\end{cases}
\]

It is routine to show \( e \in EOP_X \), \( e^2 = e \), and \( e(X) = f(X) \). By Lemma 2.3, we have \((e, f) \in R^*\). \( \square \)

In general, the semigroup \( EOP_X \) is not right abundant; that is, there may be no idempotents in some \( L^* \)-class of \( EOP_X \). In what follows we pursue a necessary and sufficient condition under which the semigroup \( EOP_X \) is abundant. For \( f \in T_X \), let \( \pi(f) \) be the partition of \( X \) induced by \( kerf \), namely

\[
\pi(f) = \{ f^{-1}(y) : y \in f(X) \},
\]

and call \( f^{-1}(y) \) a kerf-class. For each \( f \in T_E(X) \), let \( E_f = E \lor kerf \). Then \( E_f \) is the smallest equivalence relation on \( X \) containing both \( E \) and \( kerf \) and each \( E_f \)-class is a union of \( E \)-classes as well as a union of kerf-classes. Moreover, \( f(F) \subseteq A \in X/E \) for each \( E_f \)-class \( F \).

Recall that, in [1], a transformation \( f \) is said to be normal if for each \( E_f \) class \( F \), there is some \( E \)-class \( A \subseteq F \) such that \( A \cap K \neq \emptyset \) for each kerf-class \( K \subseteq F \).

**Lemma 2.5** Let \( e \in EOP_X \) be an idempotent. Then \( e \) is normal.

**Proof** The proof is similar to that of [8, Lemma 2.8] and it is omitted. \( \square \)

**Lemma 2.6** Let \( f \in EOP_X \). Then the following statements hold.

1. \( f \) is normal if and only if there is an idempotent \( e \in EOP_X \) such that \( kerf = ker \).

2. The semigroup \( EOP_X \) is abundant if and only if \( f \) is normal.

**Proof** (1) For the ‘if’ part, suppose that \( kerf = ker e \) for some idempotent \( e \in EOP_X \). It is clear that \( E_f = E_e \) and \( f \) is normal.

For the ‘only if’ part, suppose that \( f \) is normal. For each \( E_f \)-class \( F \), there is some \( E \)-class \( A \) such that \( A \cap K \neq \emptyset \) for each kerf-class contained in \( F \). Take \( k \in A \cap K \) and define \( e : K \to K \) by \( e(K) = k \). To see \( e \in EOP_X \), take \( E \)-class \( B \subseteq F \) and \( x, y \in B, x \leq y \). Obviously, \( e(B) \subseteq e(F) \subseteq A \), which implies that \((e(x), e(y)) \in E \). Now assume that \( x \in K_x \) and \( y \in K_y \) where \( K_x, K_y \in \pi(f) \). If \( K_x = K_y \), then
\[ e(x) = e(y). \] If \( K_x \neq K_y \), then \( x \neq y \) and \( f(x) < f(y) \). By the definition of \( e \), we have \( e(x) = k_x \) and \( e(y) = k_y \) where \( k_x \in A \cap K_x \) and \( k_y \in A \cap K_y \). Now we assert that \( k_x < k_y \). Indeed, if \( k_x > k_y \), then \( f(x) = f(k_x) > f(k_y) = f(y) \), which leads to a contradiction. Hence, \( k_x < k_y \) and \( e \in EOP_X \). It is routine to show that \( e^2 = e \) and \( \ker e = \ker f \).

(2) The proof is similar to that of [8, Theorem 2.10] and it is also omitted. \( \Box \)

Recall that, in [1], an equivalence relation \( E \) on \( X \) is said to be \textit{simple} if there is exactly one \( E \)-class \((\neq X)\) containing more than one point and the other \( E \)-classes are singletons, and \( E \) is said to be \textit{n-bounded} if the cardinality of each \( E \)-class is not more than \( n \).

**Lemma 2.7** Let \( E \) be an equivalence relation on \( X \). Then the following statements hold.

(1) If \( E \) is either simple or 2-bounded, then each \( f \in EOP_X \) is normal.

(2) If \( E \) is neither simple nor 2-bounded, then \( EOP_X \) is not abundant.

**Proof** (1) The proof is similar to that of Lemmas 2.12 and 2.13 of [8].

(2) Assume that \( A = \{a_1 < a_2 < \ldots < a_s\} \in X/E \) and \( B = \{b_1 < b_2 < \ldots < b_t\} \in X/E \) for \( s \geq 3, t \geq 2 \). Now define \( f : X \to X \) by

\[
 f(x) = \begin{cases} 
 a_1 & \text{if } x = a_1 \\
 a_2 & \text{if } x \in \{a_2, a_3, \ldots, a_s, b_1\} \\
 a_3 & \text{if } x \in \{b_2, b_3, \ldots, b_t\} \\
 x & \text{otherwise.}
\end{cases}
\]

It is clear that \( f \in EOP_X \) and all \( E_f \)-class are \( F = A \cup B \) and \( C \in X/E \) with \( C \neq A, C \neq B \). Moreover, there are exactly three \( \ker f \)-classes \( K_1, K_2, \) and \( K_3 \) contained in \( F \), where

\[
 K_1 = \{a_1\}, \ K_2 = \{a_2, a_3, \ldots, a_s, b_1\}, \ K_3 = \{b_2, b_3, \ldots, b_t\}.
\]

However, there is no \( E \)-class \( D \subseteq F \) such that \( D \cap K_i \neq \emptyset \) for \( i = 1, 2, 3 \), so \( f \) is not normal. Therefore, \( EOP_X \) is not abundant. \( \Box \)

Clearly, if \( |X| = 3 \), then \( E \) is both simple and 2-bounded, so the semigroup \( EOP_X \) is abundant. If \( |X| = 4 \), then \( E \) is either simple or 2-bounded and the semigroup \( EOP_X \) is also abundant. Thus, we have the main result in this paper.

**Theorem 2.8** Let \( E \) be a nontrivial equivalence on the finite totally ordered set \( X = \{1 < 2 < \ldots < n\} \) \((n \geq 3)\). Then the following statements hold.

(1) If \( |X| = 3 \) or \( |X| = 4 \), then the semigroup \( EOP_X \) is abundant.

(2) If \( |X| \geq 5 \), then the semigroup \( EOP_X \) is abundant if and only if \( E \) is either simple or 2-bounded.

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References


