Notes on cotorsion dimension of Hopf–Galois extensions

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Abstract: Let $H$ be a finite dimensional Hopf algebra over a field $k$ and $A/B$ be a right $H$-Galois extension. In this note the relationship of cotorsion dimensions between $A$ and $B$ will be studied. We prove that $r.cot.D(A) \leq r.cot.D(B) + l.D(H)$. Moreover, we give some sufficient conditions for which $r.cot.D(A) = r.cot.D(B)$. As applications, we obtain some results about cotorsion dimension of the smash product.

Key words: Hopf–Galois extension, cotorsion dimension, smash product

1. Introduction and preliminaries

Throughout this paper, $k$ denotes a fixed field, and we will always work over $k$. The tensor product $\otimes = \otimes_k$ and Hom is always assumed to be over $k$. For an algebra $A$, denote by Mod-$A$ the category of right $A$-modules. We write $M_A$ to indicate a right $A$-module. For an $A$-module $M$, let $pd(M)$ and $id(M)$ denote the projective dimension and the injective dimension of $M$, respectively. We refer the reader to [12] for details about Hopf algebras.

The definition of Hopf–Galois extension has its roots in the Chase–Harrison–Rosenberg approach to Galois theory for groups acting on commutative rings (see [2]). In 1969 Chase and Sweedler extended these ideas to coaction of a Hopf algebra $H$ acting on a commutative $k$-algebra, for $k$ a commutative ring (see [3]); the general definition appeared in [8] in 1981. Hopf–Galois extensions also generalize strongly graded algebras (here $H$ is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted enveloping algebra of a restricted Lie algebra), twisted group rings $R*G$ of a group $G$ acting on a ring $R$, and so on.

Let $H$ be a Hopf algebra over a field $k$ and $A$ be a right $H$-comodule algebra, i.e. $A$ is a $k$-algebra together with an $H$-comodule structure $\rho_A : A \rightarrow A \otimes H$ (with notation $a \mapsto a_0 \otimes a_1$) such that $\rho_A$ is an algebra map. Let $B$ be the subalgebra of the $H$-coinvariant elements, $B := A^{coH} := \{ a \in A \mid \rho_A(a) = a \otimes 1 \}$. Then the extension $A/B$ is called right $H$-Galois if the map $\beta : A \otimes_B A \rightarrow A \otimes H$, given by $a \otimes_B b \mapsto (a \otimes 1)\rho(b)$, is bijective. For more details and unexplained concepts we refer the reader to [12].

Let $R$ be a ring. For any right $R$-module $M$, the cotorsion dimension $cd(M)$ of $M$ is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}_R(F, M) = 0$ for any flat right $R$-module $F$. If there is no such
n, set \( \text{cd}(M) = \infty \). The right global cotorsion dimension \( r.\text{cot.D}(R) \) of \( R \) is defined as the supremum of the cotorsion dimensions of right \( R \)-modules (see [11]).

Recall from [6] that \( M \) is called cotorsion if \( \text{Ext}^1_R(F, M) = 0 \) for any flat right \( R \)-module \( F \), i.e. \( \text{cd}(M) = 0 \). So the cotorsion dimension of \( M \) measures how far away a module is from being cotorsion. The class of cotorsion modules contains all pure-injective (hence injective) modules. Using it, some new characterizations of right perfect rings and von Neumann regular rings can be given (see [10]).

The aim of this paper is to study the relationship of cotorsion dimensions of Hopf-Galois extensions. We will prove the following two main results:

1. Let \( A/B \) be a right \( H \)-Galois extension for a finite dimensional Hopf algebra \( H \). Then
   \[
   r.\text{cot.D}(A) \leq r.\text{cot.D}(B) + \ell.D(H).
   \]

2. Let \( H \) be a finite dimensional Hopf algebra that is semisimple as well as its dual \( H^* \) (here \( H^* = \text{Hom}(H, k) \)), and \( A/B \) be a right faithfully flat \( H \)-Galois extension. Then
   \[
   r.\text{cot.D}(A) = r.\text{cot.D}(B).
   \]

2. The main results, and their proof and corollaries

Let \( A/B \) be a right \( H \)-Galois extension. Consider the following two functors:

\[
- \otimes_B A : \text{Mod-}B \to \text{Mod-}A, \quad M \mapsto M \otimes_B A, \quad (-)_B : \text{Mod-}A \to \text{Mod-}B, \quad M_A \mapsto M_B,
\]

where \((-)_B\) is the restriction functor.

**Lemma 2.1** Let \( A/B \) be a right \( H \)-Galois extension for a finite dimensional Hopf algebra \( H \). Then \((- \otimes_B A, (-)_B)\) and \((-)_B, (-) \otimes_B A\) are both adjoint pairs.

**Proof** By adjoint isomorphism theorem, \((- \otimes_B A, (-)_B)\) is an adjoint pair. By Theorem 5 in [5], \((-)_B, (-) \otimes_B A\) is also an adjoint pair. \(\square\)

**Remark 2.2** Let \((F, G)\) be an adjoint pair of functors of abelian categories. Then \(F\) is right exact and \(G\) is left exact. If \(G\) is exact, then \(F\) preserves projective objects; if \(F\) is exact, then \(G\) preserves injective objects. Thus, by Lemma 2.1, the above functors \(- \otimes_B A\) and \((-)_B\) are both exact, and so they preserve projective objects and injective objects.

By Lemma 2.1 and the Remark, we immediately get the following lemma.

**Lemma 2.3** Let \( A/B \) be a right \( H \)-Galois extension for a finite dimensional Hopf algebra \( H \) and \( P \) be a right \( A \)-module. Then:

1. \( P_A \) being projective implies \( P_B \) and \( P \otimes_B A \) are both projective;
2. \( P_A \) being injective implies \( P_B \) and \( P \otimes_B A \) are both injective;
Lemma 2.4 (Lemma 3.1 of [9]) Let $A/B$ be a right $H$-Galois extension for a semisimple Hopf algebra $H$. Then for any right $A$-module $M$, $M$ is an $A$-direct summand of $M \otimes_B A$.

The following lemma gives another equivalent definition of the right global cotorsion dimension of a ring $R$ proved in Theorem 7.2.5 of [11].

Lemma 2.5 Let $R$ be a ring. Then

$$r.cot.D(R) = \sup \{pd(F) | F \text{ is a flat right } R\text{-module}\}.$$ 

Lemma 2.6 Let $A/B$ be a right $H$-Galois extension for a semisimple Hopf algebra $H$. Then for any flat right $A$-module $F$, $pd(F_A) = pd(F_B)$.

Proof First, by Lemma 2.2 and the Remark, any projective resolution of $F_A$ is also a projective resolution of $F_B$. It follows that $pd(F_B) \leq pd(F_A)$.

Conversely, we may assume that $pd(F_B) = n < \infty$, and let $P$ be a projective resolution of $F_B$ of length $n$. Then by Lemma 2.2 and the Remark, $P \otimes_B A$ is a projective resolution of $F \otimes_B A$ as a right $A$-module. This implies $pd((F \otimes_B A)_A) \leq pd(F_B)$. Also by Lemma 2.3, $F$ is an $A$-direct summand of $F \otimes_B A$, and it follows that $pd(F_A) \leq pd((F \otimes_B A)_A)$. Thus, $pd(F_A) \leq pd(F_B)$. The proof is completed. \hfill \Box

Combining Lemma 2.4 and Lemma 2.5, we immediately obtain the following result.

Proposition 2.7 Let $A/B$ be a right $H$-Galois extension for a semisimple Hopf algebra $H$. Then

$$r.cot.D(A) \leq r.cot.D(B).$$

Now we want to discuss when the right global cotorsion dimension of $A$ is equal to that of $B$.

First we introduce the definitions of smash products. Let $H$ be a Hopf algebra and $A$ be a left $H$-module algebra, i.e. $A$ is a $k$-algebra together with an $H$-module structure $\cdot : H \otimes A \rightarrow A$ (with notation $h \otimes a \rightarrow h \cdot a$) such that $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$, for all $a, b \in A$ and $h \in H$. Then the smash product algebra $A\#H$ is the set $A \otimes H$ as a vector space, with multiplication

$$(a \# h)(b \# k) = a(h_1 \cdot b) \# h_2 k_2$$

for $a, b \in A, h, k \in H$. Here we write $a \# h$ for the element $a \otimes h$ (see [12]).

In [4], the authors discussed the cotorsion dimension of the smash product $A\#H$. Let $H$ be a finite dimensional Hopf algebra and $A$ be a left $H$-module algebra. They proved that

$$l.cot.D(A\#H) \leq l.cot.D(A) + r.D(H),$$

where $l.cot.D(A)$ is the left global cotorsion dimension of $A$ and $r.D(H)$ is the right global dimension of $H$.

Let $A\#H$ be a smash product. It is well known that $A\#H/A$ is a right $H$-Galois extension (see [12]). In the following, we prove that the above result is also true for the Hopf-Galois extension and we give the right version.
Theorem 2.8 Let $A/B$ be a right $H$-Galois extension for a finite dimensional Hopf algebra $H$. Then

$$r.cot.D(A) \leq r.cot.D(B) + l.D(H).$$

Proof Compared to Proposition 2.6, we mainly discuss the left global dimension of $H : l.D(H)$. Since $H$ is finite dimensional, by Theorem 2.1.3 of [12] $H$ is a Frobenius algebra. It follows that the projective modules of $H$ and injective modules of $H$ coincide. So for any $H$-module $M$, $pd(M) = 0$ or $\infty$. Indeed, let $\mathcal{P}$ be a projective resolution of $M$ of length $n$, denoting

$$\mathcal{P} : 0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Set $K_i = \text{Ker} d_i$ (the kernel of $d_i$), $i = 0, \ldots, n - 1$. Consider the short exact sequence

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow K_{n-1} \rightarrow 0.$$

Since $P_n$ is injective, this sequence is split; that is, $P_{n-1} \cong P_n \oplus K_{n-1}$, and $P_{n-1}$ is also injective, so one can get that $K_{n-1}$ is injective (the category of injective modules is closed under the direct summands). Similarly, we get that $K_i, i = 0, \ldots, n - 1$ are all injective (hence projective). Consequently, one can obtain the following short exact sequence:

$$0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

with $K_0, P_0$ projective. Thus, $pd(M) = 0$.

From all of the above, $l.D(H) = 0$ or $\infty$. If $l.D(H) = 0$, then $H$ is semisimple and this theorem is just Proposition 2.6. If $l.D(H) = \infty$, then this theorem is obviously satisfied. The proof is completed. \qed

Now we give a duality theorem of Hopf-Galois extensions. Let $H$ be a finite dimensional Hopf algebra. Then a right $H$-comodule algebra $A$ corresponds to a left $H^*$-module algebra $A$ via $f \mapsto a = a_0 < f, a_1 >$ (see [12]). Thus, $A$ and $H^*$ form a smash product algebra $A \# H^*$. Let $A/B$ be a right $H$-Galois extension for a finite dimensional Hopf algebra $H$. From Theorem 8.3.3 of [12], there is a canonical isomorphism between the smash product algebra $A \# H^*$ and the endomorphism algebra $\text{End}_A B$; that is, $A \# H^* \cong \text{End}_A B$, where the right $B$-module action on $A$ is the multiplication.

Lemma 2.9 Let $A/B$ be a right $H$-Galois extension for a finite dimensional Hopf algebra $H$. If $A/B$ is faithfully flat, then $A \# H^*$ is Morita equivalent to $B$.

Proof By the above, $A \# H^* \cong \text{End}_A B$. Since $A/B$ is right faithfully flat, by [12] or the right version of Theorem 2.6 in [1], we obtain that $A$ is a right $B$-progenerator. Hence, $A \# H^*$ is Morita equivalent to $B$. \qed

Note that there are many examples of faithfully flat Hopf-Galois extensions (cf. [14]). For example, the smash product extension $A \# H/A$ is a right faithfully flat $H$-Galois extension. In [14], the author studied the representation theory of the faithfully flat Hopf-Galois extension.

Now we obtain the main result as follows.

Theorem 2.10 Let $H$ be a finite dimensional Hopf algebra that is semisimple as well as its dual $H^*$, and $A/B$ be a right faithfully flat $H$-Galois extension. Then:

1. $r.cot.D(A) = r.cot.D(B)$.
(2) A is right perfect if and only if so is B.

Proof

(1) First, by Proposition 2.6, r.cot.D(A) ≤ r.cot.D(B).

Next we consider the smash product algebra A#H*. Since A#H*/A is a right H*-Galois extension, combining the semisimplicity of H*, we have r.cot.D(A#H*) ≤ r.cot.D(A). Since A/B is faithfully flat, by Lemma 2.8, A#H* is Morita equivalent to B. It follows that r.cot.D(B) = r.cot.D(A#H*). Then

r.cot.D(A) ≤ r.cot.D(B).

Therefore, r.cot.D(A) = r.cot.D(B).

(2) It immediately follows from (1) since A is right perfect if and only if r.cot.D(A) = 0 by Corollary 7.2.7 of [11].

Let A#H be a smash product. Then A#H/A is a right faithfully flat H-Galois extension, and so we have the following corollary.

Corollary 2.11 Let H be a finite dimensional Hopf algebra that is semisimple as well as its dual H*, and A#H be a smash product. Then

r.cot.D(A#H) = r.cot.D(A).

Note that the result of the above corollary is also true for the crossed product A#H, which are generalizations of the smash products (for the definition of the crossed product, see Definition 7.1.1 of [12]), since the crossed product extension A#H/A is also a right faithfully flat H-Galois extension (see [14]).

Let A/B be a right H-Galois extension. We now give another sufficient condition for which r.cot.D(A) = r.cot.D(B) using separable functor.

Now we recall the definition of a separable functor. Let C and D be two categories and F : C → D be a covariant functor. F induces a natural transformation


We say that F is a separable functor if F splits, i.e. we have a natural transformation

P : Hom_D(F(·), F(·)) → Hom_C(·, ·)

such that

P ◦ F = 1_{Hom_C(·, ·)}

the identity natural transformation on Hom_C(·, ·). The more explicit form of the definition can be found in [13] in which separable functors were first introduced.

The terminology comes from the fact that, for a ring extension R → S, the restriction functor (−)_R is separable if and only if the extension S/R is separable.

Lemma 2.12 Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. If − ⊗_B A is separable, then for any right B-module M, M is a B-direct summand of M ⊗_B A.
Proof Consider the adjoint pair \((- \otimes_B A, (\_)_B\)). If the functor \(- \otimes_B A\) is separable, then we obtain by Proposition 5 of [7] that the natural map \(\eta_M : M_B \to (M \otimes_B A)_B\) is a split monomorphism for every \(M \in \text{Mod-} B\). □

**Proposition 2.13** Let \(A/B\) be a right \(H\)-Galois extension for a semisimple Hopf algebra \(H\). If \(- \otimes_B A\) is separable, then

\[
\text{r.cot.D}(A) = \text{r.cot.D}(B). 
\]

**Proof** First, by Proposition 2.6, \(\text{r.cot.D}(A) \leq \text{r.cot.D}(B)\).

Next we prove that \(\text{r.cot.D}(B) \leq \text{r.cot.D}(A)\). For this, by Lemma 2.2 and Lemma 2.4 we only need to show that for any flat right \(B\)-module \(F\), \(\text{pd}(F_B) = \text{pd}((F \otimes_B A)_A)\). It is clear that \(\text{pd}((F \otimes_B A)_B) \leq \text{pd}((F \otimes_B A)_A)\) and \(\text{pd}((F \otimes_B A)_A) \leq \text{pd}(F_B)\) by Lemma 2.2 and the Remark. Also by Lemma 2.11, \(F\) is a \(B\)-direct summand of \(F \otimes_B A\), and it follows that \(\text{pd}(F_B) \leq \text{pd}((F \otimes_B A)_B)\). The proof is completed. □

Finally, we remark here that the left global cotorsion dimension and the right global cotorsion dimension of a finite dimensional Hopf algebra \(H\) are both equal to 0. Indeed, since \(H\) is finite dimensional, it follows that \(H\) is left and right Noetherian and \(\text{id}_H(H) = \text{id}_H(H) = 0\) (note that \(H\) is a Frobenius algebra, and so the projective modules of \(H\) and injective modules of \(H\) coincide). Hence, \(H\) is a 0-Gorenstein algebra. By Proposition 7.2.12 of [11], \(\text{l.cot.D}(H) = \text{r.cot.D}(H) = 0\).

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**References**


