An extension of Cline’s formula for a generalized Drazin inverse

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Abstract: In this note we give an answer to a question recently posed by Zeng and Zhong, to note that Cline’s formula for a generalized Drazin inverse extends to the case when \( aba = aca \). Cline’s formula for a pseudo Drazin inverse is also presented in this case.

Key words: Cline’s formula, generalized Drazin inverse, pseudo Drazin inverse, quasinilpotent

1. Introduction

Let \( R \) be an associative ring with identity 1, and \( R^{-1} \) and \( J(R) \) denote, respectively, invertible group and Jacobson radical of \( R \). For \( a \in R \), the commutant and double commutant of \( a \) are defined by

\[
\text{comm}(a) = \{ x \in R, ax = xa \}
\]

and

\[
\text{comm}^2(a) = \{ x \in R, xy = yx \text{ for all } y \in \text{comm}(a) \}.
\]

Drazin [3] introduced the notion of a Drazin inverse in a ring in 1958. An element \( a \in R \) is said to be Drazin invertible if there exist \( b \in R \) and \( k \in \mathbb{N} \) such that

\[
b \in \text{comm}(a), \ bab = b \text{ and } a^k ba = a^k.
\]

In this case \( b \) is unique if it exists and is called a Drazin inverse of \( a \), denoted by \( b = a^D \), and the least nonnegative integer \( k \) satisfying \( a^k ba^k = a^k \) is called the Drazin index \( i(a) \) of \( a \). According to Drazin [3, Theorem 1], \( a^D \in \text{comm}^2(a) \).

Following Harte [4], an element \( a \in R \) is said to be quasinilpotent if \( 1 + ax \) is invertible for all \( x \in \text{comm}(a) \). Using this concept, Koliha and Patrício [6] introduced the notion of a generalized Drazin inverse in a ring in 2002. An element \( a \in R \) is said to be generalized Drazin invertible if there exists \( b \in R \) such that

\[
b \in \text{comm}^2(a), \ bab = b \text{ and } aba - a \text{ is quasinilpotent.} \tag{1.1}
\]
In this case $b$ is unique if it exists and is called a \textit{generalized Drazin inverse} of $a$, denoted by $b = a^{D}$. Moreover, in the Banach algebra case, the condition $b \in \text{comm}^2(a)$ in (1.1) can be weaken as $b \in \text{comm}(a)$ (see [5], Theorem 4.4)).

In 2012, Wang and Chen [9] introduced an intermedium between Drazin inverse and generalized Drazin inverse. An element $a \in R$ is said to be \textit{pseudo Drazin invertible} if there exist $b \in R$ and $k \in \mathbb{N}$ such that

$$b \in \text{comm}^2(a), \quad bab = b \quad \text{and} \quad a^k ba - a^k \in J(R).$$

In this case $b$ is unique if it exists and is called a \textit{pseudo Drazin inverse} of $a$, denoted by $b = a^{pD}$. Moreover, in the Banach algebra case, the condition $b \in \text{comm}^2(a)$ in (1.2) can be weakened as $b \in \text{comm}(a)$ (see [9], Remark 5.1)).

In 1965, Cline [1] showed that if $ab$ is Drazin invertible then so is $ba$ and in this case

$$(ba)^D = b((ab)^D)^2a.$$ 

This equation is now known as Cline’s formula. It plays an important role in finding the Drazin inverse of a sum of two elements and that of a block matrix (see [8]). Generalizations of Cline’s formula for generalized Drazin inverse and pseudo Drazin inverse were recently proved in [7] and [9], respectively. Their proof relied on the bridge “quasipolar” and “pseudopolar”, respectively.

As extensions of Jacobson’s lemma, in 2013 Corach et al. [2] firstly investigated common properties of $ac - 1$ and $ba - 1$ in the algebraic viewpoint and also obtained some interesting topological analogues under the assumption

$$aba = aca,$$

where $a, b, c \in R$. Recently, Zeng and Zhong [10] extended Cline’s formula for the Drazin inverse in a ring to the case when $aba = aca$. However, Cline’s formula for a generalized Drazin inverse in this case was established only in the setting of Banach algebra, and in a ring it left open at that time.

In this note, we establish Cline’s formula for the generalized Drazin inverse in a ring in the case when $aba = aca$, answering a question posed in [10], Question 2.15. We also present Cline’s formula for the pseudo Drazin inverse in this case. The proofs for the general case are more direct and slightly more technical than those in [7] and/or [9] for the special case $b = c$.

2. Main results

We start with the following well-known Jacobson’s lemma.

\textbf{Lemma 2.1} (Jacobson’s lemma) Let $a, b \in R$. Then

$$1 + ab \text{ is invertible } \iff 1 + ba \text{ is invertible}.$$ 

In this case, we have $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.

\textbf{Lemma 2.2} Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then

$$ac \text{ is quasinilpotent } \iff ba \text{ is quasinilpotent}.$$
Proof Suppose that \(ac\) is quasinilpotent. Then for all \(x \in \text{comm}(ac), 1+xac\) is invertible. Let \(y \in \text{comm}(ba)\). Since
\[
(ay^3bac)(ac) = (ay^3bab)(ac) = (abay^3b)(ac) = (ac)(ay^3bac),
\]
\(1+(ay^3bac)(ac)\) is invertible. Therefore, by Lemma 2.1, we have
\[
(1 + yba)(1 - yba + y^2bab) = 1 + y^3bababa = 1 + y^3bacaca
\]
is invertible. Noting that \(1+yba\) commutes with \(1-yba+y^2baba\), we see that \(1+yba\) is invertible. Consequently, \(ba\) is quasinilpotent.

For the special case \(b = c\), the previous paragraph shows that if \(ab\) is quasinilpotent, then \(ba\) is quasinilpotent. Then interchanging \(a\) and \(b\), we find an equivalence statement: \(ab\) is quasinilpotent if and only if \(ba\) is quasinilpotent, for any \(a, b \in R\).

We are now ready to answer Question 2.15 in [10] affirmatively.

Theorem 2.3 Suppose that \(a, b, c \in R\) satisfy \(aba = acb\). Then
\(ac\) is generalized Drazin invertible \(\iff ba\) is generalized Drazin invertible.

In this case, we have \((ba)^gD = b((ac)^gD)^2a\) and \((ac)^gD = a((ba)^gD)^2c\).

Proof Suppose that \(ac\) is generalized Drazin invertible and let \(d = (ac)^gD\). Then
\(d \in \text{comm}^2(ac), d(ac)d = d\) and \((ac)d(ac) - ac\) is quasinilpotent.

Put
\[e = bd^2a.\]
In order to prove that \(e = (ba)^gD\), it needs to be shown that

(i) \(e \in \text{comm}^2(ba), (ii) e(ba)e = e\) and (iii) \((ba)e(ba) - ba\) is quasinilpotent.

(i) Let \(f \in \text{comm}(ba)\). Then we have
\[
f = fbd^2a = f(b(acacd^4)a) = f(babacd^4a) = babacd^4a = b(acacfd^4a) = b(acacfd^4a).
\]
Since
\[
ac(acacfd) = ababac = afbabac = afbacac = abafcac = (acaft)cac
\]
and \(d \in \text{comm}^2(ac), (acacf)d = d(acacf).\)

Therefore, putting (2.2) into (2.1), we get
\[
f = b(acacfd^4a) = bd^4(acacfd)\]
\[
= bd^4abac = bd^4abaca
\]
\[
= bd^4afba = bd^4ababa
\]
\[
= bd^4faca = bd^2af
\]
\[
= ef.
\]
(ii) We have \( e(ba)e = bd^2a(ba)bd^2a = bd^2ababacd^3a = bd^2(ac)^3d^3a = bd^2a = e. \)

(iii) Let \( p = 1 - acd. \) Then \( pac \) is quasinilpotent. Note that

\[
ba - (ba)^2e = ba - babab^2a = ba - bababacd^2da = ba - bacacacd^2da = b(1 - acd)a = bpa.
\]

Since \( abpa = ab(1 - acd)a = ac(1 - acd)a = acpa \), \((pa)b(pa) = (pa)c(pa).\) Therefore, by Lemma 2.2, we conclude that \( ba - (ba)^2e \) is quasinilpotent. Consequently,

\[
(ba)^pD = b((ac)^pD)^2a. \tag{2.3}
\]

For the special case “\( b = c \),” (2.3) shows that

\[
(ba)^pD = b((ab)^pD)^2a \quad \text{and} \quad (ca)^pD = c((ac)^pD)^2a,
\]

for any \( a, b, c \in R. \) Then interchanging \( a \) and \( b \), and \( a \) and \( c \) in the above formulae, we find that

\[
(ab)^pD = a((ba)^pD)^2b \quad \text{and} \quad (ac)^pD = a((ca)^pD)^2c. \tag{2.4}
\]

Now interchanging \( b \) and \( c \) in (2.3), we infer that

\[
(ca)^pD = c((ab)^pD)^2a. \tag{2.5}
\]

Therefore, taking (2.4) and (2.5) together, we get

\[
(ac)^pD = a((ca)^pD)^2c = ac((ab)^pD)^2ac((ab)^pD)^2ac = aca((ba)^pD)^2ba((ba)^pD)^2ba((ba)^pD)^2bac = a((ba)^pD)^2c.
\]

This completes the proof. \( \Box \)

**Theorem 2.4** Suppose that \( a, b, c \in R \) satisfy \( aba = aca. \) Then

\[
ac \text{ is pseudo Drazin invertible } \iff \text{ba is pseudo Drazin invertible.}
\]

In this case, we have

1. \(|i(ac) - i(ba)| \leq 1; \)
2. \((ba)^pD = b((ac)^pD)^2a \) and \((ac)^pD = a((ba)^pD)^2c. \)

**Proof** Let \( d \) be the pseudo Drazin inverse of \( ac \) and let \( k = i(ac). \) Then

\[
d \in \text{comm}^2(ac), \ d(ac)d = d \quad \text{and} \quad (ac)^kd(ac) - (ac)^k \in J(R).
\]

Put

\[
e = bd^2a.
\]

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As in the proof of Theorem 2.3, we get $e \in \text{comm}^2(ba)$ and $e(ba)e = e$. Moreover, since $(ac)^k d(ac) - (ac)^k \in J(R)$,

\[(ba)^{k+1}e(ba) - (ba)^{k+1} = (ba)^{k+1}bd^2aba - (ba)^{k+1}
\]
\[= (ba)^{k+1}bd^2aca - b(ac)^k a
\]
\[= (ba)^{k+1}bacd^2 a - b(ac)^k a
\]
\[= b(ac)^{k+1}acd^2 a - b(ac)^k a
\]
\[= b(ac)^{k+1}da - b(ac)^k a
\]
\[= b((ac)^k d(ac) - (ac)^k) a \in J(R).
\]

Therefore, $ba$ is pseudo Drazin invertible, $(ba)^{pD} = b((ac)^{pD})^2 a$, and $i(ba) \leq i(ac) + 1$.

By similar arguments as above, one can show that if $ba$ is pseudo Drazin invertible, then $ac$ is pseudo Drazin invertible, $(ac)^{pD} = a((ba)^{pD})^2 c$, and $i(ac) \leq i(ba) + 1$. $\square$

References